Lesson 12: Using Probability to Make Decisions

Student Outcomes

- Students will use estimated probabilities to judge whether a given probability model is plausible.
- Students will use estimated probabilities to make informed decisions.

Lesson Notes

This is an extensive lesson. Note that there are more examples and exercises in this lesson than can be done in one class period. Teachers should work through Exercise 4, and then choose to do either Exercise 5 or 6, depending on time availability.

The examples and problem set in this lesson require students to use several devices in order to perform a simulation. Teachers should plan accordingly before the start of class.

Classwork

Example 1 (5 minutes): Number Cube

> **Example 1: Number Cube**
>
> Your teacher gives you a number cube with numbers   –   on its faces. You have never seen that particular cube before. You are asked to state a theoretical probability model for rolling it once. A probability model consists of the list of possible outcomes (the sample space) and the theoretical probabilities associated with each of the outcomes. You say
>
> that the probability model might assign a probability of – to each of the possible outcomes, but because you have never
>
> seen this particular cube before, you would like to roll it a few times. (Maybe it's a trick cube.) Suppose your teacher allows you to roll it   times and you get the following results:
>
> | Outcome | | | | | | |
> |-----------|--|--|--|--|--|--|
> | Frequency | | | | | | |

Begin by showing your students a number cube, and ask:

- What is a theoretical probability model for its six outcomes?

  - *They should respond that the outcomes are equally likely, – for each outcome.*

- Is the number cube you are showing them a fair one? Could it be a trick one?
- How could they find out whether it is fair or weighted in a way that makes it not fair?

The example suggests that they rolled the number cube _____ times and observed the following. Clearly, the number cube favors even numbers, and the estimated probabilities cause serious doubt about the conjectured equally likely model.

Exercises 1–2 (3–5 minutes)

Let students work independently on Exercises 1 and 2. Then confirm as a class.

Exercises 1–2

1. If the "equally likely" model were correct, about how many of each outcome would you expect to see if the cube is rolled _____ times?

   If the "equally likely" model were correct, you would expect to see each outcome occur about _____ times.

2. Based on the data from the _____ rolls, how often were odd numbers observed? How often were even numbers observed?

   Odd numbers were observed _____ times. Even numbers were observed _____ times.

Example 2 (5 minutes): Probability Model

Example 2: Probability Model

Two black balls and two white balls are put in a small cup whose bottom allows the four balls to fit snugly. After shaking the cup well, two patterns of colors are possible as shown. The pattern on the left shows the similar colors are opposite each other, and the pattern on the right shows the similar colors are next to or adjacent to each other.

Philippe is asked to specify a probability model for the chance experiment of shaking the cup and observing the pattern. He thinks that because there are two outcomes—like heads and tails on a coin—that the outcomes should be equally likely. Sylvia isn't so sure that the "equally likely" model is correct, so she would like to collect some data before deciding on a model.

You will need enough small cups (bathroom disposable-size) for each student to have one. You will also need enough colored balls so that each student has four, two of each color. (If you do not have access to colored balls or blocks, tape a color label on coins.) The two colors do not matter. Students need to be able to identify which of two patterns of colors emerge when the cup containing the four balls is shaken. If your students do not know the word adjacent, use the term side-by-side. When students shake the cup, be sure that they are truly shaking up and down and are not "swirling" the cup around.

Read through the example and poll the class:

- ▪ Who agrees with Philippe?
- ▪ Who agrees with Sylvia and thinks that data should be collected before making a decision?

MP. The example suggests a theoretical probability model of opposite and adjacent patterns should be equally likely since there are just two outcomes. Hopefully students will be cautious of that sort of reasoning and ask to collect some data to test the conjectured model.

Exercise 3 (10–15 minutes)

Direct students to gather empirical evidence by actually shaking their cups and recording the number of times the colors are adjacent, and the number of times the colors are opposite.

Write their totals on the board and calculate the empirical probabilities that the experiment results in an adjacent pattern or that the experiment results in an opposite pattern. If each student generates   data points, pooling the data will result in a large number of observations.

MP. When all of the results are recorded, let students estimate the probabilities and answer the question independently. Discuss and confirm as a class.

Exercise 3

Collect data for Sylvia. Carry out the experiment of shaking a cup that contains four balls, two black and two white, observing, and recording whether the pattern is opposite or adjacent. Repeat this process   times. Then, combine the data with that collected by your classmates.

3.   **Do your results agree with Philippe's equally likely model, or do they indicate that Sylvia had the right idea? Explain.**

    Answers will vary; estimated probabilities of around − for the opposite pattern and − for the adjacent pattern will emerge, casting serious doubt on the equally likely model.

As an explanation for the observations, you can have your students do a live proof of the −, − model. Choose four students to come to the front of the room, say two boys and two girls. Have them stand as though they were balls in a cup (i.e., in a kind of square array). Fix one of them in position so that rotations are not counted. Have them systematically identify the (six) possible positions of the other three students, counting that there are four patterns for which the boys and girls are adjacent and two for which the boys are opposite each other and the girls are opposite each other.

