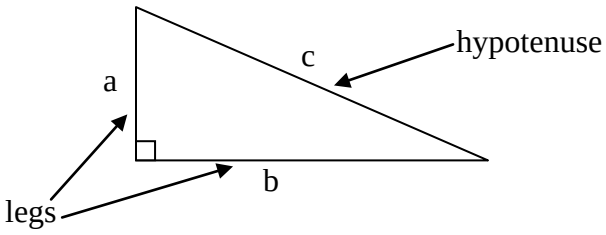
Pre-Algebra Notes – Unit 10:
The Pythagorean Theorem

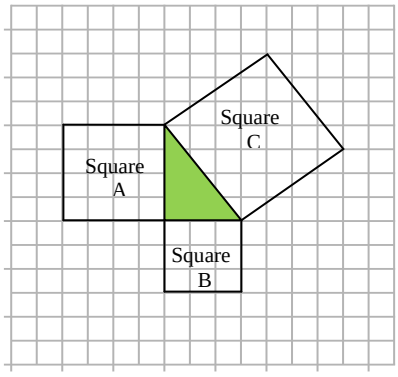
CCSS 8.G.6: Explain a proof of the Pythagorean Theorem and its converse.

In a right triangle, the side opposite the right angle is called the **hypotenuse**. The **legs** are the sides that form the right angle. We typically label the legs a and b, while the hypotenuse is c.

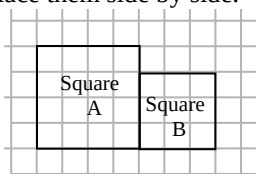


If we look at enough right triangles and experiment a little, we will notice a relationship developing.

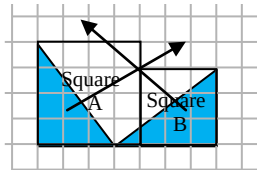
Draw a right triangle on a piece of graph paper, with leg measurements of 3 units and 4 units. Draw a square on each side of the triangle as shown.



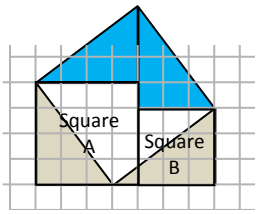
Cut out squares A and B and place them side by side.



Draw two triangles as shown; cut along the hypotenuses and slide as shown.



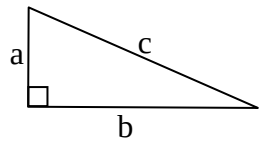
Tape the repositioned triangles. Compare your new square with square C.



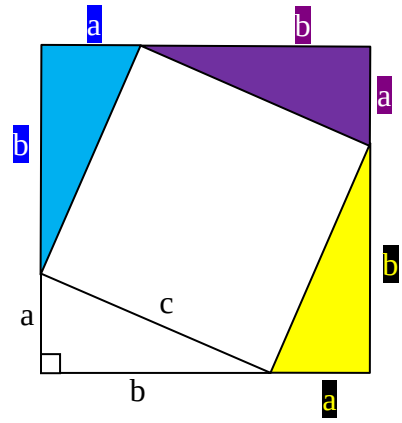
It appears that the *sum of the areas of the squares formed by the legs* is equal to *the area of the square formed by the hypotenuse.*

There is another way to look at this relationship.

Begin with a right triangle, legs a and b with hypotenuse c.



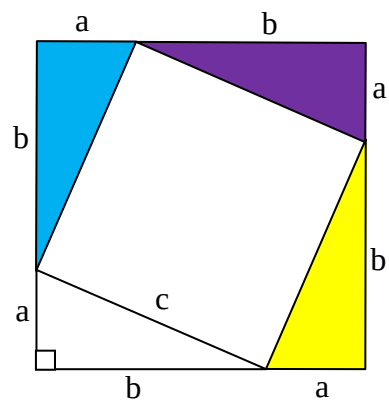
Create a square that has measure $a+b$ on a side, using 4 congruent triangles as shown.



