Pre-Algebra Notes – Unit 9: Square & Cube Roots; Irrational Numbers

Square Roots and Cube Roots

NVACS 8.EE.A.2: Evaluate square roots of small perfect squares and cube roots of small perfect cubes.

The *square root* of a number *n* is a number *m* such that $m^2 = n$. The radical sign, $\sqrt{\ }$, represents the nonnegative square root. The symbol $\pm$, read "plus or minus," refers to both the positive and negative square root. Therefore, the square roots of 36 are 6 and -6, because $6^2 = 36$ and $(-6)^2 = 36$. Also, $\sqrt{36} = 6$, $-\sqrt{36} = -6$, and $\pm\sqrt{36} = \pm 6$. We understand that $\sqrt{\ }$ will be the positive value. We refer to this as the *principal square root*.

Students should memorize the values of the squares for 1 through 20 (SBAC). These "perfect squares" (squares of integers) are listed below:

Perfect Squares:

$1^2 = 1$	$6^2 = 36$	$11^2 = 121$	$16^2 = 256$
$2^2 = 4$	$7^2 = 49$	$12^2 = 144$	$17^2 = 289$
$3^2 = 9$	$8^2 = 64$	$13^2 = 169$	$18^2 = 324$
$4^2 = 16$	$9^2 = 81$	$14^2 = 196$	$19^2 = 361$
$5^2 = 25$	$10^2 = 100$	$15^2 = 225$	$20^2 = 400$

Simplifying expressions such as the $\sqrt{25}$ and $\sqrt{64}$ are pretty straight forward.

$$\sqrt{25} = 5 \text{ and } \sqrt{64} = 8$$

Even expressions like $\sqrt{\frac{25}{81}}$ are easy to simplify once we see an example.

$$\sqrt{\frac{25}{81}} = \frac{\sqrt{25}}{\sqrt{81}} = \frac{5}{9}$$

But what if the expression is large, like $\sqrt{576}$? We will consider a method for determining this answer which extends beyond what is required by the CCSS.

Consider the perfect squares of a few multiples of 10:

$$10^2 = 100 \qquad 30^2 = 900$$
$$20^2 = 400 \qquad 40^2 = 1600 \quad \text{and so on....}$$

Now to find $\sqrt{576}$:

$$\begin{array}{cl} 400 < 576 < 900 & \textit{Identify perfect squares closest to } 576. \\ \sqrt{400} < \sqrt{576} < \sqrt{900} & \textit{Take positive square root of each number.} \\ 20 < \sqrt{576} < 30 & \textit{Evaluate square root of each perfect square that I know.} \end{array}$$

Values we now need to consider are 21, 22, 23, 24, 25, 26, 27, 28, and 29. But wait! Since we know an $even \cdot even = even$, our answer must be even. So now our answer choices are reduced to 22, 24, 26, and 28. We now need to ask ourselves which of those answer choices will give us a 6 in the one's place ($\sqrt{576}$). There are only two: 24 (since $4^2 = 16$) and 26 (since $6^2 = 36$). Since $\sqrt{576}$ is closer to $\sqrt{400}$ than it is to $\sqrt{900}$, we quickly know that the answer is 24. A quick check will identify 24 as the answer: $\sqrt{576} = 24$.

Let's look at another example, finding $\sqrt{1225}$.

$$\begin{array}{cl} 900 < 1225 < 1600 & \textit{Identify perfect squares closest to } 1225. \\ \sqrt{900} < \sqrt{1225} < \sqrt{1600} & \textit{Take positive square root of each number.} \\ 30 < \sqrt{1225} < 40 & \textit{Evaluate square root of each perfect square that I know.} \end{array}$$

The values we need to consider are 31, 32, 33, 34, 35, 36, 37, 38, and 39. Since our radicand is odd, we condense those considerations to 31, 33, 35, 37, and 39. However, if we look at the radicand 1225, we can quickly recognize that the only number whose square will give us a 5 is 35. We have our answer very quickly! $\sqrt{1225} = 35$.

In review, evaluate the following: $\sqrt{16}$, $-\sqrt{16}$, $\pm\sqrt{16}$

$\sqrt{16}$ denotes the principal (positive) square root, so the answer is 4 because $4^2 = 16$.