Exercises 4–5 (12–15 minutes)

MP. Students should work together as a class to design a simulation for this example. The simulation procedure they develop is then used on each bag. Have about an equal number of bags of the different percentages to distribute to pairs of students, one bag per pair.

Exercises 4–5

There are three popular brands of mixed nuts. Your teacher loves cashews, and in his experience of having purchased these brands, he suggests that not all brands have the same percentage of cashews. One has around   cashews, one has   , and one has   .

Your teacher has bags labeled A, B, and C representing the three brands. The bags contain red beads representing cashews and brown beads representing other types of nuts. One bag contains   red beads, another   red beads, and the third has   red beads. You are to determine which bag contains which percentage of cashews. You cannot just open the bags and count the beads.

4. Work as a class to design a simulation. You need to agree on what an outcome is, what a trial is, what a success is, and how to calculate the estimated probability of getting a cashew. Base your estimate on   trials.

   - *An outcome is the result of choosing one bead from the given bag.*

   - *A red bead represents a cashew; a brown bead represents a non-cashew.*

   - *In this problem, a trial consists of one outcome.*

   - *A success is observing a red bead. Beads are replaced between trials.   trials are to be done.*

   - *The estimated probability of selecting a cashew from the given bag is the number of successes divided by   .*

5. Your teacher will give your group one of the bags labeled A, B, or C. Using your plan from part (a), collect your data. Do you think you have the   ,   , or   cashews bag? Explain.

   Once estimates have been computed, have the class try to decide which percentage is in which bag. Agreeing on the   bag may be the easiest, but there could be disagreement concerning the   and   . If they can't decide, ask them what they should do. Hopefully they say that they need more data. Typically, about   data points in a simulation yields estimated probabilities fairly close to the theoretical ones.

Exercises 6–8 (12–15 minutes)

You may need to go over this problem carefully with your students as it could be confusing to some of them. There are two bags with the same number of slips of paper in them; on the slips of paper are numbers between   –   ; some numbers occur more than once.

Note that there are   primes and   powers of   between   and   . You may want to have   slips of paper in each bag. In the prime bag, number the slips   –   . The probability of winning would be ⎯⎯   . In the power bag, to make the probabilities of winning distinctive, you may want to put each power of   in twice, and then put any non-power of two numbers on the other   slips. The probability of winning in that bag is ⎯⎯   .

Exercises 6–8

Suppose you have two bags, A and B, in which there are an equal number of slips of paper. Positive numbers are written on the slips. The numbers are not known, but they are whole numbers between   and   , inclusive. The same number may occur on more than one slip of paper in a bag.

These bags are used to play a game. In this game, you choose one of the bags and then choose one slip from that bag. If you choose bag A, and the number you choose from it is a prime number, then you win. If you choose bag B, and the number you choose from it is a power of   , you win. Which bag should you choose?

6.   Emma suggests that it doesn't matter which bag you choose because you don't know anything about what numbers are inside the bags. So she thinks that you are equally likely to win with either bag. Do you agree with her? Explain.

   Without any data, Emma is right. You may as well toss a coin to determine which bag to choose. However, gathering information through empirical evidence will help in making an informed decision.

7.   Aamir suggests that he would like to collect some data from both bags before making a decision about whether or not the model is equally likely. Help Aamir by drawing   slips from each bag, being sure to replace each one before choosing again. Each time you draw a slip, record whether it would have been a winner or not. Using the results, what is your estimate for the probability of drawing a prime number from bag A and drawing a power of   from bag B?

   Answers will vary.

8.   If you were to play this game, which bag would you choose? Explain why you would pick this bag.

   Answers will vary.

Closing (5 minutes)

Discuss with students the Lesson Summary.

Lesson Summary

This lesson involved knowing the probabilities for outcomes of a sample space. You used these probabilities to determine whether or not the simulation supported a given theoretical probability. The simulated probabilities, or estimated probabilities, suggested a workable process for understanding the probabilities. Only   trials were conducted in some examples; however, as stated in several other lessons, the more trials that are observed from a simulation, the better.

Exit Ticket (5 minutes)

Name _____   Date_____

Lesson 12: Using Probability to Make Decisions

Exit Ticket

There are four pieces of bubble gum left in a quarter machine. Two are red and two are yellow. Chandra puts two quarters in the machine. One piece is for her and one is for her friend, Kay. If the two pieces are the same color, she's happy because they won't have to decide who gets what color. Chandra claims that they are equally likely to get the same color because the colors are either the same or they are different. Check her claim by doing a simulation.

a. Name a device to use to simulate getting a piece of bubble gum. Specify what outcome of the device represents a red piece and what outcome represents yellow.

b. Define what a trial is for your simulation.

c. Define what constitutes a success in a trial of your simulation.

d. Perform and list   simulated trials. Based on your results, is Chandra's equally likely model is correct?

