Trigonometry and Angles

Math Background

Previously, you

- Simplified linear, quadratic, radical, polynomial, rational, log and exponential functions
- Solved right triangles using trigonometry and special right triangles
- Evaluated trigonometric functions with and without technology and using the unit circle.
- Performed arithmetic operations with linear, quadratic, radical, log and exponential functions
- Identified the domain, range and x-intercepts of real-life functions
- Graphed linear, quadratic and polynomial functions
- Transformed parent functions of linear, quadratic, radical, polynomial, rational, log & exponential functions

In this unit you will

- Solve triangles using the Law of Sines and Law of Cosines
- Use Heron's formula
- Explore the graphs of trigonometric functions
- Graph the sine, cosine and tangent functions

You can use the skills in this unit to

- Solve triangles that are not right triangles and explore the ambiguous case
- Find the area of a triangle given side-angle-side information or side-side-side information
- Graph trigonometric functions with and without technology
- Describe how a trigonometric function graph is related to its parent function through transformations.
- Determine the domain and range for a sinusoidal model
- Model periodic phenomena using trigonometric functions and use the model to predict and explain events within the data set

Vocabulary

- **Altitude** – The perpendicular distance from the base of the triangle to its opposite vertex.
- **Ambiguous case of the Law of Sines –** A situation in which occurs when determining a triangle when two sides and an angle (other than the included angle) are known. It can produce no triangles, one triangle or two triangles.
- **Amplitude** – The amplitude of a sinusoidal function is one-half of the positive distance between the maximum and minimum value of a function.
- **Asymptote** – A straight line that a curve approaches more and more closely but never touches as the curve goes off to infinity in one direction.
- **Constraints –** A condition of an optimization problem that the solution must satisfy.
- **Even Function –** A function $f(x)$ that satisfies the property that $f(-x)=f(x)$.
- **Frequency** – The number of cycles the function completes in a given interval. $frequency = \frac{1}{period}$
- **Heron's Formula –** A formula for calculating the area of a triangle when the three sides a, b, and c are known.

- **Law of Cosines –** It allows us to calculate the third missing side when two sides and the angle in between are known on a triangle.

- **Law of Sines** – It states that in any triangle, the ratio of any side to the sine of the opposite angle is constant.
- **Midline** – The equation of the midline of a periodic function is the average of the maximum and minimum values of the function. It is a horizontal axis that is used as the reference line about which the graph of a periodic function oscillates.
- **Oblique –** A triangle that does not contain a right angle.
- **Odd Function** – Any function $f(x)$ that satisfies the property that $f(-x)=-f(x)$.
- **Period** – The period of a function is the horizontal length of one complete cycle.
- **Periodic function** – It is an oscillating (wave-like) function which repeats a pattern of y-values at regular intervals..
- **Phase Shift** – The amount of horizontal shift in a sinusoidal wave.
- **Sinusoid** – The name given to any curve that can be written in the form: $y = A\sin(B(x-C))+D$
- **Supplementary Angles** – Two angles whose sum is equal to 180 degrees.

Essential Questions

- How can the missing sides of a non-right triangle be determined?
- How can trigonometric functions be used to derive the area of a triangle?
- What are the characteristics of a trigonometric function?
- Why is it useful to model real-world problems with equations and graphs?
- What are periodic functions? Why is modeling with them so important?

Overall Big Ideas

Trigonometric relationships can be used to solve non-right triangles. Trigonometric functions can also be used to find the area of a triangle.

Graph a trigonometric function by the period, midline and amplitude of the function. Equations and graphs can help to predict a future outcome of a real-world problem or provide insight in to the problems past. Our world is periodic. The amount of sun light a city receives on a given day, high and low tides, are all real-life instances where sinusoids explain and model real-life phenomena. Thus, the models give us insight into daily events that are of interest to us.

Skill

To use the Law of Sines to solve a triangle.

To use the Law of Cosines to solve a triangle.

To find the area of a triangle given side-angle-side information or by using Heron's Formula.

To recognize and graph sine, cosine and tangent functions.

To recognize and graph transformations of sine, cosine and tangent functions.

To use trigonometric functions to model periodic phenomena.

Related Standards

G.SRT.D.10

Prove the Law of Sines and Cosines and use them to solve problems.

G.SRT.D.11

Understand and apply the Law of Sines and the Law of Cosines to find unknown measurements in right and non-right triangles (e.g., surveying problems, resultant forces).

G.SRT.D.9

Derive the formula $A = \frac{1}{2}ab\sin(C)$ for the area of a triangle by drawing an auxiliary line from a vertex perpendicular to the opposite side.

A.CED.A.2-2

Create equations in two or more variables to represent relationships between quantities and graph equations on coordinate axes with labels and scales. Use all types of equations. *(Modeling Standard)

A.CED.A.3-2

Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or non-viable options in a modeling context. Use all types of equations. For example, represent inequalities describing nutritional and cost constraints on combinations of different foods. *(Modeling Standard)

F.IF.B.5-2

Relate the domain of any function to its graph and, where applicable, to the quantitative relationship it describes. For example, if the function h(n) gives the number of person-hours it takes to assemble *n* engines in a factory, then the positive integers would be an appropriate domain for the function *(Modeling Standard)

F.IF.C.7e-2

Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude. *(Modeling Standard)

F.BF.B.3-2

Identify the effect on the graph of replacing $f(x)$ by $f(x) + \text{k}$, $k\,f(x)$, $f(\text{k}x)$, and $f(x + \text{k})$ for specific values of k (both positive and negative); find the value of *k* given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. Include recognizing even and odd functions from their graphs and algebraic expressions for them. Include simple radical, rational, and exponential functions, note the effect of multiple transformations on a single graph, and emphasize common effects of transformations across function types.

F.TF.B.5

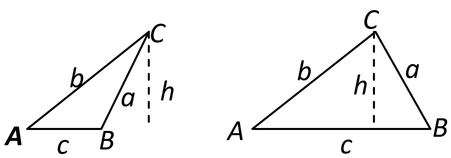
Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline. *(Modeling Standard)

Notes, Examples, and Exam Questions

<u>Unit 8.5: To use the Law of Sines to solve a triangle.</u>

We can use trigonometric ratios to solve right triangles. What if the triangle is oblique (does not contain a right angle)? We can use the Law of Sines and Law of Cosines to solve triangles that are not right triangles.

<u>Deriving the Law of Sines:</u> Consider the two triangles.



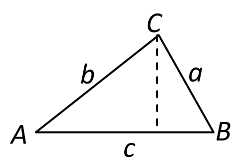
In the acute triangle, $\sin A = \dfrac{h}{b}$ and $\sin B = \dfrac{h}{a}$. In the obtuse triangle, $\sin(180 - B) = \sin B = \dfrac{h}{a}$.

Solve for h. $h = b\sin A$ and $h = a\sin B$

Substitute. $a\sin B = b\sin A$ can be rewritten as $\dfrac{\sin B}{b} = \dfrac{\sin A}{a}$

The same type of argument can be used to show that $\dfrac{\sin C}{c} = \dfrac{\sin B}{b} = \dfrac{\sin A}{a}$.

<u>Law of Sines</u>: The ratio of the sine of an angle to the length of its opposite side is the same for all three angles of any triangle.

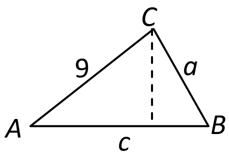


$$\boxed{\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c} \text{ or } \dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}}$$

<u>Solving a Triangle</u>: finding all of the missing sides and angles

Note: The Law of Sines can be used to solve triangles given **AAS** and **ASA**. If you are given two angles and a side or if you are given two angles and the included side, Law of Sines will solve the triangle. SAS and SSS will be dealt with in the next unit with the Law of Cosines.

Ex 1: Solve the triangle ΔABC given that $\angle B = 15^\circ$, $\angle C = 52^\circ$, and $b = 9$. (Figure not drawn to scale)



We are given AAS, so we will use the Law of Sines.

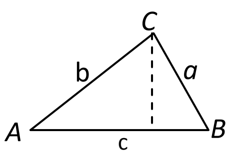
$\frac{\sin B}{b} = \frac{\sin C}{c} \Rightarrow \frac{\sin 15^\circ}{9} = \frac{\sin 52^\circ}{c}$ Solve for c: $c\sin 15^\circ = 9\sin 52^\circ \Rightarrow c = \frac{9\sin 52^\circ}{\sin 15^\circ} \approx 27.4$

Find $m\angle A$ using the triangle sum. $m\angle A = 180 - (15 + 52) = 113^\circ$

$\frac{\sin B}{b} = \frac{\sin A}{a} \Rightarrow \frac{\sin 15^\circ}{9} = \frac{\sin 113^\circ}{a}$ Solve for a: $a\sin 15^\circ = 9\sin 113^\circ \Rightarrow a = \frac{9\sin 113^\circ}{\sin 15^\circ} \approx 32$

$\boxed{\Delta ABC: m\angle A = 113^\circ,\ m\angle B = 15^\circ,\ m\angle C = 52^\circ,\ a = 32,\ b = 9,\ c = 27.4}$

Ex 2: Solve the triangle ΔABC given that $\angle B = 64^\circ$, $\angle A = 28^\circ$, and $c = 55$.



We are given ASA, so we will use the Law of Sines.

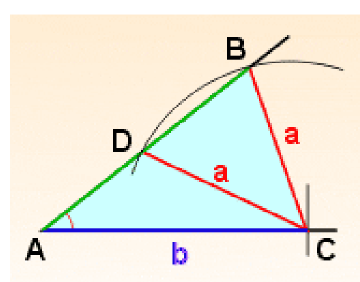
First, find $m\angle C$ using the triangle sum. $m\angle C = 180 - (64 + 28) = 88^\circ$

$\frac{\sin B}{b} = \frac{\sin C}{c} \Rightarrow \frac{\sin 64^\circ}{b} = \frac{\sin 88^\circ}{55}$ Solve for b: $b\sin 88^\circ = 55\sin 64^\circ \Rightarrow b = \frac{55\sin 64^\circ}{\sin 88^\circ} \approx 49.5$

$\frac{\sin C}{c} = \frac{\sin A}{a} \Rightarrow \frac{\sin 88^\circ}{55} = \frac{\sin 28^\circ}{a}$ Solve for a: $a\sin 88^\circ = 55\sin 28^\circ \Rightarrow a = \frac{55\sin 28^\circ}{\sin 88^\circ} \approx 25.8$

$\boxed{\Delta ABC: m\angle A = 28^\circ,\ m\angle B = 64^\circ,\ m\angle C = 88^\circ,\ a = 25.8,\ b = 49.5,\ c = 55}$

<u>**Ambiguous Case:**</u> When given **SSA**, there could be 2 triangles, 1 triangle, or no triangles that can be created with the given information.

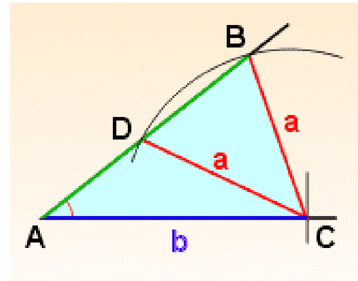


Consider a triangle in which you are given *a, b* and angle *A*. By fixing side *b* and angle *A*, you can sketch the possible positions of side *a* to figure out how many triangles can be formed.

<u>Angle A is obtuse</u>: there is either one triangle or no triangle.

if $a \le b$, no triangle can be made.

if $a > b$, exactly one triangle can be made.



