Trigonometry and Angles

Math Background

Previously, you

- Simplified linear, quadratic, radical, polynomial, rational, log and exponential functions
- Performed arithmetic operations with linear, quadratic, radical, log and exponential functions
- Identified inverses of functions
- Transformed parent functions of linear, quadratic, radical, polynomial, rational, log & exponential functions

In this unit you will

- Explore trigonometric functions
- Work with angles in degree and radian measure
- Use the unit circle to understand trigonometric relationships

You can use the skills in this unit to

- Solve right triangles
- Measure angles in standard position
- Interpret the domain and its restrictions of a real-life function.
- Evaluate inverses of trigonometric functions.
- Model and solve real-world problems with inverse trigonometric functions

Vocabulary

- **Angle of rotation –** It is the measure of degrees that a figure is rotated about a fixed point.
- **Cosine function** – The ratio of the length of the side adjacent to an acute angle of a right triangle to the length of the hypotenuse.
- **Coterminal angles –** They are angles in standard position that have a common terminal side.
- **Initial side –** The ray on the x-axis when an angle is in standard position.
- **Intercepted arc –** Corresponding to an angle, this is the portion of the circle that lies in the interior of the angle together with the endpoints of the arc.
- **Quadrantal angle –** An angle with its terminal side on the x or y axis (0, 90, 180 and 270 degrees).
- **Radian measure** – It is the standard unit of angular measure. One radian is equal to the angle created by taking the radius of a circle and stretching it along the edge of the circle. 1 rad = 180/pi
- **Reference angle –** In standard position, the reference angle is the smallest angle between the terminal side and the x-axis.
- **Sine function** – The ratio of the length of the side opposite an acute angle of a right triangle to the length of the hypotenuse.
- **Special right triangles** – Right triangles whose two acute angles are both 45 degrees or if the two acute angles are 30 degrees and 60 degrees.
- **Standard position** – An angle with its vertex on the origin and one ray (side) on the positive x-axis in the coordinate plane.
- **Tangent function** – The ratio of the length of the side opposite an acute angle of a right triangle to the length of the adjacent side of that acute angle.
- **Terminal side –** The other ray (side) of the angle that is not considered the initial side when the angle is in standard position.

- **Unit circle –** It is a circle with a center at (0, 0) and a radius of one unit. Angles are measured starting from the positive x-axis in Quadrant I and continue around the unit circle.

Essential Questions

- How can I solve a right triangle?
- What is an angle of rotation and how is it measured? Why do we need radian measure?
- How can sine, cosine and tangent functions be defined using the unit circle?
- How can inverse trig functions help me to solve real-world problems?

Overall Big Ideas

Solving right triangles has many applications in the real world. Radian measure establishes a way to measure angles with respect to arc length. The trig functions can describe rotations around the unit circle.

Skill

To solve right triangles using trigonometric functions.

To measure angles in standard position using degree and radian measure.

To find values of trigonometric functions on the unit circle.

To evaluate inverse trigonometric functions and use them to solve problems.

Related Standards

F.TF.A.1
Using radian measure of an angle as the length of the arc on the unit circle subtended by the angle.

F.TF.A.2
Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle.

F.TF.A.3
Use special triangles to determine geometrically the values of sine, cosine, tangent for $\frac{\pi}{3}, \frac{\pi}{4},$ and $\frac{\pi}{6}$, and use the unit circle to express the values of sine, cosine, and tangent for $\pi - x,\ \pi + x,$ and $2\pi - x$ in terms of their values for x, where x is any real number.

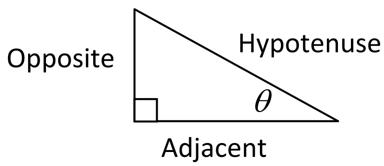
F.TF.A.4
Use the unit circle to explain symmetry (odd and even) and periodicity of trigonometric functions.

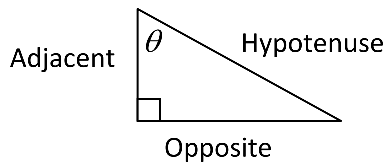
F.TF.B.6
Understand that restricting a trigonometric function to a domain on which it is always increasing or always decreasing allows its inverse to be constructed.

F.TF.B.7
Use inverse functions to solve trigonometric equations that arise in modeling contexts; evaluate the solutions using technology, and interpret them in terms of the context. *(Modeling Standard)

Notes, Examples, and Exam Questions

Unit 8.1: To solve right triangles using trigonometric functions





Note: The adjacent and opposite sides are always the legs of the right triangle, and depend upon which angle is used.

Trigonometric Ratios of θ:

Sine: $\sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}}$ Cosine: $\cos\theta = \dfrac{\text{adjacent}}{\text{hypotenuse}}$ Tangent: $\tan\theta = \dfrac{\text{opposite}}{\text{adjacent}}$

Memory Aid: SOHCAHTOA

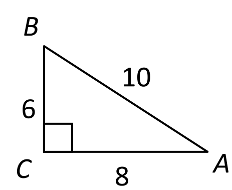
Reciprocal Functions:

Cosecant: $\csc\theta = \dfrac{\text{hypotenuse}}{\text{opposite}}$ Secant: $\sec\theta = \dfrac{\text{hypotenuse}}{\text{adjacent}}$ Cotangent: $\cot\theta = \dfrac{\text{adjacent}}{\text{opposite}}$

Think About It: Which trigonometric ratios in a triangle must always be less than 1? Why?

Sine and cosine, because the lengths of the opposite and adjacent legs are always smaller than the length of the hypotenuse.

Ex 1: For ΔABC, find the six trig ratios of $\angle A$.



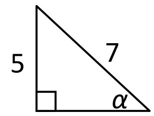
$\sin A = \dfrac{6}{10} = \dfrac{3}{5}$ $\cos A = \dfrac{8}{10} = \dfrac{4}{5}$ $\tan A = \dfrac{6}{8} = \dfrac{3}{4}$

$\csc A = \dfrac{5}{3}$ $\sec A = \dfrac{5}{4}$ $\cot A = \dfrac{4}{3}$

Teacher Note: It may help students to label the sides as Opp, Adj, and Hyp first. Have students try finding the six trig ratios for $\angle B$.

Special Relationship: $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

Ex 2: Find the other trig ratios given $\sin\alpha = \dfrac{5}{7}$.