The area of this new square can be expressed two ways:

1) $A = (a+b)^2$ (length of a side is $a+b$, so multiply sides to get area)

2) $A = 4\left(\frac{1}{2}ab\right)+c^2$ (4 triangles, each with area of $\frac{1}{2}ab$; add to the area of the square with side c)



Setting these two values equal to each other, we get

$$(a+b)^2 = 4\left(\frac{1}{2}ab\right)+c^2$$

$$a^2+2ab+b^2 = 2ab+c^2$$

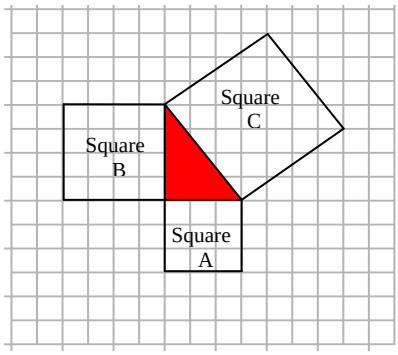
$$-2ab = -2ab$$

$$a^2+b^2=c^2$$

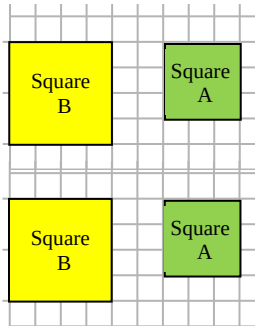
This is an important relationship in mathematics. Since it is important, we will give this relationship a name, the *Pythagorean Theorem*.

And yet another way to model the theorem:

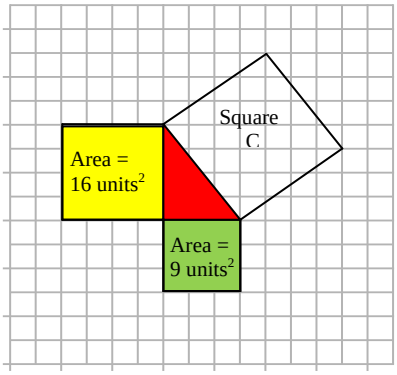
Draw a right triangle on a piece of graph paper, with leg measurements of 3 units and 4 units. Draw a square on each side of the triangle as shown.



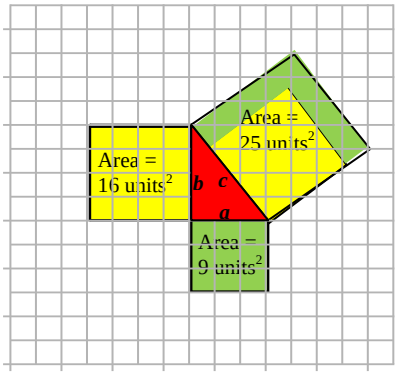
Cut out 2 squares each, congruent to A and B, in two different colors



Glue one Square A and one Square B on the original graph. Label with the area.



Have students cut and paste the remaining 2 squares (9 + 16) to form a third square C, which represents the area of 25.



Have students cut and paste the remaining 2 squares (9 + 16) to form a third square C, which represents the area of 25.

Finally, have students label the legs of the original right triangle with "a" and "b", hypotenuse "c". (This represents one possible solution.)

It appears that the *sum of the areas of the squares formed by the legs* is equal to *the area of the square formed by the hypotenuse*, or $a^2 + b^2 = c^2$.

Pythagorean Theorem:

If a triangle is a right triangle, then $a^2 + b^2 = c^2$.

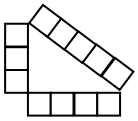
While these models are helpful, the standard does say to explain a *proof* of the theorem.
For more discussion regarding this, please go to the following document:

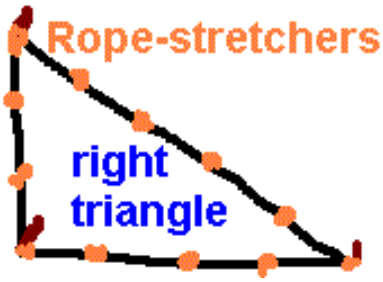
https://cpdmath.files.wordpress.com/2012/03/core-principles-8-g-6.pdf

The ***converse*** of the Pythagorean Theorem says "If $a^2 + b^2 = c^2$**,** then the triangle is a right triangle." Let's see if this seems to be true.