$-\sqrt{16}$ denotes the negative square root, so the answer is -4 because $(-4)^2 = 16$.

$\pm\sqrt{16}$ denotes the positive and negative square roots. It is read "plus or minus the square root of 16". The answer is ± 4 because $(4)(4) = 16$ and $(-4)(-4) = 16$.

Special Cases:

Square root of 0: $\sqrt{0} = 0$. Since 0 is neither positive nor negative, $\sqrt{0}$ only has one square root.

There should be mention of the square root of a negative number: negative numbers have **no real square roots**, because the square of every real number is nonnegative. Emphasize that negative numbers have no **REAL** square roots (they have imaginary roots)—not that negative numbers have no square roots.

The *cube root* of a number n is a number m such that $m^3 = n$. The radical sign, $\sqrt[3]{\ }$, represents the cube root. Therefore, the cube root of 8 is 2, because $2^3 = 8$.

Students should memorize the values of the cubes for 1 through 5. These "perfect cubes" (cubes of integers) are listed below:

Perfect Cubes:

$1^3 = 1$	$6^3 = 216$
$2^3 = 8$	$7^3 = 343$
$3^3 = 27$	$8^3 = 512$
$4^3 = 64$	$9^3 = 729$
$5^3 = 125$	$10^3 = 1000$

Therefore, $\sqrt[3]{64} = 4$ (since 4^3=64).

NOTE: Whenever there is an even number of identical factors, the product will positive because every pair of identical factors has a positive product.

$$5 \cdot 5 = 25 \qquad \text{positive} \cdot \text{positive} = \text{positive}$$

$$(-5)(-5) = 25 \qquad \text{negative} \cdot \text{negative} = \text{positive}$$

However, when there is an odd number of an identical factor, the product will have the same sign as the factors. So, while a negative square root is imaginary, a negative cube root is possible.

$$5 \cdot 5 \cdot 5 = 125 \qquad \text{positive} \bullet \text{positive} \bullet \text{positive} = \text{positive}$$

$$(-5)(-5)(-5) = -125 \qquad \text{negative} \bullet \text{negative} \bullet \text{negative} = \text{negative}$$

So $\sqrt[3]{-8} = -2$ and $\sqrt[3]{-27} = -3$. (Note: consideration of negative numbers is not included in the CCSS for this grade level.)

Simplifying Square Roots

To simplify a square root, you rewrite the radicand as a product of the largest perfect square that is a factor and some other number. You then take the square root of the perfect square.

Example: Simplify $\sqrt{50}$.

Now 50 can be written as a product of 5 and 10. Should I use those factors? Hopefully, you said no. We want to rewrite the radicand as a product of the largest perfect square, and neither 5 nor 10 are perfect squares. So, looking at my list of perfect squares, which, if any, are factors of 50?

That's right, 25 is a factor of 50 and it is a perfect square.

Simplifying, I now have

$$\begin{aligned}\sqrt{50} &= \sqrt{25\cdot 2}\\ &= \sqrt{25}\cdot\sqrt{2}\\ &= 5\sqrt{2}\end{aligned}$$

Example: Simplify $\sqrt{98x^2y}$.

$$\begin{aligned}\sqrt{98x^2y} &= \sqrt{49}\cdot\sqrt{2}\cdot\sqrt{x^2}\cdot\sqrt{y}\\ &= 7\cdot\sqrt{2}\cdot x\cdot\sqrt{y}\\ &= 7x\sqrt{2y}\end{aligned}$$

Example: Simplify $\sqrt{\frac{60}{49}}$.

$$\begin{aligned}\sqrt{\frac{60}{49}} &= \frac{\sqrt{60}}{\sqrt{49}}\\ &= \frac{\sqrt{4}\cdot\sqrt{15}}{7}\\ &= \frac{2\sqrt{15}}{7}\end{aligned}$$

Solving Equations using Square Root and Cube Root

NVACS 8.EE.A.2: Use square root and cube root symbols to represent solutions to equations of the form $x^2 = p$ and $x^3 = p$, where "p" is a positive rational number.

We are now ready to solve equations in the form $x^2 = c$.