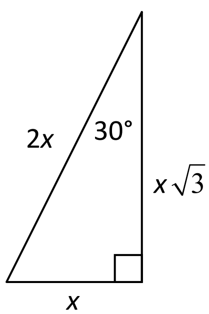
$\sin\alpha = \dfrac{5}{7} = \dfrac{\text{opp}}{\text{hyp}}$

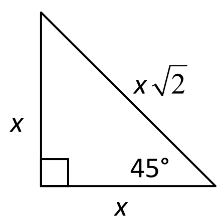
Find the missing leg (a) using the Pythagorean Theorem: $a^2 + 5^2 = 7^2 \Rightarrow a = \sqrt{24}$

This is the adjacent leg of angle α.

$$\cos\alpha = \frac{\sqrt{24}}{7},\quad \tan\alpha = \frac{5}{\sqrt{24}},\quad \csc\alpha = \frac{7}{5},\quad \sec\alpha = \frac{7}{\sqrt{24}},\quad \cot\alpha = \frac{\sqrt{24}}{5}$$

Special Right Triangles: 30-60-90 & 45-45-90





θ (Degrees)	**30°**	**45°**	**60°**
θ (Radians)	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$
sinθ	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
cosθ	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$
tanθ	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$

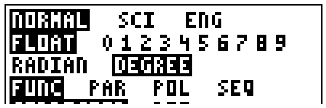
Teacher Note: Have students fill in the table (un-bolded values). This table must be memorized!

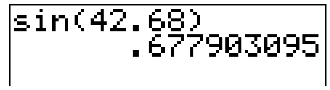
Evaluating Trigonometric Ratios on the Calculator

Note: Check the MODE on your calculator and be sure it is correct for the question asked (radian/degree). Unless an angle measure is shown with the degree symbol, assume the angle is in radians. The default mode for the TI-84 is radians.

Ex 3: Evaluate the following using a calculator.

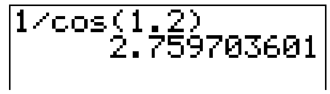
1. $\sin 42.68^\circ$ Mode: degree





$\sin 42.68^\circ \approx$ 0.678

2. $\sec 1.2$ Mode: radian



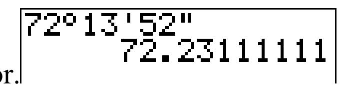
Note: There is not a key for secant. We must use $1/(\cos x)$ because secant is the reciprocal of cosine.

$\sec 1.2 \approx$ 2.760

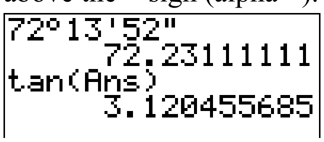
Also recall: $\csc x = \dfrac{1}{\sin x}$ and $\cot x = \dfrac{1}{\tan x}$

3. $\tan 72^\circ 13' 52''$ Mode: degree

Convert DMS to decimal degrees on calculator.



Note: The degree and minute symbols can be found in the ANGLE menu. The seconds symbol can be found above the + sign (alpha +).

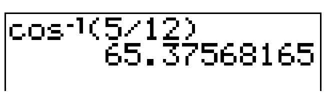


$\tan 72^\circ 13' 52'' \approx$ 3.120

Inverse Trigonometric Functions: use these to find the angle when given a trig ratio

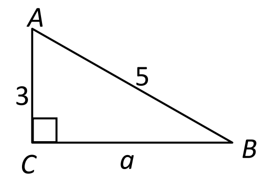
Ex 4: Find θ (in degrees) if $\cos\theta = \dfrac{5}{12}$.

$\theta = \cos^{-1}\left(\dfrac{5}{12}\right)$ Mode: degrees



$\theta \approx 65.376^\circ$

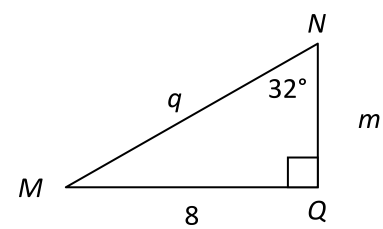
Solving a Triangle: Find the missing angles and sides with given information



Ex 5: Solve the triangle (find all missing sides and angles).

$\cos A = \frac{3}{5} \Rightarrow A = \cos^{-1}\left(\frac{3}{5}\right) \approx 53.130^\circ$ $\quad m\angle B \approx 90 - 53.130 = 36.87^\circ$

$a = 4$ (Pythagorean Triple: 3-4-5; or use the Pythagorean Thm.) $\boxed{m\angle A \approx 53.13^\circ,\ m\angle B \approx 36.87^\circ,\ a = 4}$



Ex 6: Solve the triangle.

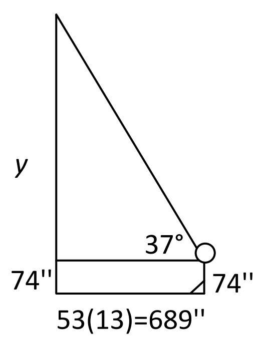
$m\angle M = 90 - 32 = 58^\circ$

$\sin 32 = \frac{8}{q} \Rightarrow q = \frac{8}{\sin 32} \Rightarrow q \approx 15.097$

$\tan 58 = \frac{m}{8} \Rightarrow 8 \cdot \tan 58 = m \Rightarrow m \approx 12.80$ $\quad \boxed{m\angle M = 58^\circ, q \approx 15.097, m \approx 12.80}$

Application Problem

Ex 7: A 6 ft 2 in man looks up at a 37° angle to the top of a building. He places his heel to toe 53 times and his shoe is 13 in. How tall is the building?



Draw a picture: The height of the building is (y + 74) inches.

Use trig in the right triangle to solve for y: $\tan 37^\circ = \frac{y}{689} \Rightarrow y = 689 \tan 37^\circ \approx 519.199$

Height of the building = $y + 74 = 519.199 + 74 = 593.199$ inches

Convert to feet: The building is approximately $\frac{593.199}{12} \approx \boxed{49.433\text{ ft}}$ tall.

QOD: Explain how to find the inverse cotangent of an angle on the calculator.

SAMPLE EXAM QUESTIONS

Use ΔABC for questions 1 and 2.

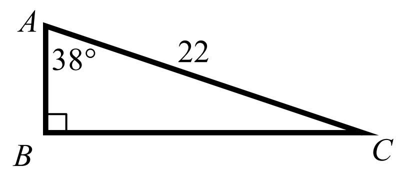
1. **Which of the following is equal to $\cos 38°$?**

A. $\sin 38°$

B. $\tan 52°$

C. $\cos 52°$

D. $\sin 52°$



Ans: D

2. **Which expression represents the length of $\overline{AB}$?**

A. $22\sin 38°$ **B.** $22\cos 38°$ **C.** $22\tan 38°$

Ans: B

3. **If $\tan\theta = \frac{3}{4}$, what is $\sin\theta$?**

A. $\frac{4}{5}$

B. $\frac{3}{5}$

C. $\frac{5}{3}$

D. $\frac{4}{3}$

Ans: B

4. **Find $\sin\theta \cdot \cos^2\theta$ when $\sin\theta = \frac{1}{4}$ and θ is in Quadrant I.**

A. $\frac{\sqrt{15}}{4}$

B. $\frac{15}{64}$

C. $\frac{15}{256}$

D. $\frac{3}{16}$

Ans: B

<u>Unit 8.2: To measure angles in standard position using degree and radian measure.</u>

a) Convert 1.2 hours into hours and minutes.
Solution: 1 hour + (0.2)(60) = 1 hour and 12 minutes

b) Convert 3 hours and 20 minutes into hours.
Solution: 3 hours + $\frac{20}{60}$ hours = 3.3 hours

<u>Babylonian Number System</u>: based on the number 60; 360 approximates the number of days in a year. Circles were divided into 360 degrees.