Angle A is acute: there can be 0, 1 or 2 solutions. First, find the shortest length possible for a, by finding the altitude: $h = b\sin A$

if $h > a$, no triangle can be made.

if $h = a$, exactly one triangle can be made - a right triangle

if $a \ge b$, exactly one triangle can be made

if $h < a < b$, two triangles (one acute, one obtuse) can be made.

Note the two triangles formed in the diagram: Triangle ABC and Triangle ADC.

Ex 3: Solve the triangle ΔABC (if possible) when $m\angle C = 54^\circ$, $a = 10$, $c = 7$.

Given **SSA**, use Law of Sines. Check to see how many possible solutions there are.

Find h. $h = a\sin C = 10\sin(54) = 8.09$

Since $h = 8.09$ and $h > 7$, no triangle can be made.

or $\dfrac{\sin C}{c} = \dfrac{\sin A}{a} \Rightarrow \dfrac{\sin 54^\circ}{7} = \dfrac{\sin A}{10}$

Solve for A. $10\sin 54^\circ = 7\sin A \Rightarrow \sin A = \dfrac{10\sin 54^\circ}{7} \Rightarrow A = \sin^{-1}\left(\dfrac{10\sin 54^\circ}{7}\right) = \sin^{-1}(1.1557) = \varnothing$

Solution: There is **no triangle** with the given information because there is no angle with a sine greater than 1.

Ex 4: Solve the triangle ΔABC (if possible) when $m\angle C = 31^\circ$, $b = 46$, $c = 29$.

Find h. $h = b\sin C = 46\sin(31) = 23.69$

Since $h < c < b$, $(23.69 < 29 < 46)$, two triangles can be made.

Given SSA, use Law of Sines. $\dfrac{\sin C}{c} = \dfrac{\sin B}{b} \Rightarrow \dfrac{\sin 31^\circ}{29} = \dfrac{\sin B}{46}$

Solve for ***B***. $46\sin 31^\circ = 29\sin B \Rightarrow \sin B = \dfrac{46\sin 31^\circ}{29} \Rightarrow B = \sin^{-1}\left(\dfrac{46\sin 31^\circ}{29}\right) \approx 54.8^\circ$

Note that the calculator only gives the **acute** angle measure for B. There does exist an **obtuse** angle B with the same sine. $m\angle B = 180 - 54.8 = 125.2^\circ$ This is also an appropriate measure of an angle in a triangle, so there are **2 triangles** that can be formed with the given information.

Triangle 1

$m\angle A = 180 - (54.8 + 31) = 94.2^\circ$

$\dfrac{\sin C}{c} = \dfrac{\sin A}{a} \Rightarrow \dfrac{\sin 31^\circ}{29} = \dfrac{\sin 94.2^\circ}{a}$

$a\sin 31^\circ = 29\sin 94.2^\circ \Rightarrow a \approx 56.2$

Triangle 2

$m\angle A = 180 - (125.2 + 31) = 23.8^\circ$

$\dfrac{\sin C}{c} = \dfrac{\sin A}{a} \Rightarrow \dfrac{\sin 31^\circ}{29} = \dfrac{\sin 23.8^\circ}{a}$

$a\sin 31^\circ = 29\sin 23.8^\circ \Rightarrow a \approx 22.7$

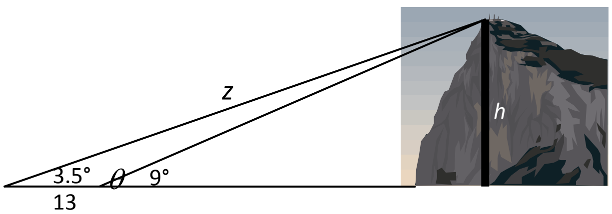
Triangle 1: $\boxed{m\angle A = 94.2^\circ,\ m\angle B = 54.8^\circ,\ m\angle C = 31^\circ,\ a = 56.2,\ b = 46,\ c = 29}$

Triangle 2: $\boxed{m\angle A = 23.8^\circ,\ m\angle B = 125.2^\circ,\ m\angle C = 31^\circ,\ a = 22.7,\ b = 46,\ c = 29}$

Application Problem

1. Draw a picture!
2. Use the Law of Sines to solve for what is asked in the problem.

Ex 5: The angle of elevation to a mountain is 3.5°. After driving 13 miles, the angle of elevation is 9°. Approximate the height of the mountain.



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First, find θ: $\theta = 180 - 9 = 171^\circ$. Therefore, the third angle in the small triangle = 5.5°.

Using the law of sines, we know that $\frac{13}{\sin 5.5} = \frac{z}{\sin 171} \Rightarrow \sin 5.5z = 13\sin 171 \Rightarrow z \approx 21.22$.

Now we can use right triangle trig to find h: $\sin 3.5 = \frac{h}{21.22} \Rightarrow \boxed{h \approx 1.3\,\text{miles}}$

QOD: Explain why SSA is the ambiguous case when solving triangles.

SAMPLE EXAM QUESTIONS

1. In ΔRST, $r = 11.2$, $s = 9.8$, $m\angle T = 38^\circ$, and $m\angle S = 60^\circ$. Which expression can be used to find the length of side t?

A. $\frac{9.8\sin 82^\circ}{\sin 60^\circ}$

B. $\frac{11.2\sin 82^\circ}{\sin 60^\circ}$

C. $\frac{11.2\sin 60^\circ}{\sin 38^\circ}$

D. $\frac{9.8\sin 38^\circ}{\sin 60^\circ}$

Ans: D

2. Solve ΔABC, given that $A = 47°, B = 52°$, and $b = 78$.

$$\frac{\sin 47°}{a} = \frac{\sin 52°}{78} = \frac{\sin C}{c}$$

$$\boxed{a = 72.39, c = 97.76, C = 81°}$$

Given ΔABC with $a = 10, b = 13$, and $A = 19°$, find *c*. Round your answer to two decimal places.

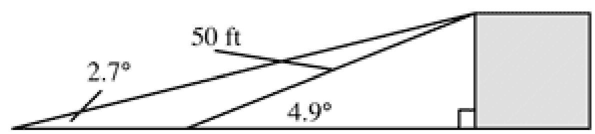
$$\frac{\sin 19°}{10} = \frac{\sin B}{13} = \frac{\sin C}{c}$$

$\angle B = 25°$ or $155°$, **there are 2 possible triangles, therefore** $\boxed{c = 21.35 \text{ or } 3.23}$.

3. Solve ΔABC with $A = 110°$, $a = 5$, and $b = 7.3$. **No Triangles possible.**

4. A 50 foot ramp makes an angle of $4.9°$ with the horizontal. To meet new accessibility guidelines, a new ramp must be built so it makes an angle of $2.7°$ with the horizontal. What will be the length of the new ramp?

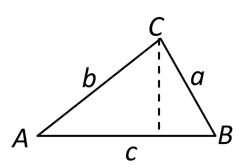
Apply the Law of Sines to the obtuse triangle.



$\frac{\sin 2.7°}{50} = \frac{\sin 175.1}{\text{ramp}}$, ramp = 90.66 feet

<u>**Unit 8.6: To use the Law of Cosines to solve a triangle.**</u>

<u>Law of Cosines:</u> For any triangle, *ABC*



$$\boxed{\begin{aligned} c^2 &= a^2 + b^2 - 2ab\cos C \\ b^2 &= a^2 + c^2 - 2ac\cos B \\ a^2 &= b^2 + c^2 - 2bc\cos A \end{aligned}}$$

<u>Note</u>: In a right triangle, $c^2 = a^2 + b^2 - 2ab\cos 90^\circ \Rightarrow c^2 = a^2 + b^2 - 2ab(0) \Rightarrow c^2 = a^2 + b^2$ (Pythagorean Theorem)

The Law of Cosines can be used to solve triangles when given SAS or SSS. This is when two sides and the include angle are known or when all three sides are known.

Ex 6: Solve the triangle ABC when $m\angle A = 49^\circ$, $b = 42$, & $c = 15$.

Note: The given information is SAS. Use $\boldsymbol{a^2 = b^2 + c^2 - 2bc\cos A}$.

$$a^2 = 42^2 + 15^2 - 2(42)(15)\cos 49^\circ \Rightarrow a^2 = 1162.36 \Rightarrow a \approx 34.09$$

Now that we know all three sides and one angle, we can use the Law of Cosines OR the Law of Sines to find the measure of the second angle. If using the Law of Sines, always make sure to solve for the smallest angle as the Law of Sines will not find any obtuse angles. Since side b is the largest side, it is possible that Angle B is obtuse. Therefore, we will solve for angle *C*. Using the Law of Sines:

$$\frac{a}{\sin A} = \frac{c}{\sin C} \Rightarrow \frac{34.09}{\sin 49^\circ} = \frac{15}{\sin C} \Rightarrow \sin C = \frac{15\sin 49^\circ}{34.09} \Rightarrow \sin C \approx 0.332 \Rightarrow C \approx \sin^{-1}(0.332) \approx 19.4^\circ$$

Solve for $m\angle B$ using the triangle sum: $B = 180 - (49 + 19.4) = 111.6$

$\boxed{m\angle A = 49^\circ,\ m\angle B = 111.6^\circ,\ m\angle C = 19.4^\circ,\ a = 34.09,\ b = 42,\ c = 15}$

Ex 7: Solve the triangle ABC when $a = 31$, $b = 52$, & $c = 28$.

Note: The given information is SSS. Use $\boldsymbol{a^2 = b^2 + c^2 - 2bc\cos A}$. (Or any of them!)

$$31^2 = 52^2 + 28^2 - 2(52)(28)\cos A \Rightarrow \cos A = \frac{2527}{2912} \Rightarrow A \approx 29.8^\circ$$

Now that we have a matching pair of a side and angle, we can use the Law of Sines. Solve for the acute angle which would be opposite *c*. If there is an obtuse angle, it would be opposite the largest side, *b*.

$$\frac{a}{\sin A} = \frac{c}{\sin C} \Rightarrow \frac{31}{\sin 29.8^\circ} = \frac{28}{\sin C} \Rightarrow \sin C = \frac{28\sin 29.8^\circ}{31} \Rightarrow \sin C \approx 0.449 \Rightarrow C \approx \sin^{-1}(0.449) \approx 26.7^\circ$$

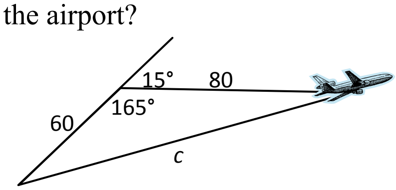
Now find $m\angle B$ using the triangle sum: $B = 180 - (29.8 + 26.7) = 123.5^\circ$

$\boxed{m\angle A = 29.8^\circ,\ m\angle B = 123.5^\circ,\ m\angle C = 26.7^\circ,\ a = 31,\ b = 52,\ c = 28}$

<u>Application Problem</u>

1. Draw a picture!
2. Use the Law of Cosines to solve for what is asked in the problem.

Ex 8: A plane takes off and travels 60 miles, then turns 15° and travels for 80 miles. How far is the plane from the airport?



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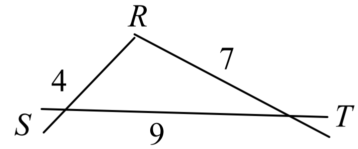
Using the picture, we can find the angle in the triangle to give us SAS.