Example: Solve $x^2 = 9$.

$$x^2 = 9$$
$$\sqrt{x^2} = \sqrt{9}$$
$$x = \pm 3$$

Solve $225 = p^2$

$$225 = p^2$$
$$\sqrt{225} = \sqrt{p^2}$$
$$\pm 15 = p$$

Example: Solve $x^3 = 8$

$$x^3 = 8$$
$$\sqrt[3]{x^3} = \sqrt[3]{8}$$
$$x = 2$$

Solve $125 = r^3$

$$125 = r^3$$
$$\sqrt[3]{125} = \sqrt[3]{r^3}$$
$$5 = r$$

Approximating Square Roots

NVACS 8.NS.A.2: Estimate the value of irrational expressions (e.g., π^2)

You can use perfect squares to approximate the square root of a number.

Example: Approximate $\sqrt{53}$ to the nearest integer.

First, find the perfect square that is closest but less than 53. That would be 49. The perfect square closest to 53 but greater than 53 is 64. So, 53 is between 49 and 64.

$49 < 53 < 64$	*Identify perfect squares closest to* 53.
$\sqrt{49} < \sqrt{53} < \sqrt{64}$	*Take positive square root of each number.*
$7 < \sqrt{53} < 8$	*Evaluate square root of each perfect square.*

Because 53 is closer to 49, $\sqrt{53}$ is closer to 7. Therefore, $\sqrt{53} \approx 7$.

Students can approximate square roots by iterative processes.

Example: Approximate the value of $\sqrt{5}$ to the nearest hundredth.

Solution: Students start with a rough estimate based upon perfect squares. $\sqrt{5}$ falls between 2 and 3 because 5 falls between $2^2 = 4$ and $3^2 = 9$. The value will be closer to 2 than to 3. Students continue the iterative process with the tenths place value. $\sqrt{5}$ falls between 2.2 and 2.3 because 5 falls between $2.2^2 = 4.84$ and $2.3^2 = 5.29$. The value is closer to 2.2. Further iteration shows that the value of $\sqrt{5}$ is between 2.23 and 2.24 since 2.23^2 is 4.9729 and 2.24^2 is 5.0176.

Another way to make the estimate is shown below:

Example: Approximate $\sqrt{53}$.

Start as we did with the previous example.

$$\sqrt{49} < \sqrt{53} < \sqrt{64}$$

$$\left.\begin{array}{l}\left.\begin{array}{l}\sqrt{49}\\ \sqrt{53}\end{array}\right\} 4\\ \sqrt{64}\end{array}\right\} 15$$

Determine the difference between 49 (smaller perfect square) and 53 (the radicand). The difference is 4.

Then find the difference between 49 (smaller perfect square) and 64 (the larger perfect square). The difference is 15.

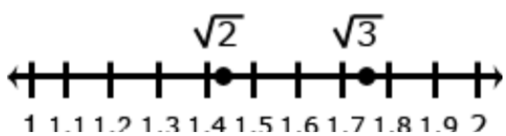
We know that $\sqrt{53}$ falls between 7 and 8. To approximate the decimal, take $4 \div 15 = 0.2\overline{6}$. We would estimate $\sqrt{53} \approx 7.27$. Using a calculator, we find $\sqrt{53} \approx 7.28$, which is very close to our estimate!

NVACS 8.NS.A.2: Locate irrational numbers approximately on a number line diagram.

Example: Compare $\sqrt{2}$ and $\sqrt{3}$ by estimating their values, plotting them on a number line, and making comparative statements.

Solution: Statements for the comparison *could* include:

$\sqrt{2}$ is approximately 1.4
$\sqrt{2}$ is between the whole numbers 1 and 2
$\sqrt{3}$ is between 1.7 and 1.8



Ordering Real Numbers

NVACS 8.NS.A.2: ***Use rational approximations of irrational numbers to compare the size of irrational numbers.***

Now is a good time to return to the concept of ordering real numbers, including irrational numbers.

Example: Order the numbers from least to greatest: $\sqrt{40},\ \frac{22}{5},\ -4.1,\ -\sqrt{23}$

One way to solve this would be to convert all terms into decimal form (or approximate decimal form).

$\sqrt{40}$

$36 < 40 < 49$ — *Identify perfect squares closest to* 40.