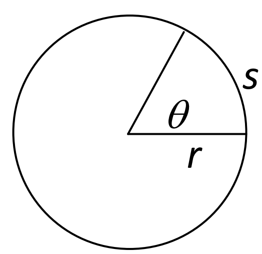
A degree can further be divided into 60 **minutes (60')**, and each minute can be divided into 60 **seconds (60'')**.

<u>Converting from Degrees to DMS (Degrees – Minutes – Seconds):</u> **multiply** by 60

Ex 8: Convert 114.59° to DMS.
114.59° = 114° + 0.59(60)' = 114° 35.4' = 114° 35' +0.4(60)" = $\boxed{114^\circ 35' 24''}$

<u>Converting to Degrees (decimal):</u> **divide** by 60

Ex 9: Convert 72° 13' 52" to decimal degrees.
72° 13' 52" = $72 + \frac{13}{60} + \frac{52}{3600} = \boxed{72.23^\circ}$



<u>Radian</u>: a measure of length; $\theta = 1$ radian when $r = s$
r = radius, s = length of arc

<u>Recall</u>: Circumference of a Circle; $C = 2\pi r$

So, there are 2π radians around the circle 2π radians = 360°, or $\boxed{\pi \text{ radians} = 180^\circ}$

<u>Converting Degrees (*D*) to Radians (*R*)</u>: $D \cdot \frac{\pi}{180^\circ} = R$ (Note: $\frac{\pi}{180^\circ} = 1$)

Ex 10: Convert 540° to radians.

$$540^\circ \cdot \frac{\pi}{180} = \boxed{3\pi \text{ radians}}$$

Note: We multiply by $\frac{\pi}{180^\circ}$ so that the degrees cancel. This will help you remember what to multiply by.

Converting from Radians (R) to Degrees (D): $D = R \cdot \frac{180^\circ}{\pi}$

Or use the proportion: $\frac{\theta_d}{180^\circ} = \frac{\theta_r}{\pi}$

Ex 11: Convert $\frac{3\pi}{4}$ radians to degrees.

$$\frac{3\pi}{4} \cdot \frac{180^\circ}{\pi} = \boxed{135^\circ}$$

Note: We multiply by $\frac{180^\circ}{\pi}$ so that the π's cancel.

Special Angles to Memorize (Teacher Note: Have students fill this in for practice.)

Degrees	30°	45°	60°	90°	120°	135°	150°	180°
Radians	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π

Arc Length: $s = r\theta$ (θ must be measured in **radians**)

Ex 12: A circle has an 8 inch diameter. Find the length of an arc intercepted by a 240° central angle.

Step One: Convert to radians. $\theta = 240^\circ \cdot \frac{\pi}{180^\circ} = \frac{4\pi}{3}$

Step Two: Find the radius. $r = \frac{d}{2} = \frac{8}{2} = 4$

Step Three: Solve for s. $s = 4 \cdot \frac{4\pi}{3} = \boxed{\frac{16\pi}{3} \text{ in}}$

Linear Speed (example: miles per hour): $l = \frac{s}{t} = \frac{r\theta}{t} = \frac{\text{length}}{\text{time}}$

Angular Speed (example: rotations per minute): $A = \frac{\theta}{t} = \frac{\text{angle}}{\text{time}}$

Ex 13: Find the speed in mph of 36 in diameter wheels moving at 630 rpm (revolutions per minute).

Unit Conversion: $\frac{630 \text{ rev}}{1 \text{ min}} \cdot \frac{60 \text{ min}}{1 \text{ hr}} \cdot \frac{36\pi \text{ in}}{1 \text{ rev}} \cdot \frac{1 \text{ ft}}{12 \text{ in}} \cdot \frac{1 \text{ mi}}{5280 \text{ ft}} = \frac{945\pi}{44} \approx \boxed{67.473 \text{ mph}}$

1 statute (land) mile = 5280 feet Earth's radius $\approx$ 3956 miles
1 **nautical mile** = 1 minute of arc length along the Earth's equator

Ex 14: Find the linear and angular speed (per second) of a 10.2 cm second hand.

Note: Each revolution generates 2π radians.

Linear Speed: A second hand travels half the circumference in 30 seconds: $l = \frac{10.2\pi}{30} = 0.34\pi \approx \boxed{1.068\text{ cm/sec}}$

or $\frac{10.2(2\pi)}{60} = 0.34\pi \approx \boxed{1.068\text{ cm / sec}}$

Angular Speed: $A = \frac{2\pi}{60} = \boxed{\frac{\pi}{30}\text{radians/sec}}$ or $\frac{180^\circ}{30} = \boxed{6^\circ\text{ / sec}}$

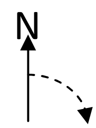
Note: The angular speed does not depend on the length of the second hand!

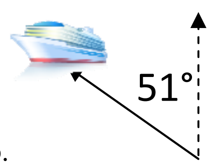
For example: The riders on a carousel all have the same angular speed yet the riders on the outside have a greater linear speed than those on the inside due to a larger radius.

Ex 15: How many statute miles are there in a nautical mile?

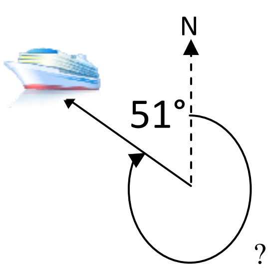
$s = r\theta$ $\quad r = 3956$ $\quad \theta = \frac{\pi}{180^\circ} \cdot \frac{1}{60}^\circ = \frac{\pi}{10800}$ $\quad s = 3956\left(\frac{\pi}{10800}\right) \approx \boxed{1.151\text{ miles}}$

<u>Bearing</u>: the **course** of an object given as the angle measured clockwise from due north





Ex 16: Use the picture to find the bearing of the ship.



$? = 360 - 51 = \boxed{309^\circ}$

<u>QOD</u>: How are radian and degree measures different? How are they similar?