Use the Law of Cosines: $\boldsymbol{c^2 = a^2 + b^2 - 2ab\cos C}$

$c^2 = 60^2 + 80^2 - 2(60)(80)\cos 165 \Rightarrow c^2 = 19272.89 \Rightarrow \boxed{c \approx 138.8\,\text{miles}}$

SAMPLE EXAM QUESTIONS

1. Which equation would you use to find $m\angle S$?

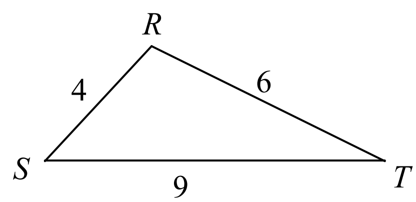


A. $7^2 = 4^2 + 9^2 - 2(4)(9)\cos S$

B. $4^2 = 7^2 + 9^2 - 2(7)(9)\cos S$

Ans: A

2. Which expression can be used to find $m\angle R$?



A. $\cos^{-1}\left(-\frac{48}{29}\right)$

B. $\cos^{-1}\left(\frac{81}{4}\right)$

C. $\cos\left(-\frac{29}{48}\right)$

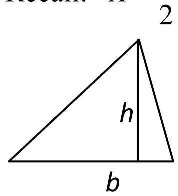
D. $\cos^{-1}\left(-\frac{29}{48}\right)$

Ans: D

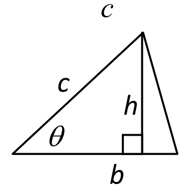
<u>Sec. 8.7 To find the area of a triangle given side-angle-side information or by using Heron's Formula.</u>

<u>Area of a Triangle</u>

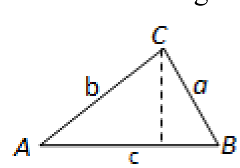
Recall: $A = \frac{1}{2}bh$



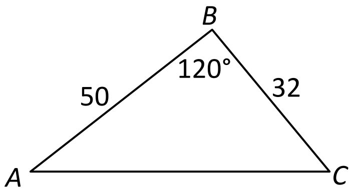
$\sin\theta = \frac{h}{c} \Rightarrow h = c\sin\theta$; $\therefore A = \frac{1}{2}bc\sin\theta$



<u>Formula for the Area of a Triangle Given SAS:</u> $\boxed{A = \frac{1}{2}ab\sin C}$ **or** $\boxed{A = \frac{1}{2}bc\sin A}$ **or** $\boxed{A = \frac{1}{2}ac\sin B}$ dependent on the information given in for Triangle ABC.



Ex 9: Find the area of triangle *ABC* shown.



We will use $\boxed{A = \frac{1}{2}ac\sin B}$.

$A = \frac{1}{2}(32)(50)\sin 120^\circ \Rightarrow A = (16)(50)\left(\frac{\sqrt{3}}{2}\right) \Rightarrow A = \boxed{400\sqrt{3}\text{ sq units}}$

Ex 10: Find the area of triangle *ABC* if $b = 15,\ c = 13$ and $A = 57^\circ$.

We will use $\boxed{A = \frac{1}{2}bc\sin A}$.

$A = \frac{1}{2}(15)(13)\sin 57^\circ \Rightarrow A = (7.5)(50)(0.839) \Rightarrow A \approx \boxed{81.8\text{ sq units}}$

Heron's Area Formula

Semi-Perimeter: $s = \frac{a+b+c}{2}$

Area of a Triangle Given SSS: $\boxed{A = \sqrt{s(s-a)(s-b)(s-c)}}$

Ex 11: Find the area of the triangle with side lengths 5 m, 6 m, and 9 m.

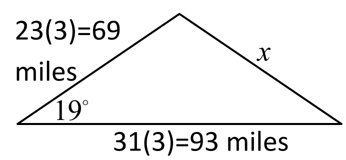
Semiperimeter: $s = \frac{a+b+c}{2} \Rightarrow s = \frac{5+6+9}{2} \Rightarrow s = 10$

$A = \sqrt{s(s-a)(s-b)(s-c)}$

$A = \sqrt{10(10-5)(10-6)(10-9)} \Rightarrow A = \sqrt{10(5)(4)(1)} \Rightarrow A = \sqrt{200} \Rightarrow A = \boxed{10\sqrt{2}\text{ m}^2}$

Ex 12: Two ships leave port with a 19° angle between their planned routes. If they are traveling at 23 mph and 31 mph, how far apart are they in 3 hours?

First find the missing length. With SAS, we use Law of Cosines.



$x^2 = 69^2 + 93^2 - 2(69)(93)\cos(19^\circ)$

$x^2 = 13410 - 12134.7854$

$x^2 = 1275.2146$

$\boxed{x \approx 35.7\text{ miles}}$

Now, calculate the area of the triangle formed by the ships routes and their distance apart after three hours.

Semiperimeter: $s = \dfrac{69 + 93 + 35.7}{2} \Rightarrow s \approx 98.85$

$$\boldsymbol{A = \sqrt{s(s-a)(s-b)(s-c)}}$$

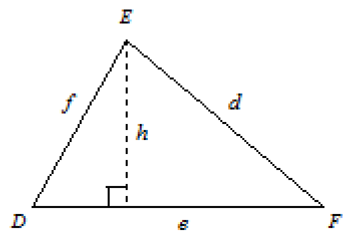
$$A = \sqrt{98.85(98.85-69)(98.85-93)(98.85-35.7)} \Rightarrow A = \sqrt{98.85(29.85)(5.85)(63.15)}$$

$$A = \sqrt{1090059.565} \Rightarrow A \approx \boxed{1044.06 \text{ m}^2}$$

QOD: Can there be an ambiguous case when using the Law of Cosines? Explain why or why not.

SAMPLE EXAM QUESTIONS

1. Give an expression for the height *h* of ΔDEF, and use the expression to write a formula for the height of the triangle in terms of the variables shown by replacing *h* in the formula $A = \dfrac{1}{2}bh$.



A. $h = f \sin D, A = \dfrac{1}{2} ef \sin D$

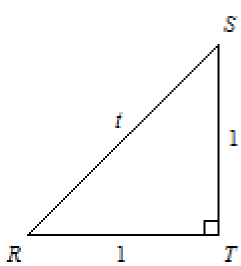
B. $h = \dfrac{1}{2} f \sin D, A = \dfrac{1}{2} e\left(\dfrac{1}{2} f \sin D\right)$

C. $h = d \sin D, A = \dfrac{1}{2} ed \sin D$

D. $h = e \sin D, A = \dfrac{1}{2} e^2 \sin D$

Ans: A

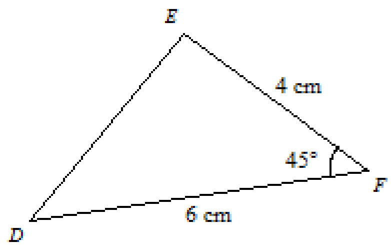
2. ΔRST is an isosceles right triangle.



Part A: Determine the exact value of *t*. Use radical notation if necessary, and do not approximate. Show your work.

Part B: Use ΔRST to determine the exact value of $\sin 45°$. Use radical notation if necessary, and do not approximate. Show your work.

Refer to ΔDEF to answer Parts C and D.



Part C: Use your answer to Part B to determine the exact value for the area of ΔDEF .

Part D: Using a calculator, determine the area of ΔDEF to the nearest tenth of a cm^2.

Part A: Because $\triangle RST$ is a right triangle, the Pythagorean Theorem can be used to determine the value of t.

$t^2 = 1^2 + 1^2 = 1 + 1 = 2$

$t = \sqrt{2}$

Part B: Since the angles opposite the congruent sides of an isosceles triangle are congruent, and since the three angle measures of a triangle must sum to 180°, $m\angle R + m\angle S = \dfrac{180° - 90°}{2} = 45°$

Therefore $\sin R = \dfrac{\text{opp}}{\text{hyp}} = \dfrac{1}{t} = \dfrac{1}{\sqrt{2}} = \dfrac{1}{\sqrt{2}} \cdot \dfrac{\sqrt{2}}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$

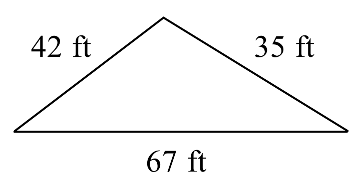
Part C: The area of triangle DEF is:

$A = \dfrac{1}{2} \cdot DF \cdot EF \cdot \sin(45°)$

$= \dfrac{1}{2} \cdot 4\text{ cm} \cdot 6\text{ cm} \cdot \dfrac{\sqrt{2}}{2} = 6\sqrt{2}\text{ cm}^2$

Part D: $6\sqrt{2}\text{ cm}^2 \approx 8.5\text{ cm}^2$

3. Which expression represents the area of the triangle in square feet?



A. $\sqrt{48(6)(13)(19)}$

B. $\sqrt{72(30)(37)(5)}$

C. $\sqrt{144(42)(35)(67)}$

D. $\sqrt{144(102)(109)(77)}$

Ans: B

Sec. 8.8/8.9 To recognize and graph sine, cosine and tangent functions and their transformations.

Teacher Note: Have students fill out the table as quickly as they can. Discuss patterns they can use to be able to memorize these.

Θ	0°	30°	45°	60°	90°	120°	135°	150°	180°	210°	225°	240°	270°	360°
radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	2π
sinθ	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1	0
cosθ	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0	1
tanθ	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	und	$-\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	und	0

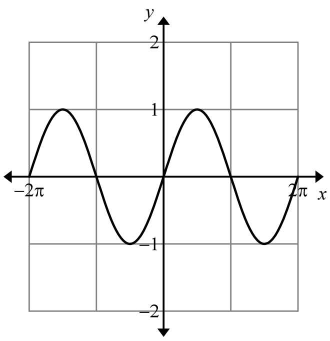
Sinusoid: a function whose graph is a sine or cosine function; can be written in the form $y = a\sin(bx+c)+d$

Sinusoidal Axis: the horizontal line that passes through the middle of a sinusoid

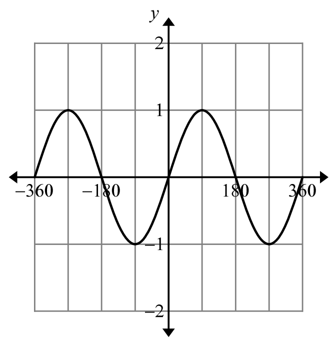
Graph of the Sine Function

Use the table above to sketch the graph. Sine is an ODD function. This means that its graph remains unchanged after rotation of 180 degrees about the origin. Note its rotational symmetry.

Radians: $y = \sin x$



Degrees: $y = \sin\theta$



Sine is **periodic**, so we can extend the graph to the left and right.

Characteristics:

Domain: $(-\infty,\infty)$ Range: $[-1,1]$ y-intercept: $(0,0)$ x-intercepts: $(0+k\pi,0)$; $k\in\mathbb{Z}$

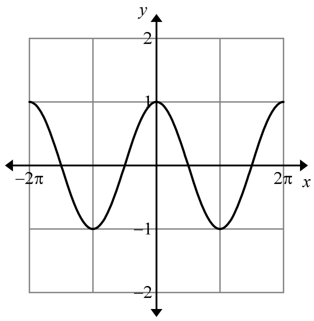
Absolute Max = 1   Absolute Min = -1 Decreasing: $\left(\frac{\pi}{2},\frac{3\pi}{2}\right)+2\pi$

Increasing: $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)+2\pi$ Period = 2π Sinusoidal Axis (Midine): $y=0$

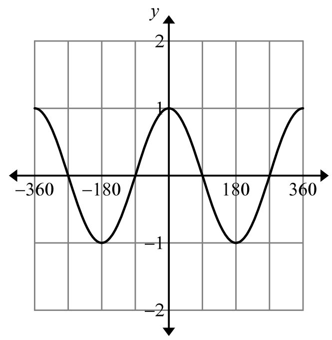
Graph of the Cosine Function

Use the table on the previous page to sketch the graph. Cosine is an EVEN function. This means that the graph of cosine is symmetric with respect to the y-axis.