$\sqrt{36} < \sqrt{40} < \sqrt{49}$ — *Take the positive square root of all terms.*

$6 < \sqrt{40} < 7$ — *Simplify the perfect squares that I know.*

$\sqrt{40}$ will fall between 6 and 7, closer to 6.

$\frac{22}{5}$

$\frac{22}{5} = 4\frac{2}{5} = 4.4$

$-\sqrt{23}$

$16 < 23 < 25$ — *Identify perfect squares closest to* 23.

$\sqrt{16} < \sqrt{23} < \sqrt{25}$ — *Take the positive square root of all terms.*

$4 < \sqrt{23} < 5$ — *Simplify the perfect squares that I know.*

$\sqrt{23}$ will fall between 4 and 5, closer to 5. Thus $-\sqrt{23}$ will fall closer to -5 than -4.

So I would order these numbers from least to greatest:

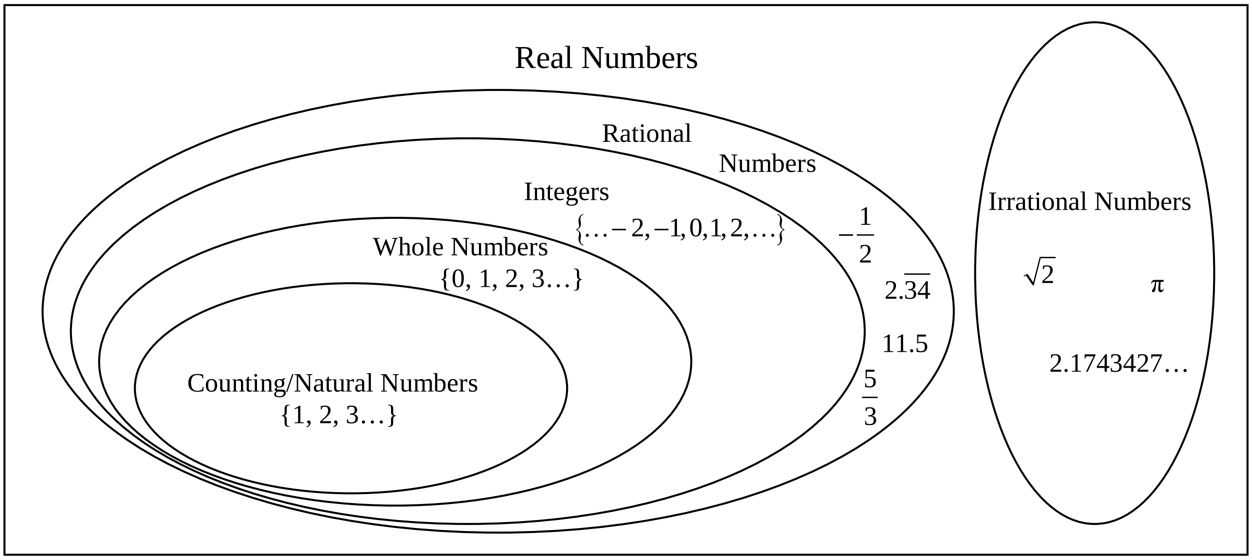
closer to -5, -4.1, 4.4, closer to 6.

That translates to $-\sqrt{23}, -4.1, \frac{22}{5}, \sqrt{40}$.

Real Numbers

NVACS 8.NS.A.1: Know that numbers that are not rational are called irrational.
NVACS 8.EE.A.2: Know that $\sqrt{2}$ is irrational.

The set of **real numbers** consists of all rational and irrational numbers. This relationship can be shown in a Venn diagram.