Fill in the first empty column in the chart below. Then, use grid strips cut to one square width to create the triangle with the 3 side lengths given (see example below the chart). Does it appear to be a right triangle?

a	**b**	**c**	**Is $a^2 + b^2 = c^2$ true?**	**Does this appear to be a right triangle?**
3	4	5	$3^2 + 4^2 = 25$ $25 = 25$ true	Yes, see below.
5	12	13		
7	24	25		
8	15	17		

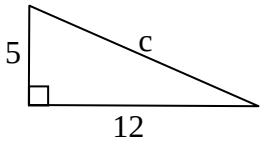
3-4-5  Looks like a right triangle.



To model the converse, use the example of the Egyptian rope-stretchers. Some ancient tombs depict scribes carrying ropes tied with 12 equally spaced knots. It turns out that if the rope was pegged to the ground in the dimensions of 3-4-5, a right triangle would emerge. This enabled the Ancient Egyptian builders to lay the foundations for their buildings accurately. To model this, decide on a unit length and make 13 knots (representing this unit's measure) in a piece of rope or twine. This way you can hold the first and last knot together. Using 3 people you can stretch the rope into a 3-4-5 triangle and model where/how the right angle occurred.

CCSS 8.G.7: Apply the Pythagorean Theorem to determine unknown side lengths in right triangles in real-world and mathematical problems in two dimensions.

Example: Find the unknown length *c* in simplest form.



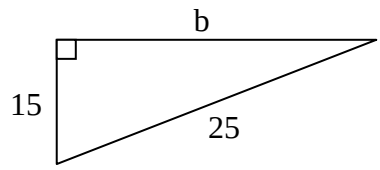
$$a^2 + b^2 = c^2$$
$$5^2 + 12^2 = c^2$$
$$25 + 144 = c^2$$
$$169 = c^2$$
$$\sqrt{169} = c$$
$$13 = c$$

Please note that this 5-12-13 is one of our "Pythagorean triples". A *Pythagorean triple* is a set of three positive integers, *a*, *b*, and *c* such that $a^2 + b^2 = c^2$. Another triple worth noting is 3-4-5.

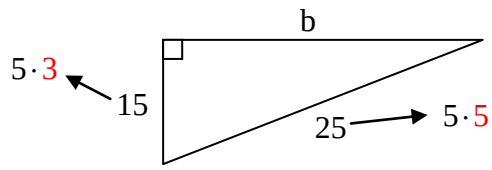
You can create other Pythagorean triples by multiplying the original triple by a factor. For instance, 3-4-5 becomes a 6-8-10 by multiplying all the sides by 2. One might create another, 10-24-26, by multiplying the 5-12-13 by 2.

This skill is very valuable when working with problems on the Nevada CRT and NV High School Proficiency Exam. It can save the student time and frustration!

Example: Find the unknown length.



If we recognize the common factor of 5, we can think:



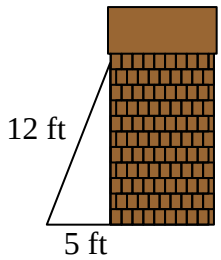
This is a leg and the hypotenuse of the 3-4-5. So the missing side would be $5 \cdot 4$, which is 20. We had a 3-4-5 that was multiplied by a factor of 5 to produce a 15-20-25. So $b = 20$.

Example: A 12-foot ladder is placed against a building. The foot of the ladder is 5 feet from the base of the building. How high up the side of the building does the ladder reach? Give your answer to the nearest foot.

$$a^2 + b^2 = c^2$$
$$5^2 + b^2 = 12^2$$
$$25 + b^2 = 144$$
$$b^2 = 119$$
$$b = \sqrt{119}$$

$$100 < 119 < 121$$
$$\sqrt{100} < \sqrt{119} < \sqrt{121}$$
$$10 < \sqrt{119} < 11$$

Since 119 is closer to 121, $\sqrt{119}$ is closer to 11. Therefore, the ladder reaches approximately 11 feet up the side of the building.