SAMPLE EXAM QUESTIONS

1. **Convert $-60°$ to radians.**

   A. $-\frac{\pi}{3}$ radians

   B. $-\frac{\pi}{6}$ radians

   C. $\frac{\pi}{3}$ radians

   D. 3π radians

Ans: A

2. **An analog watch had been running fast and needed to be set back. In resetting the watch, the minute hand on the watch subtended an arc of $\frac{5\pi}{3}$ radians.**

   Part A: Suppose the radius of the watch is 1 unit. What is the length of the arc on the outside of the watch that the angle subtends?
   Part B: If the watch was at 10:55 before being reset, what is the new time on the watch?

   A. Part A: $\frac{10\pi}{3}$ units

   Part B: 9:05

   B. Part A: $\frac{5\pi}{3}$ units

   Part B: 10:05

   C. Part A: $\frac{5\pi}{3}$ units

   Part B: 9:20

   D. Part A: $\frac{10\pi}{3}$ units

   Part B: 8:40

Ans: B

3. **Convert $\frac{3\pi}{8}$ radians to degrees.** $\frac{3\pi^{rad}}{8}\cdot\frac{180°}{\pi^{rad}}=\frac{135°}{2}=67.5°$

4. **Convert $\frac{16\pi}{3}$ radians to degrees.** $\frac{16\pi^{rad}}{3}\cdot\frac{180°}{\pi^{rad}}=960°$

5. Suppose each paddle on the wall of a clothes dryer makes 80 revolutions per minute.

Part A: What angle does one paddle subtend in 10 seconds? Give your answer in radians.

$$\frac{80\ revolutions}{1min} * \frac{1min}{60\ sec} = \frac{{}^{80}\!/_{6}\ revolutions}{10\ sec}$$

So in 10 seconds you have 80/6 revolutions, which in radians is

$$\frac{80 revolutions}{6} * \frac{2\pi^{rad}}{1 revolution} = \frac{80\pi^{rad}}{3}$$

Part B: Write an algebraic expression to determine the measure in radians of the subtended angle after *x* seconds. Show how the units simplify in your expression.

$$\frac{80 \text{ revolutions}}{1 \text{ minute}} \times \frac{1 \text{ minute}}{60 \text{ seconds}} = \frac{{}^{4}\!/_{3} \text{ revolutions}}{1 \text{ second}}$$

$$\frac{{}^{4}\!/_{3} \text{ revolutions}}{1 \text{ second}} \times \frac{2\pi \text{ radians}}{1 \text{ revolution}} \times (x \text{ seconds}) = \frac{8}{3}\pi x \text{ radians}$$

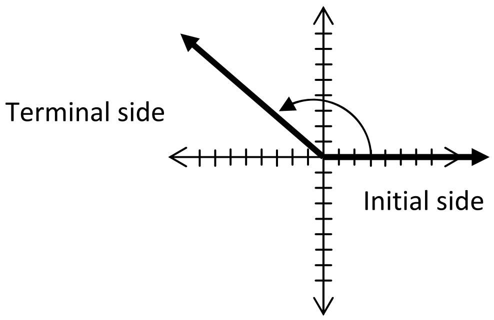
Part C: You are interested in determining the total distance a point on the drum travels in a 20-minute drying cycle. Can you use your expression from Part B? What other information, if any, is needed? Explain.

In 20 minutes, which is 1200 seconds, the measure of the subtended angle is

$\frac{8\pi * 1200^{rad}}{3} = 3200\pi^{rad}$**, which happens in 1600 revolutions**

The distance a point on the drum travels in a full revolution is the circumference of the circle $C = 2\pi r$, so the formula for 1600 revolutions would be d=1600*2πr. To be able to come up with the distance traveled we need to know the radius of the circle.

<u>Sec. 8.3 To find values of trigonometric functions on the unit circle.</u>



<u>Standard Position</u> (of an angle): initial side is on the positive x-axis; positive angles rotate counter-clockwise; negative angles rotate clockwise

<u>Coterminal Angles</u>: angles with the same initial and terminal sides, for example $\pm 360^\circ$ or $\pm 2\pi$

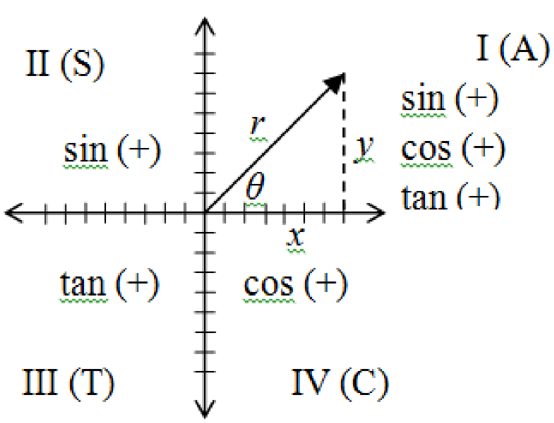
Ex 17: Find a positive and negative coterminal angle for each.

1. 130° Positive: $130^\circ + 360^\circ = \boxed{490^\circ}$ Negative: $130^\circ - 360^\circ = \boxed{-230^\circ}$
2. $\frac{17\pi}{2}$

   Positive: $\frac{17\pi}{2} - 2\pi = \boxed{\frac{13\pi}{2}} - 2\pi = \boxed{\frac{9\pi}{2}} - 2\pi = \boxed{\frac{\pi}{2}}$ (Note: Any of these are acceptable answers.)

   Negative: $\frac{\pi}{2} - 2\pi = \boxed{-\frac{3\pi}{2}}$

<u>Trigonometric Functions of any Angle</u>



$$x^2 + y^2 = r^2 \Rightarrow r = \sqrt{x^2 + y^2}$$

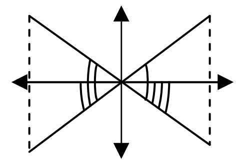
$$\sin\theta = \frac{y}{r} \qquad \csc\theta = \frac{r}{y}$$

$$\cos\theta = \frac{x}{r} \qquad \sec\theta = \frac{r}{x}$$

$$\tan\theta = \frac{y}{x} \qquad \cot\theta \frac{x}{y}$$

<u>Memory Aid</u>: To remember which trig functions are positive in which quadrant, remember **A**wesome **S**illy **T**rig **C**lass. A – all are positive in QI, S – sine (and cosecant) is positive in QII, T – tangent (and cotangent) is positive in QIII, C – cosine (and secant) is positive in QIV.

<u>Reference Angle</u>: every angle has a **reference angle**, with initial side as the x-axis. In standard position, the reference angle is the smallest angle between the terminal side and the x-axis.