Radians: $y=\cos x$



Degrees: $y=\cos\theta$



Cosine is **periodic**, so we can extend the graph to the left and right.

Characteristics:

Domain: $(-\infty,\infty)$ Range: $[-1,1]$ y-intercept: $(0,1)$ x-intercepts: $\left(\dfrac{\pi}{2}+k\pi,0\right)$; $k\in\mathbb{Z}$

Absolute Max = 1 Absolute Min = −1 Decreasing: $(\pi,2\pi)+2\pi$

Increasing: $(0,\pi)+2\pi$ Period = 2π Sinusoidal Axis (Midline): $y=0$

Note: We will work mostly with the graphs in radians, since it is the graph of the function in terms of *x*.

Recall: $f(x)=a(x-h)^2+k$ is a transformation of the graph of $f(x)=x^2$. a represents the vertical stretch/shrink, h is a horizontal shift and k is a vertical shift.

Transformations of Sinusoids:

General Form $y=a\sin\left[b(x-h)\right]+k$

a: vertical stretch and/or reflection over x-axis **amplitude** = $|a|$

h: horizontal shift **phase shift** = h k: vertical shift b: horizontal stretch/shrink

period = $\dfrac{2\pi}{|b|}$ **frequency** = $\dfrac{|b|}{2\pi}$ $|b|$ = number of cycles completed in 2π

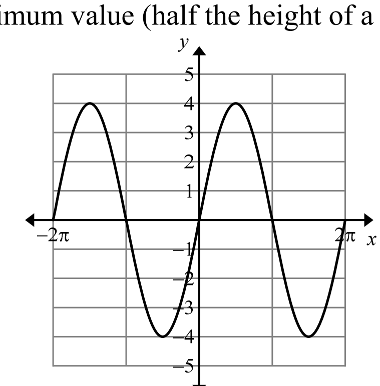
Amplitude: the distance from the sinusoidal axis to the maximum value (half the height of a wave)

Ex 13: Sketch the graph of $y=4\sin x$.

Vertical stretch: 4

Amplitude = 4

Period: 2π



Period: the length of time taken for one full cycle of the wave $P = \dfrac{2\pi}{|b|}$

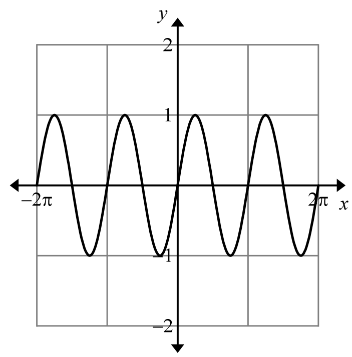
Frequency: the number of complete cycles the wave completes per unit of time freq $= \dfrac{|b|}{2\pi}$

Ex 14: Sketch the graph of $y = \sin(2x)$.

Amplitude: 1 Period $= \dfrac{2\pi}{2} = \pi$

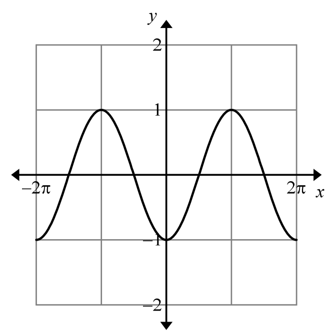
There is one complete cycle between 0 and π.
Since $b>1$, there is a horizontal shrink of a factor of 2.

Frequency $= \dfrac{2}{2\pi} = \dfrac{1}{\pi}$



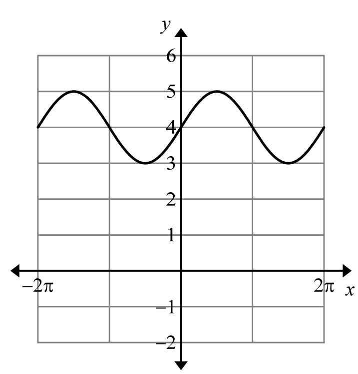
Reflection: If $a < 0$, the sinusoid is reflected over the x-axis.

Ex 15: Sketch the graph of $y = -\cos x$.
Note: Period, amplitude, etc. all stay the same.

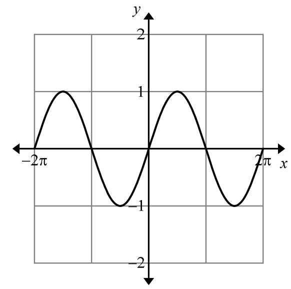


Vertical Translation: in the sinusoid $y = a\sin\left[b(x-h)\right]+k$, the line $y = k$ is the sinusoidal axis (midline)

Ex 16: Sketch the graph of $y = 4 + \sin x$.
There is a vertical shift of 4 units up.
Sinusoidal Axis: $y = 4$



<u>Phase Shift:</u> the horizontal translation, h, of a sinusoid $y = a\sin\left[b\left(x-h\right)\right]+k$



Ex 17: Sketch the graph of $y=\cos\left(x-\dfrac{\pi}{2}\right)$.

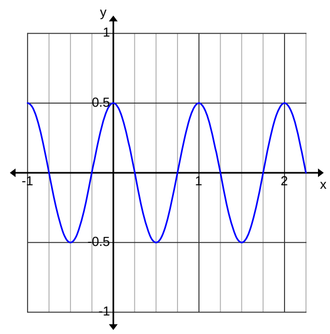
This graph has a horizontal translation to the right.

Shift $y=\cos x$ to the right $\dfrac{\pi}{2}$.

Does this graph look familiar? It is $y=\sin x$!

<u>Note:</u> Every cosine function can be written as a sine function using a phase shift.

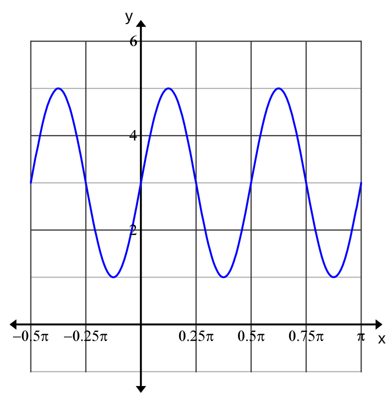
$$y=\cos x \iff y=\sin\left(x+\frac{\pi}{2}\right)$$



Ex 18: Sketch the graph of $y=\dfrac{1}{2}\cos 2\pi x$.

The amplitude is $a=\dfrac{1}{2}$ and the period is $\dfrac{2\pi}{b}=\dfrac{2\pi}{2\pi}=1$.

There is one complete cycle from 0 to 1.



Ex 19: Sketch the graph of $y=2\sin 4x+3$.

The amplitude is $a=2$ and the period is $\dfrac{2\pi}{b}=\dfrac{2\pi}{4}=\dfrac{\pi}{2}$.

There will be one complete cycle from 0 to $\dfrac{\pi}{2}$.

There is a vertical shift of 3 up and no horizontal shift.

The midline of the graph is at $y=3$.

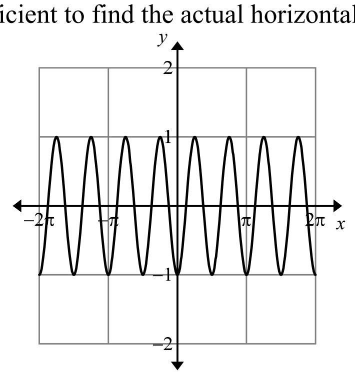
Ex 20: Sketch the graph of $y=\cos\left(4x+\pi\right)$. Then, rewrite the function as a sine function.

The coefficient of x must equal 1. Factor out any other coefficient to find the actual horizontal shift.

$$y=\cos\left(4x+\pi\right) \Rightarrow y=\cos\left[4\left(x+\frac{\pi}{4}\right)\right]$$

Phase Shift: $-\dfrac{\pi}{4}$ (left)

Period: $\dfrac{2\pi}{4}=\dfrac{\pi}{2}$



Writing the Equation of a Sinusoid

Ex 21: Write the equation of a sinusoid with amplitude 4 and period $\frac{\pi}{3}$ that passes through $(6,0)$.

Amplitude: $a=4$; Period: $\frac{\pi}{3}=\frac{2\pi}{b} \Rightarrow b=6$;

Phase Shift: normally passes thru $(0,0)$, so shift right 6 units: $y=4\sin(6x-6)$ or $\boxed{y=4\sin\left[6(x-1)\right]}$

Ex 22: Describe the transformations of $y=7\sin\left(2x+\frac{\pi}{4}\right)-3$. Then, sketch the graph.

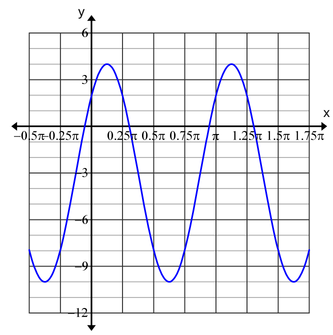
Factor out 2, so the coefficient of the x is one. $y=7\sin 2\left(x+\frac{\pi}{8}\right)-3$

The amplitude is $a=7$ and the period is $\frac{2\pi}{b}=\frac{2\pi}{2}=\pi$.

There will be one complete cycle from 0 to π.

There is a vertical shift of 3 down and a horizontal phase shift of $\frac{\pi}{8}$ to the left.

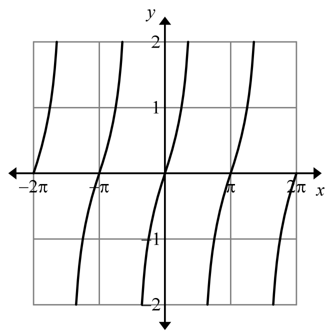
The midline of the graph is at $y=-3$.



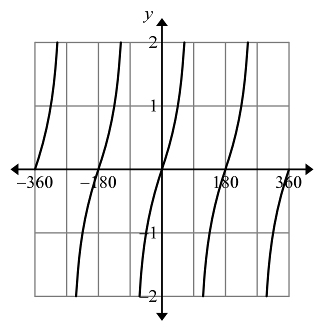
Graph of the Tangent Function

Use your knowledge of the unit circle to sketch the graph.

Radians: $y=\tan x$



Degrees: $y=\tan\theta$



Characteristics:

Domain: $x \neq \frac{\pi}{2}+k\pi,\ k\in\mathbb{Z}$ Range: $(-\infty,\infty)$ Intercepts: $(k\pi,0)$ Always Increasing

Period: π Vertical Asymptotes: At odd multiples of $x=\frac{\pi}{2b}$

Note: $\tan x=\frac{\sin x}{\cos x}$, so the zeros of sine are the zeros of tangent, and the zeros of cosine are the vertical asymptotes of tangent.

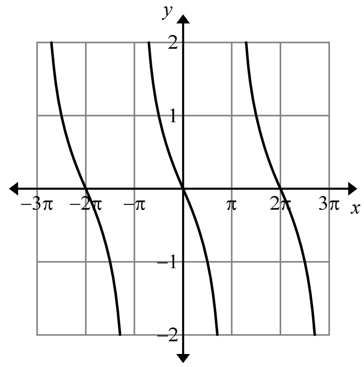
Ex 23: Sketch the graph of $y = -\tan\left(\frac{x}{2}\right)$.