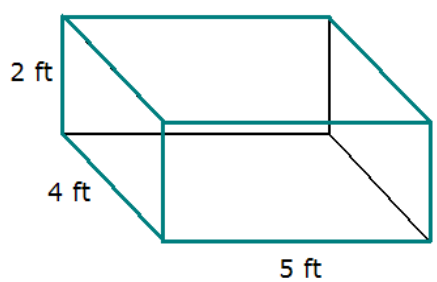


Example: You use four stakes and string to mark the foundation of a house. You want to make sure the foundation is rectangular. The four sides measure 30 ft, 72 ft, 30 ft and 72 ft. You then measure a diagonal to be 78 ft. Is your foundation rectangular?

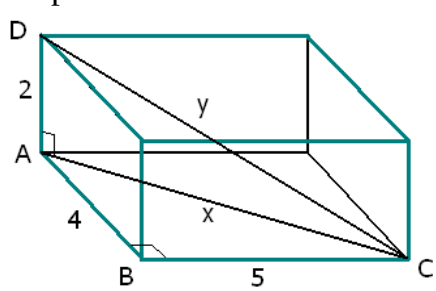
The diagonal divides the foundation into 2 triangles. If the triangles are right triangles, $a^2 + b^2 = c^2$ will be a true statement. Because $30^2 + 72^2 = 78^2$ (900 + 5184 = 6084), you can conclude that the triangles are right and your foundation is rectangular.

CCSS 8.G.7: Apply the Pythagorean Theorem to determine unknown side lengths in right triangles in real-world and mathematical problems in three dimensions.

Example: Find the length of the longest line segment (called the main diagonal) in the rectangular prism shown on the picture below.



Solution: We will apply the Pythagorean Theorem twice. First, label the points and sides that will used on the picture.



We will find x using the Pythagorean Theorem in triangle ABC. Then we can find y using the Pythagorean Theorem in triangle ACD.

For $\triangle ABC$, $a^2 + b^2 = c^2$

$$4^2 + 5^2 = x^2$$

$$16 + 25 = x^2$$

$$41 = x^2$$

We do not have to simplify x^2 as we need it in that form in the Pythagorean Theorem:

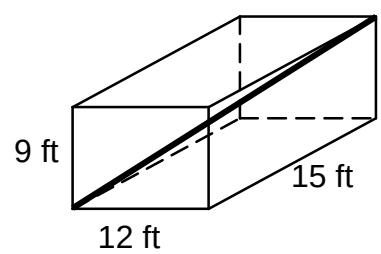
For $\triangle ACD$, $a^2 + b^2 = c^2$

$$2^2 + 41 = y^2$$
$$45 = y^2$$
$$\sqrt{45} = \sqrt{y^2}$$
$$\sqrt{45} = y$$

The length of the main diagonal would be $\sqrt{45}$ *feet (or approximately 6.7 feet).*

Note: Our result is actually $\sqrt{2^2 + 4^2 + 5^2}$. Indeed, we can see that the length of the main diagonal in a rectangular prism with sides x, y, and z is $L = \sqrt{x^2 + y^2 + z^2}$. This is sometimes called the *3-dimensional Pythagorean Theorem*.

Another Rectangular Box problem: You have a huge rectangular box with dimensions $12\text{ft} \times 15\text{ft} \times 9\text{ft}$. What is the length of the longest rod you could fit in your box?

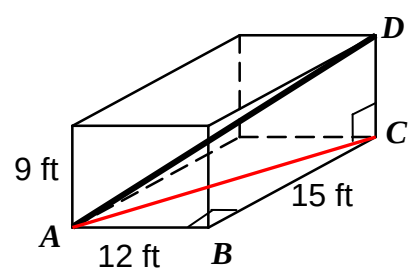


Solution: We will apply the Pythagorean Theorem twice. First, label the points and sides that will used on the picture.

We will find $\overline{AC}$ using the Pythagorean Theorem in triangle ABC. Then we can find $\overline{AD}$ using the Pythagorean Theorem in triangle ACD.