How to find reference angles:

For Quadrant II: Degrees: $180^\circ - \theta$ Radians: $\pi - \theta$

For Quadrant III: Degrees: $\theta - 180^\circ$ Radians: $\theta - \pi$

For Quadrant IV: Degrees: $360^\circ - \theta$ Radians: $2\pi - \theta$

Ex 18: Find the reference angle for the given angles.

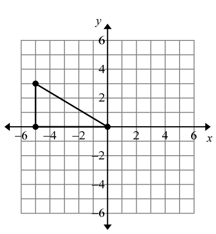
1. $\theta = 120^\circ$ Note that the terminal side of θ lies in Quadrant II. So, $180^\circ - 120^\circ = \boxed{60^\circ}$
2. $\theta = -130^\circ$ Note that θ is coterminal with 230°, whose terminal side lies in Quadrant III. So, $230^\circ - 180^\circ = \boxed{50^\circ}$
3. $\theta = \dfrac{5\pi}{3}$ The terminal side of θ lies in Quadrant IV. So, $2\pi - \dfrac{5\pi}{3} = \boxed{\dfrac{\pi}{3}}$

<u>Evaluating Trig Functions Given a Point</u>

- Use the ordered pair as x and y
- Find r: $r = \sqrt{x^2 + y^2}$
- Check the signs of your answers by the quadrant

Ex 19: Find the six trig functions of θ in standard position whose terminal side contains the point $(-5, 3)$.

Note: The point is in QII, so sine will be positive.



$x = -5,\ y = 3,\ r = \sqrt{(-5)^2 + 3^2} = \sqrt{34}$

$\sin\theta = \dfrac{3}{\sqrt{34}}$

$\cos\theta = -\dfrac{5}{\sqrt{34}}$

$\tan\theta = -\dfrac{3}{5}$

<u>Evaluating Trig Functions Given an Angle</u>

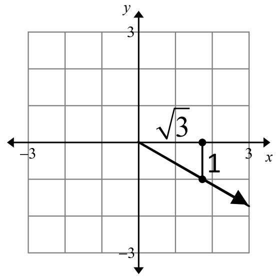
- Find the reference angle
- Label the sides (if the reference angle creates a special right triangle)
- Check the signs of your answers by the quadrant

Ex 20: Find the 3 trig functions of 330°.

To find the reference angle, start at the positive x-axis and go counter-clockwise 330°.

Reference Angle = 30° Since the angle is in QIV, cosine is positive.

$x=\sqrt{3},\ y=-1,\ r=2$ (30-60-90 triangle)



$$\sin 330^\circ = -\frac{1}{2}$$

$$\cos 330^\circ = \frac{\sqrt{3}}{2}$$

$$\tan 330^\circ = -\frac{1}{\sqrt{3}}$$

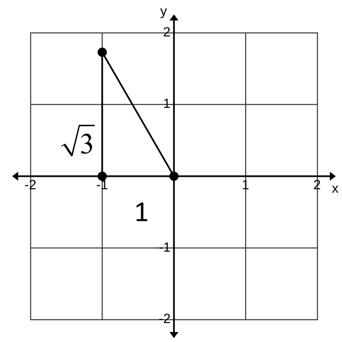
Ex 21: Find the 3 trig functions of $\frac{8\pi}{3}$.

To find the reference angle, start at the positive x-axis and clockwise $\frac{6\pi}{3}=2\pi$. The coterminal angle is then $\frac{2\pi}{3}$ which is in the second quadrant, so the reference angle is $\pi-\frac{2\pi}{3}=\frac{\pi}{3}$.

Reference Angle = 60° Since the angle is in QII, sine is positive.

$x=-1,\ y=\sqrt{3},\ r=2$ (30-60-90 triangle)

$$\sin\frac{8\pi}{3}=\frac{\sqrt{3}}{2} \qquad \cos\frac{8\pi}{3}=-\frac{1}{2} \qquad \tan\frac{8\pi}{3}=-\frac{\sqrt{3}}{1}=-\sqrt{3}$$



<u>Evaluating Trig Functions Given One Trig Function</u>

- Determine which quadrant by the signs of the trig functions given
- Label the sides corresponding to the given trig functions
- Check the signs of your answers by the quadrant

Ex 22: Find the other two trig functions if $\sin\theta=\frac{5}{7}$ and $\tan\theta>0$.

Both sine and tangent are positive so we know that θ is in the first quadrant. $\sin\theta=\frac{5}{7}=\frac{\text{opp}}{\text{hyp}}=\frac{y}{r}$

$$r^2=x^2+y^2 \Rightarrow 7^2=x^2+5^2 \Rightarrow x=\sqrt{24}$$

$$\sin\theta=\frac{5}{7} \qquad\qquad \sin\theta=\frac{5}{7}$$

$$\cos\theta=\frac{\sqrt{24}}{7} \quad \text{or} \quad \cos\theta=\frac{2\sqrt{6}}{7}$$

$$\tan\theta=\frac{5}{\sqrt{24}} \qquad\qquad \tan\theta=\frac{5}{2\sqrt{6}}$$

Quadrantal Angles: angles with the terminal side on the axes, for example, $0^\circ, 90^\circ, 180^\circ, 270^\circ$

Degrees	**0°/360°**	**30°**	**45°**	**60°**	**90°**	**180°**	**270°**
Radians	$0/2\pi$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
sinθ	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0	-1
cosθ	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1	0
tanθ	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	und	0	und

Teacher Note: Have students fill in the table (unbolded cells). Needs to be memorized!

Ex 23: Find $\csc 13\pi$.

Find the reference angle: $13\pi = 12\pi + \pi = 6(2\pi) + \pi \Rightarrow \pi$

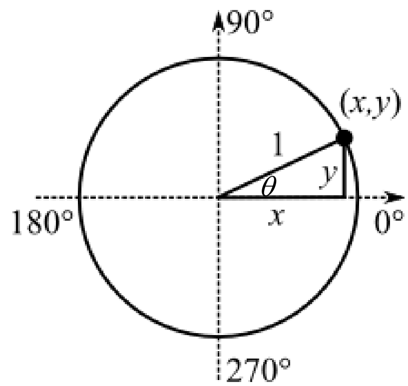
$$\csc 13\pi = \csc \pi = \frac{1}{\sin \pi} = \frac{1}{0} \Rightarrow \boxed{\text{undefined}}$$

The Unit Circle:

The circle $x^2 + y^2 = 1$, which has the center (0, 0) and a radius of 1, is called the **unit circle**. The values of $\sin\theta$ and $\cos\theta$ are simply the y-coordinate and x-coordinate, respectively, of the point where the terminal side of θ intersects the unit circle. It is convenient to use the unit circle to find trigonometric functions of quadrantal angles and the special right triangle angles. The quicker values on the unit circle are memorized, the better! After memorizing Quadrant I, note the pattern!!