Period: $P = \frac{\pi}{1/2} = 2\pi$

Asymptotes: $\frac{\pi}{2\left(\frac{1}{2}\right)} = \pi$ Odd multiples of π

x-intercepts: $\frac{\pi}{b} = 2\pi$ - all multiples of 2π

Reflect over x-axis

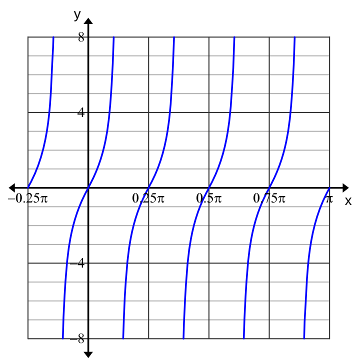


Ex 24: Sketch the graph of $y = 2\tan 4x$

Period: $\frac{\pi}{4}$

Asymptotes: Odd multiples of $\frac{\pi}{8}$

x-intercepts: $\frac{\pi}{b} = \frac{\pi}{4}$ - all multiples of $\frac{\pi}{4}$



SAMPLE EXAM QUESTIONS

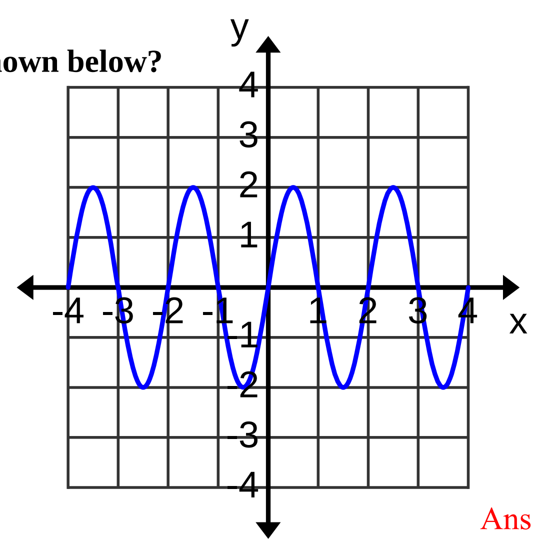
1. Which is the equation of the graph shown below?

A. $f(x) = 2\sin \pi x$

B. $f(x) = 2\sin 2\pi x$

C. $f(x) = \frac{1}{2}\sin \pi x$

D. $f(x) = \frac{1}{2}\sin \frac{\pi}{2} x$



Ans: A

2. The graph of which function has a period of π and an amplitude of π?

A. $y = \frac{1}{\pi}\sin 2x$

B. $y = \frac{1}{\pi}\sin \frac{1}{2}x$

C. $y = \pi \sin 2x$

D. $y = \pi \sin \frac{1}{2}x$

Ans: C

3. Which function has an amplitude of 2 and a period of π?

A. $f(x) = 2\cos 2x$

B. $f(x) = 2\cos \pi x$

C. $f(x) = \frac{1}{2}\cos 2x$

D. $f(x) = \frac{1}{2}\cos \pi x$

Ans: A

4. Which function has an amplitude of π and a period of $\frac{2}{\pi}$?

A. $f(x) = \frac{\pi}{2}\cos 2\pi x$

B. $f(x) = \pi \cos \frac{1}{x}$

C. $f(x) = \frac{2}{\pi}\cos \frac{2}{\pi} x$

D. $f(x) = \pi \cos \pi^2 x$

Ans: D

5. Which of the following is a vertical asymptote of the graph of $f(x) = -\tan\left(x + \frac{\pi}{4}\right)$?

A. $x = \frac{\pi}{4}$

B. $x = \frac{\pi}{2}$

C. $x = \pi$

D. $x = \frac{4}{\pi}$

E. $x = \frac{3\pi}{4}$

Ans: A

6. What is the equation for the graph shown?

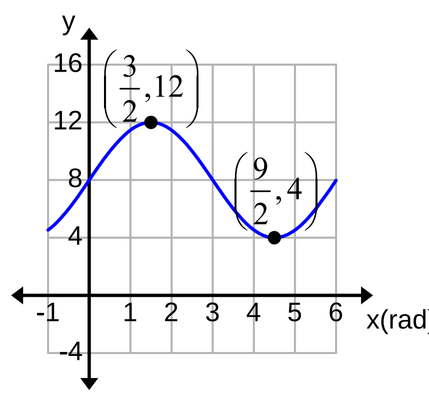
A. $y = 4\sin \frac{\pi}{3} x + 8$

B. $y = 6\sin \frac{\pi}{3} x + 4$

C. $y = 4\cos \frac{\pi}{3} x + 8$

D. $y = 6\cos \frac{\pi}{3} x + 4$

E. $y = 2\sin \frac{\pi}{3} x + 12$



Ans: A

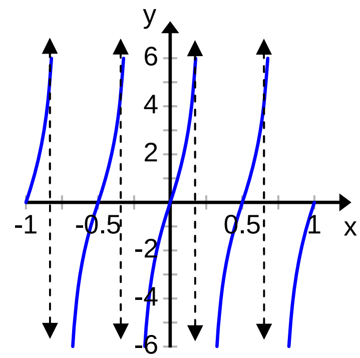
7. Which function has an amplitude of 3 and a period of $\frac{1}{2}$?

A. $y = -3\sin 4\pi x$ **C.** $y = -3\cos 4x$

B. $y = 3\sin\left(\frac{\pi}{2}\right)x$ **D.** $y = \frac{1}{2}\cos 3\pi x$

Ans: A

8. Which function is represented by the graph shown?



A. $y = 3\tan 2x$

B. $y = 3\tan\left(\frac{1}{2}\right)x$

C. $y = 3\tan \pi x$

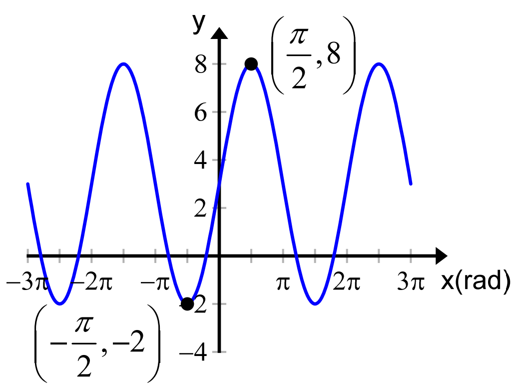
D. $y = 3\tan 2\pi x$

Ans: D

9. Write an equation of the form $y = a\sin bx$, where $a > 0$, $b > 0$, with amplitude $\frac{2}{3}$ and period 12.

$$y = \frac{2}{3}\pi \sin\frac{\pi}{6}x$$

10. Write a function for the sinusoid.



The period is the length of a full cycle. The given points represent the end points of half cycle so by subtracting their x values we get the length of half cycle $\frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi$. That result is doubled to get the length of the full cycle, which is $P = 2\pi$. Knowing that $P \cdot b = 2\pi$, we get $b = \frac{2\pi}{2\pi} = 1$.

The amplitude $A = \frac{max - min}{2} = \frac{8-(-2)}{2} = 5$

The sinusoidal curve was shifted 3 units up so the simplest form of the equation is: $y = 5\sin\theta + 3$

11. Is the function $y = -2\sin\left(2x - \frac{\pi}{2}\right) + 3$ in the form $y = a\sin b(x-h)+k$? Why or why not?

How does the amplitude and period of the function compare to the amplitude and period of $y = \sin x$? How does the graph of the function compare to the graph of $y = 2\sin 2x$?

In order to put the equation in standard form we need to factor out the coefficient of x which is 2.

$$y = -2\sin\left(2x - \frac{\pi}{2}\right) + 3 = -2\sin\left[2\left(x - \frac{\pi}{4}\right)\right] + 3$$

The amplitude is $A = |-2| = 2$ and the period is $P = \frac{2\pi}{b} = \frac{2\pi}{2} = \pi$

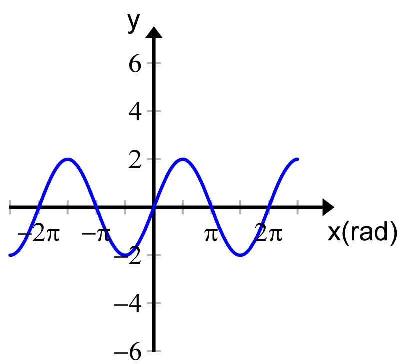
How does the graph of the function compare to the graph of $y = 2\sin 2x$?

$$y = -2\sin\left[2\left(x - \frac{\pi}{4}\right)\right] + 3$$

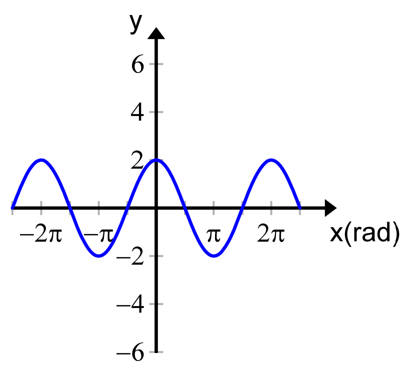
The phase shift $\frac{\pi}{4}$ tells us to shift the graph of the function $y = 2\sin 2x$ to the right by $\frac{\pi}{4}$, then to reflect it across the x axis because of the negative coefficient -2 and then shift it vertically 3 units up.

12. Graph $y = 2\sin x$.

A.



B.



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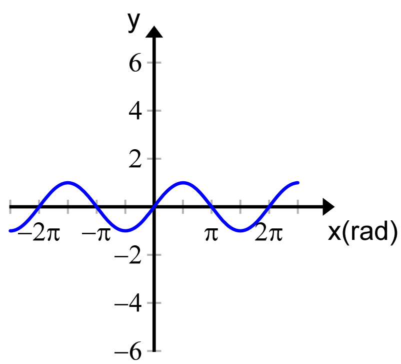

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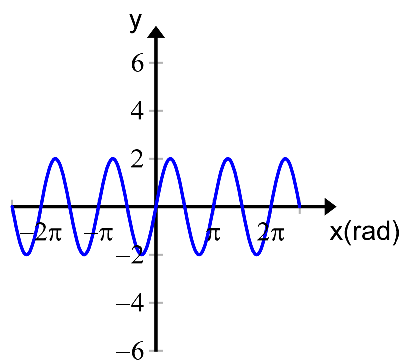
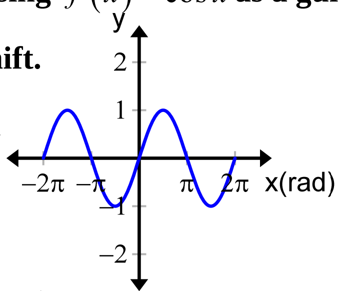


Ans: A

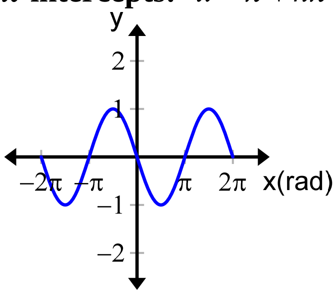
13. Using $f\left(x\right)=\cos x$ as a guide, graph $g\left(x\right)=\cos\left(x-\frac{\pi}{2}\right)$. Identify the *x*-intercepts and phase shift.

A.



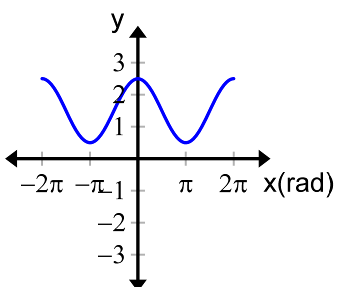
***x*-intercepts:** $x=\pi+n\pi$ **where *n* is an integer; phase shift:** $\frac{\pi}{2}$ **units to the right**

B.