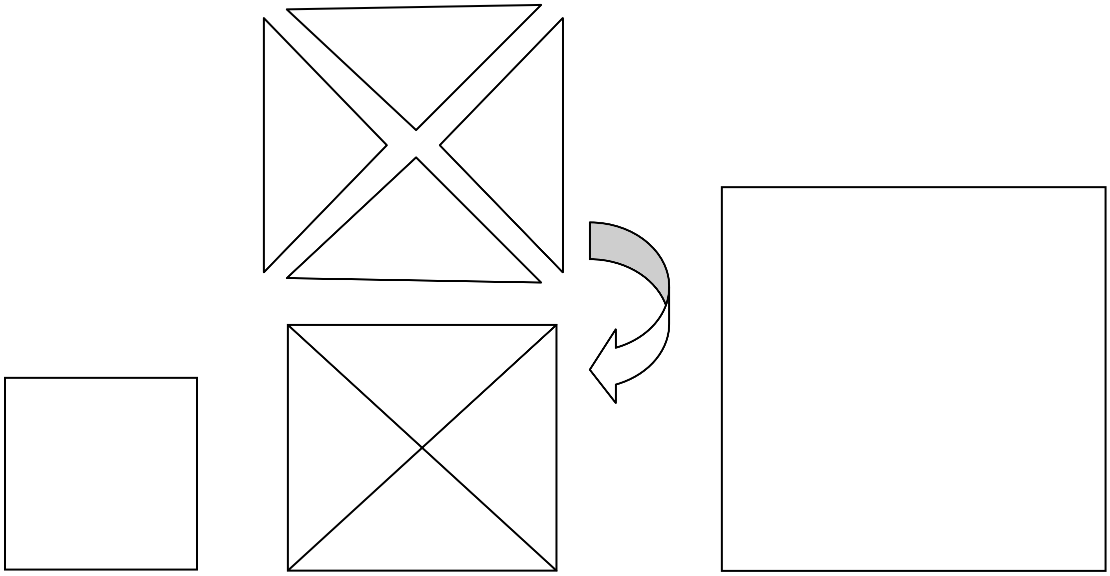


A **rational number** is a number that can be written as a quotient of two integers. The decimal form repeats or terminates.

An **irrational number** is a number that cannot be written as a quotient of two integers. The decimal form neither terminates nor repeats.

You can illustrate an irrational number by having students try to "Think of a number, when multiplied by itself, equals 2." Perhaps we are finding the length of the side of a square when we know the area is 2. Or for a more visual representation, use tangrams as discussed below.

The task is to compare the three squares and try to estimate the side length of the middle square.



side = 1 unit
area = 1 square unit

side = ?
area = 2 square units

side = 2 units
area = 4 square units

You can tell that the small square is 1 by 1, so its area is 1 square unit. The biggest square has 2 units on a side, so its area is 4 square units because it is 2 by 2. But the middle square is made from 4 half-squares, so it must have an area of 2 square units. What is its side length?

1 • 1 = 1 and 2 • 2 = 4. But ___ • ___ = 2 ?

By matching the sides to each other, students can see that the unknown side is between 1 and 2, perhaps 1 and a half. Have students get out calculators and try 1.5 times 1.5 and make a table, with a large gap between 1 and 2.

side length	**area**
1	1
1.5	2.25
2	4

Students will find 1.5 to be too large (2.25) and 1.4 to be too small (1.96). What number "squared" will make 2? It must be between 1.4 and 1.5.

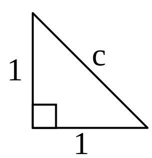
Further investigation:

side length	area
1	1
1.4	1.96
1.41	1.9881
1.415	2.002225
1.42	2.0164
1.5	2.25

This is an excellent form of interpolation that teachers can build upon to clarify place value in the decimals. Let students go for as long as they want within reason, getting ever closer by never reaching exactly 2. Review some repeating decimals….

By now students are ready to hear that there are some special numbers that will never repeat as decimals, basically ratios that go on to infinity with no repeating pattern. We were looking for one of them (1.41421…) and it does not repeat. In fact, these decimal places will only round to 2 when squared—raised to the second power. Instead of remembering the decimal and saying it is "2 when squared", we have a better way to deal with this. The side length, that "when squared is 2", we call the *square root of 2* and write it $\sqrt{2}$. It can never be expressed as a repeating decimal because it is not the ratio of two numbers, like 1/6 or 3/8. Therefore it is called *irrational*. (You might have students explore the square root key on the calculator to see how many decimal places they get.)

Another way to model $\sqrt{2}$ is to wait until you have introduced the Pythagorean Theorem. Draw a right triangle with each leg measuring 1 unit. Find the length of the hypotenuse of the right triangle by using the Pythagorean Theorem.



$$
\begin{aligned}
a^2+b^2&=c^2\\
1^2+1^2&=c^2\\
1+1&=c^2\\
\sqrt{2}&=\sqrt{c^2}\\
\sqrt{2}&=c
\end{aligned}
$$