$$\sin\theta = \frac{y}{r} = \frac{y}{1} = y$$

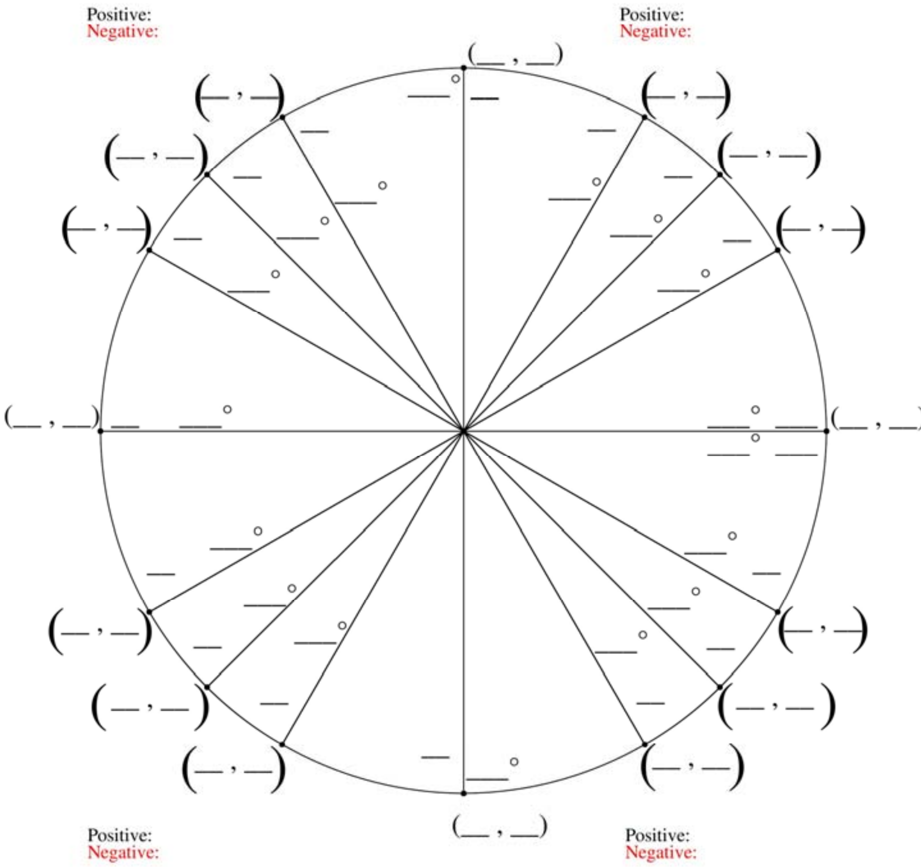
$$\cos\theta = \frac{x}{r} = \frac{x}{1} = x$$



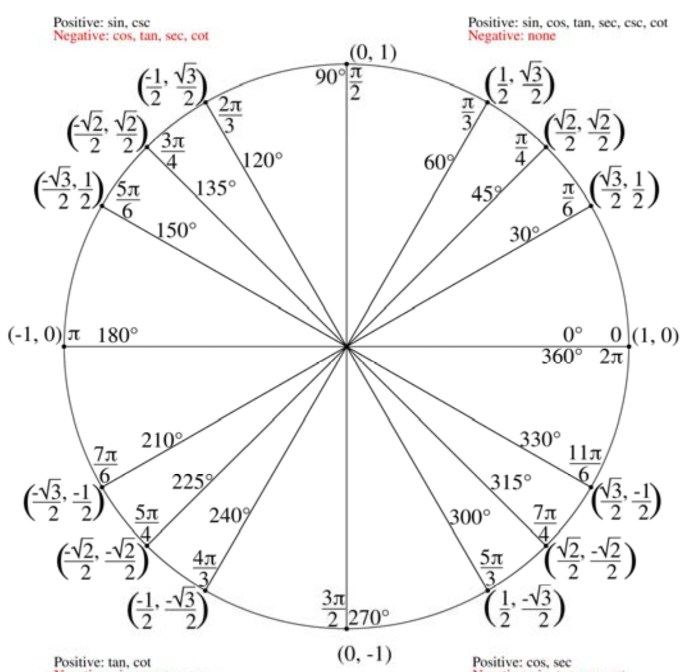
Teacher Note: Have students fill out the table as quickly as they can. Discuss patterns they can use to be able to memorize these.

θ	**0°**	**30°**	**45°**	**60°**	**90°**	**120°**	**135°**	**150°**	**180°**	**210°**	**225°**	**240°**	**270°**	**360°**
radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	2π
sin θ	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	−1	0
cos θ	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	−1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0	1
tan θ	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	und	$-\sqrt{3}$	−1	$-\frac{1}{\sqrt{3}}$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	und	0

Fill in The Unit Circle



The Unit Circle



Ex 24: Find the following without a calculator.

1. $\cos 990^\circ$ Find the coterminal angle. $990^\circ - 360^\circ - 360^\circ = 270^\circ$. Using the unit circle, we see that this is a quadrantal angle and the x-coordinate at 270° is 0, so $\boxed{\cos 990^\circ = 0}$

2. $\tan\frac{5\pi}{3}$ Using the unit circle, $\tan\frac{5\pi}{3} = \frac{y}{x} = \frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}} = -\frac{\sqrt{3}}{1} = \boxed{-\sqrt{3}}$

3. $\sin 225°$ Using the unit circle, $\sin 225^\circ = y = \boxed{-\frac{\sqrt{2}}{2}}$

QOD: How is the unit circle used to evaluate the trigonometric functions? Explain.

SAMPLE EXAM QUESTIONS

1. What is the exact value of $\tan\left(\frac{5\pi}{4}\right)$**?**

A. 1

B. -1

C. $-\frac{\sqrt{2}}{2}$

D. $\frac{\sqrt{2}}{2}$

Ans: A

2. Which expression has the same value as $\tan\left(-\frac{\pi}{3}\right)$**?**

A. $-\tan\left(\frac{\pi}{3}\right)$

B. $\tan\left(\frac{\pi}{3}\right)$

C. $-\cos\left(\frac{\pi}{3}\right)$

D. $-\sin\left(\frac{\pi}{3}\right)$

Ans: A

3. What is the reference angle corresponding to $\frac{7\pi}{4}$?

A. $\frac{5\pi}{2}$

B. $\frac{4\pi}{7}$

C. $\frac{\pi}{4}$

D. $\frac{3\pi}{2}$

Ans: C

4. A regular hexagon is inscribed in the unit circle. One vertex of the hexagon is at the point $\left(-\frac{\sqrt{3}}{2},\frac{1}{2}\right)$. A diameter of the circle starts from that vertex and ends on another vertex of the hexagon. What are the coordinates of the other vertex?