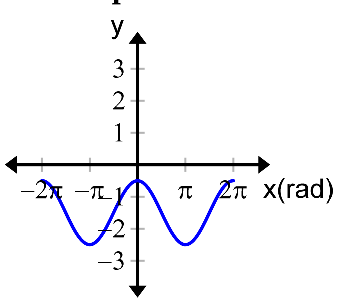
***x*-intercepts:** $x=\pi+n\pi$ **where *n* is an integer; phase shift:** $\frac{\pi}{2}$ **units to the left**

C.



***x*-intercepts:** $x=\pi+n\pi$ **where *n* is an integer; phase shift:** $\frac{\pi}{2}$ **units up**

D.



***x*-intercepts:** $x=\pi+n\pi$ **where *n* is an integer; phase shift:** $\frac{\pi}{2}$ **units down**

Ans: A

14. Find the amplitude and period of the graph of $y=-3\cos\pi x$.

$$A=\left|-3\right|=3$$

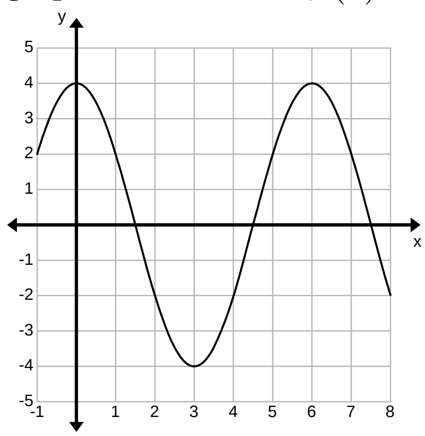
$$P=\frac{2\pi}{b}=\frac{2\pi}{\pi}=2$$

15. Find the amplitude and period of the graph of $y=-2\cos 6x$.

$$A=|-2|=2$$
$$P=\frac{2\pi}{b}=\frac{2\pi}{6}=\frac{\pi}{3}$$

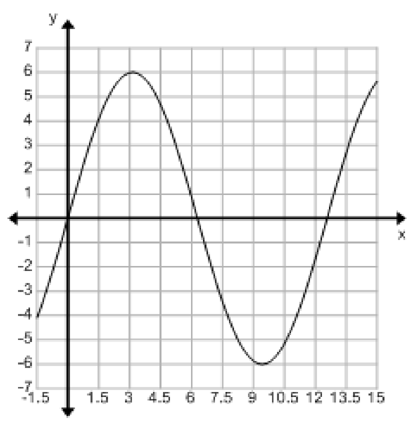
16. Graph one cycle of the graph of the function $f(x)=4\cos\frac{\pi x}{3}$.

$$A=|4|=4$$
$$P=\frac{2\pi}{b}=\frac{2\pi}{\pi/3}=6$$

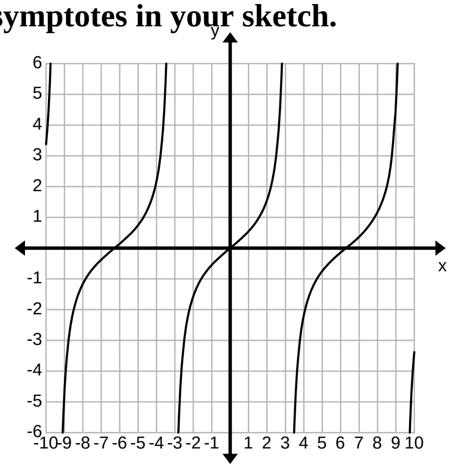


17. Graph one cycle of the graph of the function $f(x)=6\sin\frac{x}{2}$

$$A=|6|=6$$
$$P=\frac{2\pi}{b}=\frac{2\pi}{1/2}=4\pi$$



18. Graph $y=\tan\frac{x}{2}$. Include vertical asymptotes in your sketch.



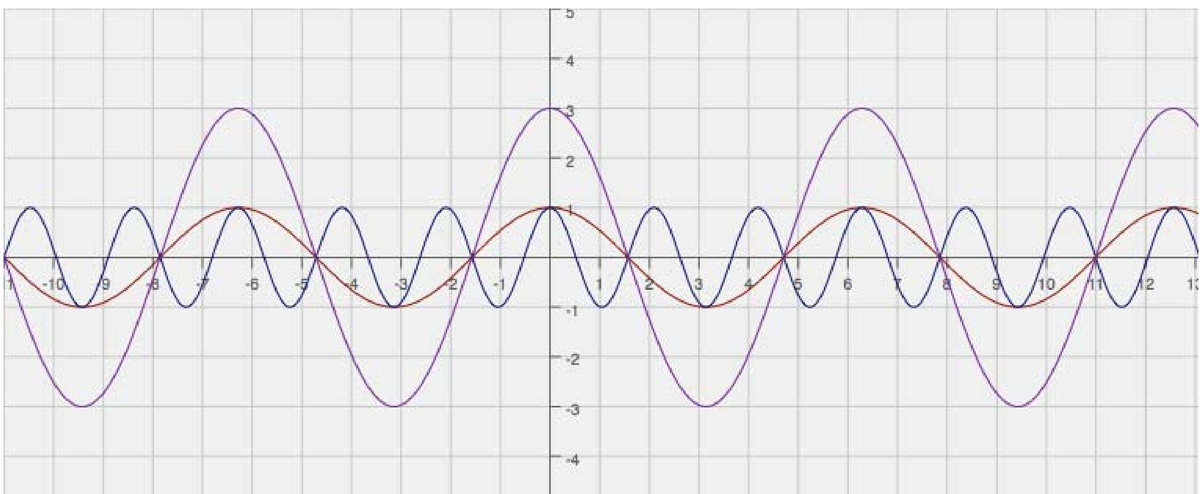
19. The graph of a sine function has amplitude 5, period $72°$, and a vertical translation 4 units down. Write an equation for the function.

$$P = 72^{\circ} = 72^{\circ}\frac{\pi}{180^{\circ}} = \frac{2\pi}{5}^{rad} \qquad b = \frac{2\pi}{P} = \frac{2\pi}{2\pi/5} = 5 \qquad \text{so } y = 5\sin(5x) - 4$$

20. The graph of a cosine function has amplitude 4, period $90°$, and a vertical translation 3 units down. Write an equation for the function. Then sketch the graph without using graphing technology.

$$P = 90^{\circ} = 90^{\circ}\frac{\pi}{180^{\circ}} = \frac{\pi}{2}^{rad} \qquad b = \frac{2\pi}{P} = \frac{2\pi}{\pi/2} = 4 \qquad \text{so } y = 4\cos(4x) - 3$$

21. Sketch the graphs of $y = \cos x, y = \cos 3x,$ and $y = 3\cos x$. Tell how the graphs are alike and how they are different.



Both functions $y = \cos x$ *and* $y = \cos(3x)$ have the same amplitudes A=1,

The function $y = 3\cos x$ has amplitude of 3. Both functions $y = \cos x$ *and* $y = 3\cos x$ have the same periods $P = 2\pi$, while the function $y = \cos(3x)$ has a period $P = 2\pi/3$

22. Consider the related equations $y = \sin x, y = 2\sin x,$ and $y = \sin 2x$. Explain the effect that the coefficient 2 has on the graphs of $y = 2\sin x$ and $y = \sin 2x$ when compared to the graph of $y = \sin x$.

When compared to the parent function, the graph of $y = 2\sin x$ has an amplitude of 2 which means that the parent function has been stretched by a factor of 2.

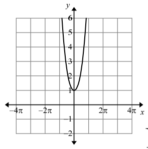
When it comes to the function y=sin (2x) the 2 gives the number of complete cycles on an x-interval of length 2π , so since b=2 the period is π.

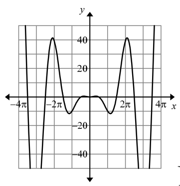
Sec. 8.10 To use trigonometric functions to model periodic phenomena.

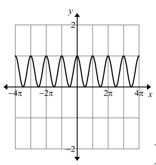
Trigonometric Functions Combined with Algebraic Functions

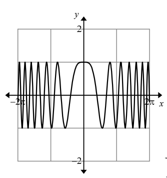
Periodic Functions: A function $y = f(t)$ is **periodic** if there is a positive number c such that $f(t+c) = f(t)$ for all values of t in the domain of f.

Ex 25: Graph the following and determine which function(s) are periodic.

1. $y = \cos x + x^2$  Not Periodic

2. $y = x^2 \cos x$  Not Periodic

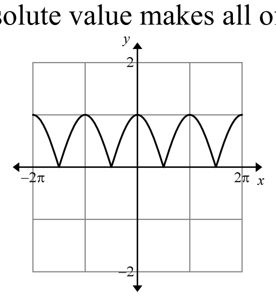
3. $y = (\cos x)^2$  Periodic

4. $y = \cos(x^2)$  Not Periodic

Absolute Value of Trig Functions

Ex 26: Sketch the graph of $f(x) = |\cos x|$.

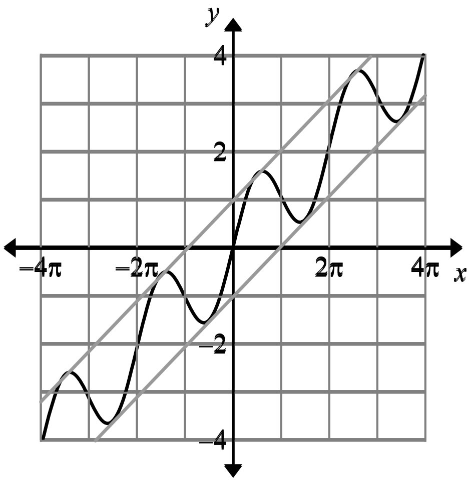
The absolute value makes all of the y-values of the function positive.



Note: The period of $f(x) = |\cos x|$ is π.

Adding a Sinusoid to a Linear Function

Ex 27: Analyze the graph of $y=\frac{1}{3}x+\sin x$. Is it periodic?



The function oscillates between the lines $y=\frac{1}{3}x-1$ and $y=\frac{1}{3}x+1$.

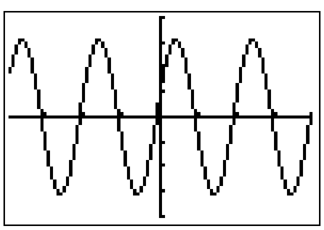
It is NOT periodic.

Are the Sums of Sinusoid Function Sinusoids?

Teacher Note: Have students graph various sums to determine a pattern.

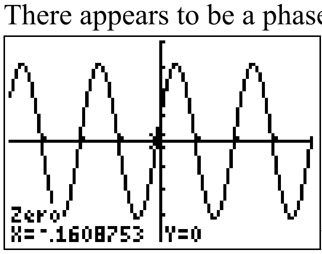
Conclusion: The sum of two sinusoids is a sinusoid if and only if the two sinusoids being added have the same period. The sum has this same period as well.

Ex 28: Graph the function $f(x)=3\sin 2x+\cos 2x$. Write a sinusoid function that estimates $f(x)$.

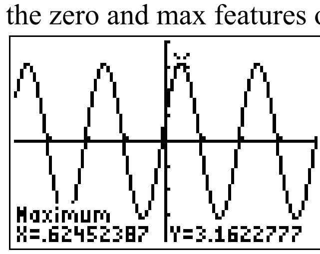


The period of $f(x)$ is the same as each of the addends.

There appears to be a phase shift and an amplitude. Use the zero and max features on the calculator.