For $\triangle ABC$, $12^2 + 15^2 = x^2$

$$144 + 225 = x^2$$
$$369 = x^2$$



For $\triangle ACD$, $9^2 + 369 = y^2$

$$450 = y^2$$
$$\sqrt{450} = \sqrt{y^2}$$
$$\sqrt{450} = y$$

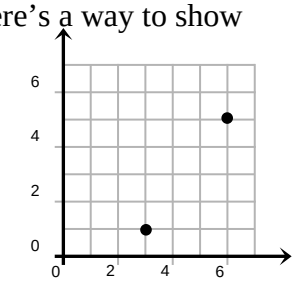
The length of the longest rod would be $\sqrt{450}$ *feet (or approximately 21.2 feet).*

The Distance Formula

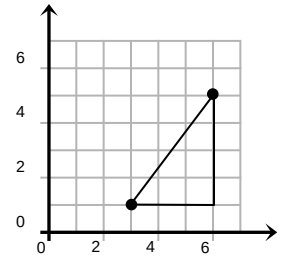
CCSS 8.G.8: Apply the Pythagorean Theorem to find the distance between two points in a coordinate system.

Take the time to link the distance formula with the Pythagorean Theorem. Here's a way to show it.

You are given two points, (3, 1) and (6, 5). You are asked to find the distance between them. Plot the points.



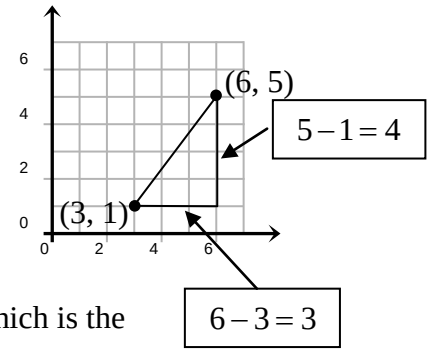
You can draw a right triangle, using these points as two of the vertices, as shown.



Find the lengths of the horizontal and vertical sides of the right triangle by subtracting the *x*- and *y*- values.

or

You could simply count the units from one point to the next.



We now will use the Pythagorean Theorem to find the missing length, which is the hypotenuse.

$$c^2 = a^2 + b^2$$
$$c^2 = (6-3)^2 + (5-1)^2$$
$$c^2 = 3^2 + 4^2$$
$$c^2 = 9 + 16$$
$$c^2 = 25$$
$$\sqrt{c^2} = \sqrt{25}$$

$c = \pm 5$; however, since we are looking at a measurement, only the positive value will be considered.

Now, instead of using the ordered pairs (3, 1) and (6, 5), we use a pair of points, (x_1, y_1) and (x_2, y_2). Substituting into the Pythagorean Theorem once again:

$$c^2 = a^2 + b^2$$
$$c^2 = (x_2 - x_1)^2 + (y_2 + y_1)^2$$
$$c = \sqrt{(x_2 - x_1)^2 + (y_2 + y_1)^2}$$

Since we are determining distance, we replace the *c* with *d, and we will only consider the positive value.*

So we now have the **distance formula**:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Example: Find the distance between the points $A(2,-1)$ and $B(-2,\ 5)$.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$d = \sqrt{[(-2) - 2]^2 + [5 - (-1)]^2}$$
$$d = \sqrt{(-4)^2 + 6^2}$$
$$d = \sqrt{16 + 36}$$
$$d = \sqrt{52}$$
$$d = \sqrt{4} \cdot \sqrt{13}$$
$$d = 2\sqrt{13}$$

($\sqrt{52}$ is between 7 and 8, but closer to 7)

The distance between the points $A(2,-1)$ and $B(-2,\ 5)$ is $\sqrt{52}$ or $2\sqrt{13}$ units.

Since $\sqrt{49} < \sqrt{52} < \sqrt{64}$ and $\frac{3}{15} = .2$ then $\sqrt{52} \approx 7.2$ units

$7 < \sqrt{52} < 8$