A. $\left(\frac{\sqrt{3}}{2},-\frac{1}{2}\right)$

B. $\left(\frac{1}{2},-\frac{\sqrt{3}}{2}\right)$

C. $\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right)$

D. $\left(-\frac{1}{2},\frac{\sqrt{3}}{2}\right)$

Ans: A

5. For what angles *x* in $[0,2\pi)$ does the $\cos(x)$ have the same value as $\sin\left(\frac{3\pi}{4}\right)$?

A. $\cos\left(\frac{3\pi}{4}\right)$ and $\cos\left(\frac{5\pi}{4}\right)$

B. $\cos\left(\frac{3\pi}{4}\right)$ and $\cos\left(\frac{7\pi}{4}\right)$

C. $\cos\left(\frac{\pi}{4}\right)$ and $\cos\left(\frac{7\pi}{4}\right)$

D. $\cos\left(\frac{\pi}{4}\right)$ and $\cos\left(\frac{5\pi}{4}\right)$

Ans: C

6. For which radian measures *x* will tan *x* be negative?

A. $\frac{5\pi}{4}, \frac{7\pi}{4}, \frac{13\pi}{4}, \frac{15\pi}{4}, \frac{21\pi}{4}, \frac{23\pi}{4}, \ldots$

B. $\frac{3\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{4}, \frac{15\pi}{4}, \frac{19\pi}{4}, \frac{23\pi}{4}, \ldots$

C. $\frac{3\pi}{4}, \frac{5\pi}{4}, \frac{11\pi}{4}, \frac{13\pi}{4}, \frac{19\pi}{4}, \frac{21\pi}{4}, \ldots$

D. $\frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \frac{13\pi}{4}, \frac{17\pi}{4}, \frac{21\pi}{4}, \ldots$

Ans: B

7. The diameter of a bicycle tire is 20 in. A point on the outer edge of the tire is marked with a white dot. The tire is positioned so that the white dot is on the ground, then the bike is rolled so that the dot rotates clockwise through an angle of 16.75π radians.

Part A: To the nearest tenth of an inch, how high off the ground is the dot when the wheel stops? Show your work.

Let's find how many degrees the dot rotates clockwise:

$$16.75\pi^{rad} * \frac{180^\circ}{\pi^{rad}} = 3015^\circ$$

If the angle is greater than 360 degrees, you subtract 360 degrees from it until the angle is less than 360 degrees or you could use this shortcut for very large positive angles:

Divide the angle by 360. Take the integer part of the result and multiply 360 by that. Subtract the result from the angle.

Example:
Your angle is 3015. Divide by 360 to get 8.375. Multiply 360 by 8 to get 2880.
Subtract 2880 from 3015 to get 135.

135 is the angle you need to work with to get your reference angle from. So this dot moves from the ground, clockwise 135 degrees, so it is going to end in the second quadrant. The reference angle will be the acute angle between the terminal side and the x axis which will be 45 degrees.

So the height of the dot will be $10 + 10\sin 45^\circ = 10 + 10 * \frac{\sqrt{2}}{2} = 17.07(in)$

Part B: What distance was the bicycle pushed? Round your answer to the nearest foot.

For every full rotation the dot moved a distance equal to the circumference of the circle $2\pi r, so\ 3015^\circ * \frac{2\pi * 10}{360^\circ} = 526.22(in) =$ 44 feet

Part C: Would changing the size of the tire (value of *r*) change either of the answers found in Parts A or B? Explain your reasoning.

In both answers the radius is present as a factor, so if you would triple the original value for the radius, both answers would triple their values.

8. **A ribbon is tied around a bicycle tire at the standard position $0°$. The diameter of the wheel is 26 inches. The bike is then pushed forward 20 feet from the starting point. In what quadrant is the ribbon? Explain how you obtained your answer.**

If the bike is pushed 20 ft from the starting point, then the ribbon moved 20 ft from the starting point.

20 ft = 240 in

The circumference of a circle is the length of the curve that encloses that circle, which is $C = 2\pi r$. Another way to think of the curve that encloses a circle is through the 360 degree arc of that curve. Thus, the circumference of a circle is the length of the 360 degree arc of that circle. So there is a "unit" fraction that we could use:

$$240in * \frac{360^\circ}{2\pi * 13in} = 1057.77^\circ$$

Through its circular movement the ribbon will define a curve whose measure in degrees is 1057.77

1057.77-360-360=337.77 which tells us that the ribbon is in the 4th quadrant.

9. **Find $\tan\theta$ when $\sin\theta = -\cos\theta$ and θ is in Quadrant IV.**

A. $\frac{\sqrt{2}}{2}$

B. $-\frac{\sqrt{2}}{2}$

C. 1

D. -1

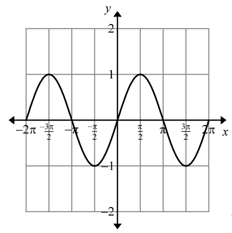
Ans: D

Sec. 8.4 To evaluate inverse trigonometric functions and use them to solve problems.

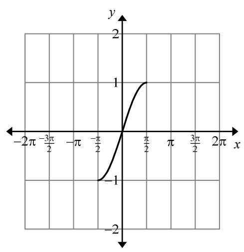
Recall: In order for a function to have an inverse function, it must be **one-to-one** (must pass both the horizontal and vertical line tests).

Notation: The inverse of $f(x)$ is labeled as $f^{-1}(x)$.

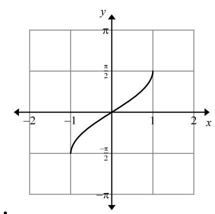
Inverse of the Sine Function

Graph of $f(x) = \sin x$  Domain: $(-\infty, \infty)$ Range: $[-1, 1]$

In order for $f(x) = \sin x$ to have an inverse function, we must restrict its domain to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.



To graph the inverse of sine, reflect about the line $y = x$.



Domain of $f^{-1}(x) = \sin^{-1} x$: $[-1, 1]$ Range of $f^{-1}(x) = \sin^{-1} x$: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (Quadrant I and IV)

Notation: Inverse of Sine $f^{-1}(x) = \sin^{-1} x$ or $y = \arcsin x$ (arcsine)

Note: $y = \sin^{-1} x$ denotes the inverse of sine (arcsine). It is NOT the reciprocal of sine (cosecant).

Evaluating the Inverse Sine Function

Ex 25: Find the exact values of the following.

1. $\arcsin \frac{\sqrt{2}}{2}$ What value of x makes the equation $\sin x = \frac{\sqrt{2}}{2}$ true? $\boxed{x = \frac{\pi}{4} \text{ or } 45^\circ}$

   Note: The range of arcsine is restricted to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, so $\frac{\pi}{4}$ is the only possible answer.