Phase Shift: left 0.161



Amplitude: $a=3.162$

Sinusoid: $\boxed{f(x)=3.162\sin 2(x+0.161)}$

Note: Check your answer by graphing.

SAMPLE EXAM QUESTIONS

1. **A sound wave models a sinusoidal function.**

Part A: If the wave reaches its maximum at $\left(\frac{\pi}{2},12\right)$ and its minimum at $\left(\frac{3\pi}{2},0\right)$, what are the shift, amplitude, and period of the function?

The amplitude $A = \frac{max - min}{2} = \frac{12-0}{2} = 6$

Between max and min we have half cycle so the difference between the x values represents the length of half cycle.

$$\frac{3\pi}{2} \quad \frac{\pi}{2} = \pi$$

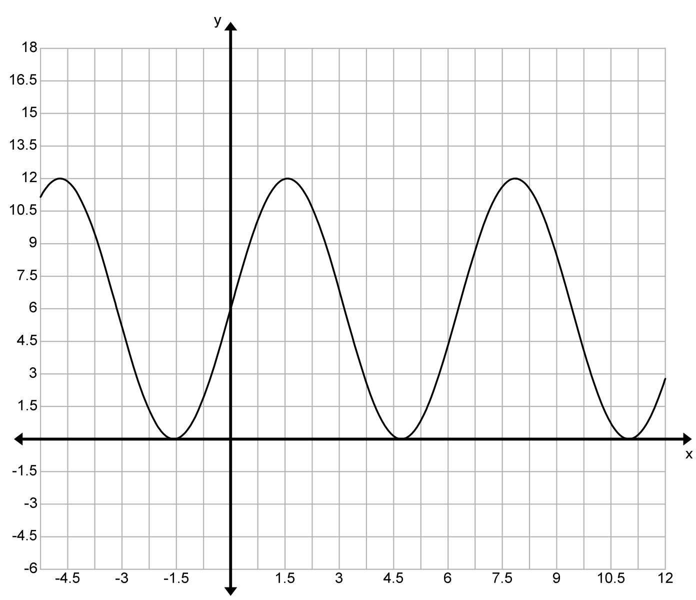
Since the period is the length of a full cycle, the period is $P = 2\pi$, so b=1.

Since the max value is 12 and min is 0 is clear that the sinusoidal curve was shifted vertically 6 units up.

Part B: Write the function that models this sound wave.

$$y = 6\sin x + 6$$

Part C: Graph the function.



2. **Tides can be modeled by periodic functions. Suppose high tide at the city dock occurs at 2:22 AM at a depth of 35 meters and low tide occurs at 9:16 AM at a depth of 9 meters. Write an equation that models the depth of the water as a function of time after midnight. When will the next high and low tides occur?**

The period P is twice the time between the low and the high tide, so

P=2*(9:16-2:22)=2*(6h 54min)=13 h 44min = $13\frac{44}{60}h = 13.73h$ **, so** $b = \frac{2\pi}{13.73}$

- **The amplitude** $A = \frac{1}{2}(35-9) = 13\text{m}$

- **If they asked us to write an equation that models the depth of the water as a function of time after 2:22 AM (when the maximum occurs) the equation would have been:**

$$h(t) = 13\cos\frac{2\pi}{13.73}(t-2.37)$$

$$2h22\min = 2\frac{22}{60} = 2.37h$$

$$y(t) = r\sin bt$$

but it has to be written after midnight so we need to accommodate a shift in time of

$$2h22\min = 2\frac{22}{60} = 2.37h$$

so the equation that models the depth of the water as a function of time after midnight is:

$$h(t) = 13\cos\left[\frac{2\pi}{13.73}(t-2.37)\right]$$

- **When will the next high and low tides occur?**

We know that the high tide occurs at 2:22 AM and the period is P=13h 44min, so the next high tide occurs 13h44min later from the time the high tide occurred, which is @4:06AM the next day. The low tide occurs at 9:16 AM and the period is 13h44min, so 13h44min from 9:16 AM will be @ 11:00AM the next day.

3. A Ferris wheel with a radius of 25 feet is rotating at a rate of 3 revolutions per minute. When $t = 0$, a chair starts at the lowest point on the wheel, which is 5 feet above ground. Write a model for the height *h* (in feet) of the chair as a function of the time *t* (in seconds).

Let's think about the Ferris wheel as a trigonometric circle. The y-coordinate of any point on the trigonometric circle tells you how far the point is above or below the x-axis and the function that gives the y-coordinate is the sine function. Since the wheel rotates at a rate of 3 revolutions per minute, but the problems asks us to write the height *h* (in feet) of the chair as a function of the time *t* (in seconds), we need to do the following transformation:

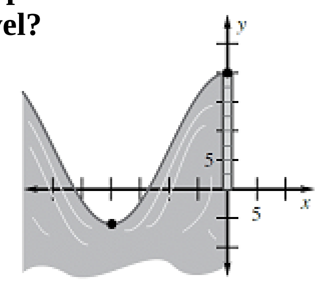
$$\frac{3 revolutions}{1\min} = \frac{3 revolutions}{60\sec} = \frac{.05 revolutions}{1\sec}$$

which gives us the number of revolutions/sec, so we can conclude that b = .05

The x-axis is located at the wheel's center, not at ground level, so the wheel's center is 30 ft above ground (the lowest point on the wheel is 5 feet above ground +25 the radius). We need to accommodate this situation by adding the constant k=30, so the equation that describes height *h* (in feet) of the chair as a function of the time *t* (in seconds) is:

$$h(t) = r\sin(bt) + k$$
$$h(t) = 25\sin(.05t) + 30$$

4. Storm surge from a hurricane causes a large sinusoidal wave pattern to develop near the shore. The highest wave reached the top of a wall 20 feet above sea level. The low point immediately behind this wave was 6 feet below sea level, and was 20 feet behind the peak. What is the amplitude of the sinusoid? What is the vertical shift of the sinusoid from a wave at ground level?



The amplitude is defined as half the distance between the minimum and maximum values of the range, so:

$$A = \frac{20-(-6)}{2} = 13$$

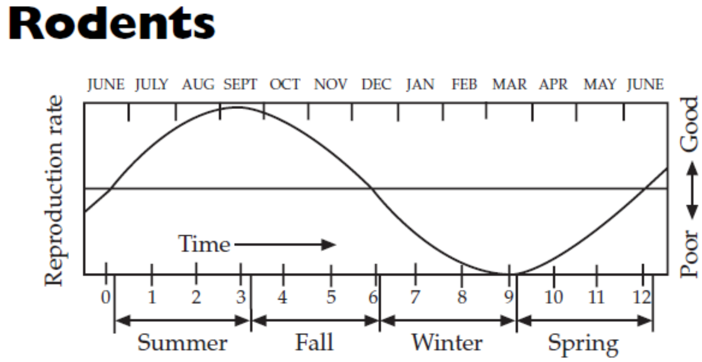
The vertical shift of the sinusoid from a wave at ground level is 6 feet down.

P=2*20=40minutes, so $b = \frac{2\pi}{40}$

So the equation would be

$$h(t) = 13\cos\left(\frac{2\pi t}{40}\right) + 7$$

5. The graph below shows how the reproductive rate of rodents varies depending on the season. On the *x*-axis, the months are grouped by season and on the *y*-axis, the reproductive rate is represented on a scale from poor to good.



a) When is the reproduction of the rodents at the lowest? When is it at the highest?

The reproduction of the rodents is at the lowest in March and at its highest in September.

b) Put 0.2 and 2 as the minimum and maximum values on the *y*-axis. Design a formula that describes the graph. Explain how you determined your formula.

$$A=\frac{\max-\min}{2}=\frac{2-0.2}{2}=\frac{1.8}{2}=0.9$$

P=12 months, so $b=\frac{2\pi}{12}$

To obtain the min value of 0.2 we need to do a vertical shift on the parent function cos(t) by 1.1(0.9+0.2=1.1 because we need the cosine 0.2 above the x-axis)

Since the max value is obtain in March (not in January) there is a horizontal shift 3 units to the right, which in the equation is translated by (t - 3)

$$y=0.9\cos\left(\frac{2\pi}{12}(t-3)\right)+1.1$$

c) Suppose the reproductive rate were put on a scale from 0, for extremely poor, to 10 for extremely good. In this case, use 0 and 10 as the minimum and maximum values on the *y*-axis. Design a formula that describes the corresponding graph. What changes did you have to make to your formula form part (b)?

The formula that describes the corresponding graph is:

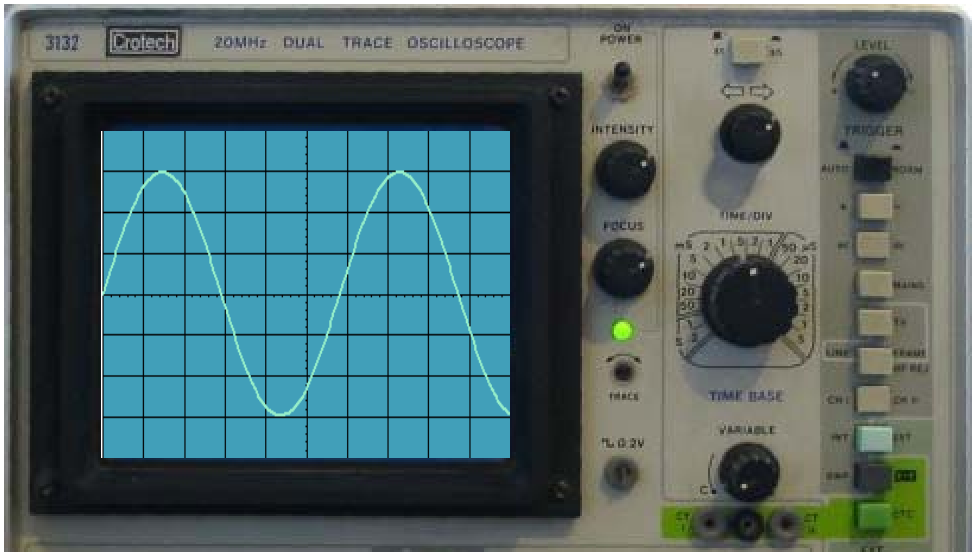
$y=5\cos\left(\frac{2\pi}{12}(t-3)\right)+5$ The necessary changes are in the amplitude and the vertical shift.

The amplitude is different since the minimum and maximum are changed:

$$A=\frac{\max-\min}{2}=\frac{10-0}{2}=5$$

Halfway between the min and max values we have the midline which tells us that there is a vertical shift 5 units up.

6. **An oscilloscope is a machine that measures the magnitude of fluctuating voltages by displaying a graph of the voltage over time. The figure below shows the shape of the fluctuating voltage. The horizontal axis displays time, *t*, and the vertical axis shows voltage. The person who is working with scope can see from the buttons that in this case, one step on the horizontal-axis scale is 0.5 sec and one step on the vertical-axis scale is 0.2 V.**



Design the formula for this fluctuating voltage. Explain how you determined your formula.

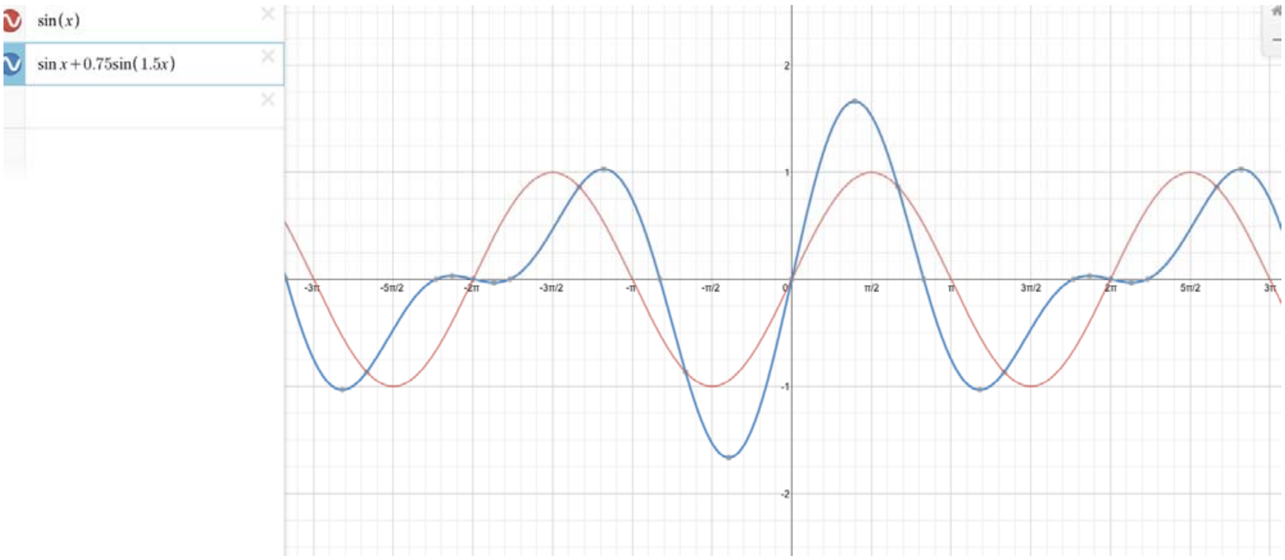
$$V(t) = 0.6\sin\left(\frac{2\pi}{3}(t-0.4)\right)$$

From the picture we can see that the sinusoidal is symmetric with respect to the x axis so the amplitude is 3units on the y-axis and since the vertical scale is 0.2 we get the amplitude A=3*0.2=0.6

The period P is defined as the amount of time it takes to complete a full revolution so the distance between the maximum points is <u>about</u> 6 units on the x-axis and since we know that the horizontal scale is 0.5, the period P=6*0.5=3, so $b=\frac{2\pi}{3}$. The graph is shifted horizontally 4/5 of a horizontal unit =4/5*0.5=0.4 to the right, so we need (t-0.4) factored in the equation.

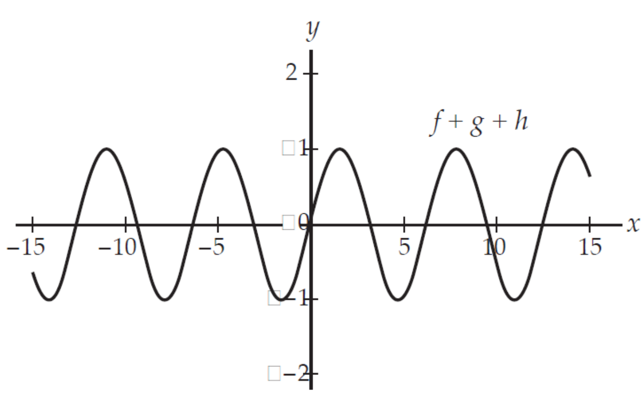
7. **One modern application of the addition and subtraction of functions appears in the field of audio engineering. To help understand the idea behind this use, consider the following simplified example. Suppose that the function $f(x) = \sin(x)$ represents the sound of music to which you are listening. Unfortunately, there is background noise. Let $g(x) = 0.75\sin(1.5x)$ represent that noise.**

a) Sketch a graph of f and the sum $f + g$, on a single set of axes. Your graph represents both what you want to hear and what you actually do hear. They clearly are not the same.



b) Now apply some mathematics to engineer away the noise. You can add a microphone to your headphones. In turn, the microphone picks up the background noise, $g(x)$, then plays back through our headphones an altered version, $h(x)$. Thus what you now hear in the headphones is the sum of three functions: f (what you want to hear), g (the noise you don't want), and h (the correction for the noise). Suppose the compensating function is $h(x) = 0.75\sin(1.5x - \pi)$. Graph the new sum, $f + g + h$, to see what you hear now. Explain the idea behind the adjustment.

The function h is exactly the opposite of g, so $g + h$ is the constant function $y = 0$. Hence, $f + g + h$, is what you want to hear.



c) Another engineer suggests adding $h(x) = -0.75\sin(1.5x)$ instead of the *h* defined in part (b). Discuss the merits of that solution.

This is also a good solution. Once possible drawback, though is that it requires playing the solution at the same time the background noise is heard. This means that the detection by the microphone and the construction of the solution curve must occur in the amount of time that the original noise travels from the microphone to the headphone. That may be possible, but it sure is fast!

8. The price of oranges fluctuates, depending on the season. The average price during a complete year is $2.25 per kilogram (about 2.2 pounds). The lowest price will be paid in mid-February ($1.60 per kilo); the highest price is paid in mid-August. Assume the price fluctuations are sinusoidal in nature.

a) Design a formula using the cosine to describe the price of oranges by months during a year. Let $t = 0$ represent January 1, $t = 1$, February, and so forth. Explain how you determined your answer.

- A full cycle of the cosine function starts with a maximum, followed by an intercept, minimum, intercept, maximum; they are known as the five key points. In our case we have a minimum in February, so since we start with a minimum we need to reflect the cosine with respect to the x axis, so the coefficient of the cosine will be <u>negative.</u>

- The midline is the <u>horizontal</u> line who divides the sinusoidal function in two equal parts. The average price 2.25 tells us the equation of the midline y=2.25. The distance between the midline and the minimum is the amplitude A=2.25-1.60=0.65

- The distance between the midline and the x-axis is the vertical shift that is done with respect to the x-axis(because the x-axis is the midline of the parent function), so we calculate this distance by adding the minimum to the amplitude VS=1.60+0.65=2.25. It would have been easier to observe that the midline is 2.25 above the x-axis and that number will represent the vertical shift.

- The period is twice the time between the minimum and the maximum. The minimum happened in mid February and the maximum in mid August so the period P=6months, so $b = \dfrac{2\pi}{6}$

- There is a horizontal shift of the parent function 1 unit to the right since the minimum happens in February and not in January so the argument of the cosine will be $(t-1)$

- So the formula that describes the price of oranges by months is:

$$C(t) = -0.65\cos\left(\frac{2\pi}{6}(t-1)\right) + 2.25$$

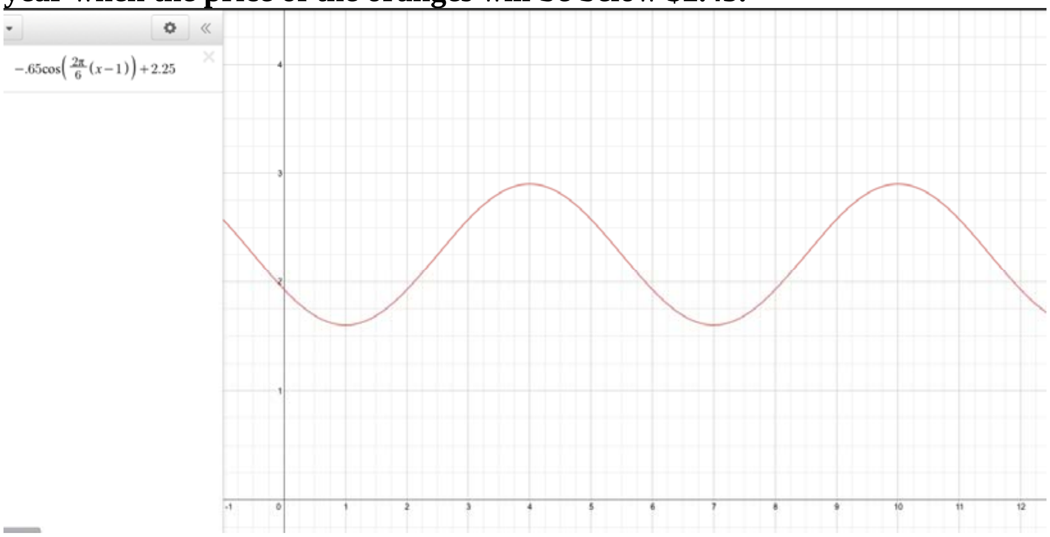
b)How would your formula change if you used the sine function instead of cosine? Explain how you arrived at your answer.

If the sine function is shifted left by $\frac{\pi}{2}$ we get the cosine function. Remember that a horizontal shift to the left changes the argument to $\left(\theta+\frac{\pi}{2}\right)$ so the equation is:

$\cos\theta=\sin\left(\theta+\frac{\pi}{2}\right)$ **. Let's apply this to our function:**

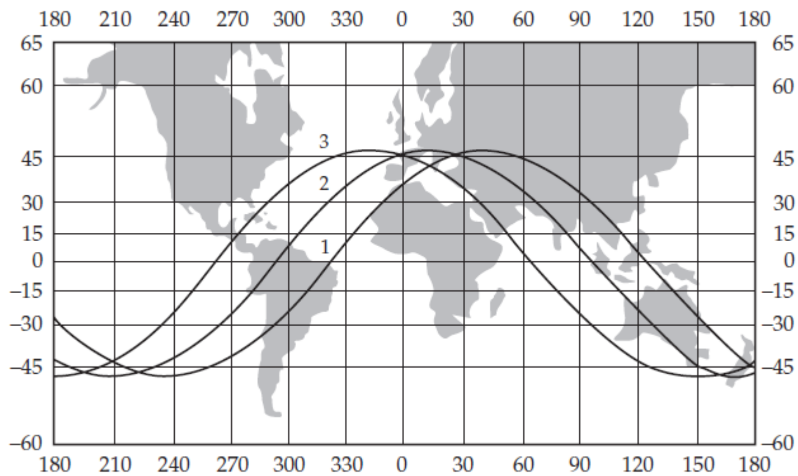
$$C(t)=-0.65\cos\left(\frac{2\pi}{6}(t-1)\right)+2.25=-0.65\sin\left(\frac{2\pi}{6}(t-1)+\frac{\pi}{2}\right)+2.25$$

c) Sketch the graph of your model from part (a). Then read from your graph the times of the year when the price of the oranges will be below $2.45.



From the graph we see that the price of the oranges will be below $2.45 in: January, February, the first three weeks in March, the last three weeks in June, July, August, the first three weeks in September and the last three weeks in December.

9. A satellite is circling west-to-east around the earth. Below are the projections of three orbits of the satellite, labeled as curves 1, 2, and 3. The projections are given in terms of longitude and latitude readings (both are in degrees). Curve 1, can be expressed as a sinusoidal function of the form $y = A\sin(B(x-C))+D$.



a) Determine values for *A, B, C,* and *D* to produce a sinusoidal model that you think best describes Curve 1.

Since A = 47, B = 0.5, C = -40 and D = 0, the equation describing curve 1 will be:

$$y = 47\sin\left(0.5\left(x-\left(-40\right)\right)\right)+0$$
$$y = 47\sin\left(0.5\left(x+40\right)\right)$$
$$y = 47\sin\left(0.5\left(x+68\right)\right)$$
$$y = 47\sin\left(0.5\left(x+100\right)\right)$$

b) What constants in your sinusoidal model for (a) will you need to modify in order to describe Curves 2 and 3? What are your models describing these curves?

$$y = 47\sin\left(0.5\left(x+68\right)\right)$$
$$y = 47\sin\left(0.5\left(x+100\right)\right)$$