2. $\sin^{-1}\left(\frac{1}{2}\right)$ What value of x makes the equation $\sin x = \frac{1}{2}$ true? Since the range of arcsine is restricted to Quadrants I and IV, the only possible answer is $\boxed{\frac{\pi}{6} \text{ or } 30^\circ}$.

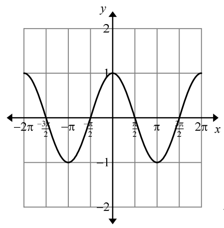
3. $\sin^{-1} 3$ What value of x makes the equation $\sin x = 3$ true?

   No solution, because 3 > 1 and the domain of arcsine = the range of sine = $[-1, 1]$

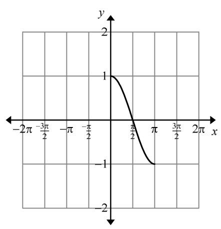
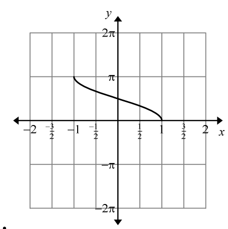
4. $\arcsin(-1)$ What value of x makes the equation $\sin x = -1$ true? Looking at Quadrant I and IV, the only possible answer is $\boxed{-\frac{\pi}{2} \text{ or } -90^\circ}$.

5. $\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$ Taking the inverse sine of the sine function results in the argument. $\boxed{\frac{2\pi}{3}}$

Inverse of the Cosine Function

Graph of $f(x) = \cos x$  Domain: $(-\infty, \infty)$ Range: $[-1, 1]$

In order for $f(x) = \cos x$ to have an inverse function, we must restrict its domain to $[0, \pi]$.

 To graph the inverse of cosine, reflect about the line $y = x$. 

Domain of $f^{-1}(x) = \cos^{-1} x$: $[-1, 1]$ Range of $f^{-1}(x) = \cos^{-1} x$: $[0, \pi]$ (Quadrant I and II)

Notation: Inverse of Cosine $f^{-1}(x) = \cos^{-1} x$ or $y = \arccos x$ (arccosine)

Note: $y = \cos^{-1} x$ denotes the inverse of cosine (arccosine). It is NOT the reciprocal of cosine (secant).

Evaluating the Inverse Cosine Function

Ex 26: Find the exact values of the following.

1. $\arccos\left(-\frac{\sqrt{2}}{2}\right)$ What value of x makes the equation $\cos x = -\frac{\sqrt{2}}{2}$ true? $\boxed{x = \frac{3\pi}{4} \text{ or } 135^\circ}$

   Note: The range of arcsine is restricted to $[0, \pi]$, so $\frac{3\pi}{4}$ is the only possible answer.

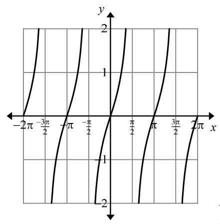
2. $\sin\left(\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)\right)$ $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$, so $\sin\left(\frac{\pi}{6}\right) = \boxed{\frac{1}{2}}$

3. $\arccos\left(-\frac{1}{2}\right)$ What value of x makes the equation $\cos x = -\frac{1}{2}$ true? Since the range of arccosine is restricted to Quadrants I and II, $\boxed{\frac{2\pi}{3} \text{ or } 120^\circ}$ is the only possible answer.

4. $\cos^{-1}\left(\cos\frac{11\pi}{6}\right)$ Taking the inverse cosine of the cosine function results in the argument. $\boxed{\frac{11\pi}{6}}$

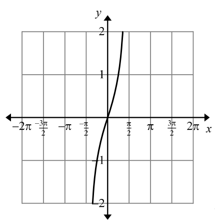
Inverse of the Tangent Function

Graph of $f(x)=\tan x$

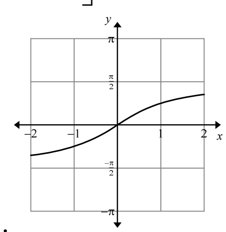


Domain: $x \neq \frac{\pi}{2}+k\pi$ Range: $(-\infty,\infty)$

In order for $f(x)=\tan x$ to have an inverse function, we must restrict its domain to $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$.



To graph the inverse of tangent, reflect about the line $y=x$.



Domain of $f^{-1}(x)=\tan^{-1}x$: $(-\infty,\infty)$ Range of $f^{-1}(x)=\tan^{-1}x$: $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ (Quadrant I and IV)

Notation: Inverse of Tangent $f^{-1}(x)=\tan^{-1}x$ or $y=\arctan x$ (arctangent)

Note: $y=\tan^{-1}x$ denotes the inverse of tangent (arctangent). It is NOT the reciprocal of tangent (cotangent).

Evaluating the Inverse Tangent Function

Ex 27: Find the exact values of the following.

1. $\arctan\left(-\sqrt{3}\right)$ What value of x makes the equation $\tan x=-\sqrt{3}$ true? Since the range of arctangent is restricted to Quadrants I and IV, the only possible answer is $\boxed{-\frac{\pi}{3} \text{ or } -60^\circ}$.

2. $\sin\left(\tan^{-1}\frac{\sqrt{3}}{3}\right)$ $\sin\left(\tan^{-1}\frac{\sqrt{3}}{3}\right)=\sin\left(\frac{\pi}{6}\right)=\boxed{\frac{1}{2}}$

Note: The range of arctangent is restricted to $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, so $\frac{\pi}{6}$ is the only possible answer for $\tan^{-1}\frac{\sqrt{3}}{3}$.

3. $\cos\left(\tan^{-1}1\right)$ $\cos\left(\tan^{-1}1\right)=\cos\left(\frac{\pi}{4}\right)=\boxed{\frac{1}{\sqrt{2}}}$

4. $\arccos\left(\tan\frac{\pi}{3}\right)$ $\arccos\left(\tan\frac{\pi}{3}\right)=\arccos\left(\sqrt{3}\right)$ No Solution, because $\sqrt{3}>1$.

<u>You Try</u>: Evaluate $\cos^{-1}\left(\cos\left(-\frac{\pi}{4}\right)\right)$. Be careful!

<u>QOD</u>: Explain how the domains of sine, cosine, and tangent must be restricted in order to create an inverse function for each.

SAMPLE EXAM QUESTIONS

1. If $0 \le \theta \le \pi$ and $\cos^{-1}(\sin\theta) = \frac{\pi}{3}$, then what is the value of θ?

A. $\frac{3\pi}{2}$

B. $\frac{\pi}{2}$

C. $\frac{5\pi}{6}$

D. $\frac{\pi}{6}$

Ans: D