Exit Ticket Sample Solutions

There are four pieces of bubble gum left in a quarter machine. Two are red and two are yellow. Chandra puts two quarters in the machine. One piece is for her and one is for her friend, Kay. If the two pieces are the same color, she's happy because they won't have to decide who gets what color. Chandra claims that they are equally likely to get the same color because the colors are either the same or they are different. Check her claim by doing a simulation.

a. Name a device to use to simulate getting a piece of bubble gum. Specify what outcome of the device represents a red piece and what outcome represents yellow.

There are several ways to simulate the bubble gum outcomes. For example, two red and two yellow disks could be put in a bag. A red disk represents a red piece of bubble gum, and a yellow disk represents a yellow piece.

b. Define what a trial is for your simulation.

A trial is choosing two disks (without replacement).

c. Define what constitutes a success in a trial of your simulation.

A success is if two disks are of the same color; a failure is if they differ.

d. Perform and list   simulated trials. Based on your results, is Chandra's equally likely model is correct?

simulated trials will produce a probability estimate of about − for same color and − for different colors.

Note: Some students may believe that the model is equally likely, but hopefully they realize by now that they should make some observations to make an informed decision. Some may see that this problem is actually similar to Example 3.

Problem Set Sample Solutions

1. Some M&M's are "defective." For example, a defective M&M may have its "m" missing, or it may be cracked, broken, or oddly shaped. Is the probability of getting a defective M&M higher for peanut M&M's than for plain M&M's?

Gloriann suggests the probability of getting a defective plain M&M is the same as the probability of getting a defective peanut M&M. Suzanne doesn't think this is correct because a peanut M&M is bigger than a plain M&M, and, therefore, has a greater opportunity to be damaged.

a. Simulate inspecting a plain M&M by rolling two number cubes. Let a sum of   or   represent a defective plain M&M, and the other possible rolls represent a plain M&M that is not defective. Do   trials and compute an estimate of the probability that a plain M&M is defective. Record the   outcomes you observed. Explain your process.

A simulated outcome for a plain M&M involves rolling two number cubes. A trial is the same as an outcome in this problem. A success is getting a sum of   or   , either of which represents a defective plain M&M.   trials should produce somewhere around   successes. A side note is that the theoretical probability of

getting a sum of   or   is —   .

b.   Simulate inspecting a peanut M&M by selecting a card from a well-shuffled deck of cards. Let a one-eyed face card and clubs represent a defective peanut M&M, and the other cards represent a peanut M&M that is not defective. Be sure to replace the chosen card after each trial and to shuffle the deck well before choosing the next card. Note that the one-eyed face cards are the King of Diamonds, Jack of Hearts, and Jack of Spades. Do _____ trials and compute an estimate of the probability that a peanut M&M is defective. Record the list of _____ cards that you observed. Explain your process.

A simulated outcome for a peanut M&M involves choosing one card from a deck. A trial is the same as an outcome in this problem. A success is getting a one-eyed face card or a club, any of which represents a defective peanut M&M. _____ trials of choosing cards with replacement should produce somewhere around six successes.

c.   For this problem, suppose that the two simulations provide accurate estimates of the probability of a defective M&M for plain and peanut M&M's. Compare your two probability estimates and decide whether Gloriann's belief that the defective probability is the same for both types of M&M's is reasonable. Explain your reasoning.

Estimates will vary; the probability estimate for finding a defective plain M&M is approximately _____ , and the probability estimate for finding a defective peanut M&M is about _____ . Gloriann could possibly be right, as it appears more likely to find a defective peanut M&M than a plain one, based on the higher probability estimate.

2.   One at a time, mice are placed at the start of the maze shown below. There are four terminal stations at A, B, C, and D. At each point where a mouse has to decide in which direction to go, assume that it is equally likely for it to choose any of the possible directions. A mouse cannot go backwards.

In the following simulated trials, L stands for Left, R for Right, and S for Straight. Estimate the probability that a mouse finds station C where the food is. No food is at A, B, or D. The following data were collected on _____ simulated paths that the mice took.

LR RL RL LL LS LS RL RR RR RL

RL LR LR RR LR LR LL LS RL LR

RR LS RL RR RL LR LR LL LS RR

RL RL RL RR RR RR LR LL LL RR

RR LS RR LR RR RR LL RR LS LS
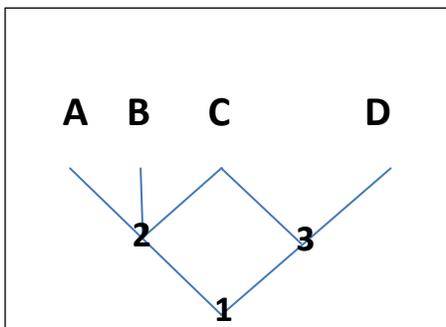
a.   What paths constitute a success and what paths constitute a failure?

An outcome is the direction chosen by the mouse when it has to make a decision. A trial consists of a path (two outcomes) that leads to a terminal station. The paths are LL (leads to A), LS (leads to B), LR (leads to C), RL (leads to C), and RR (leads to D). A success is a path leading to C, which is LR or RL; a failure is a path leading to A, B, or D, which is LL, LS, or RR.

    b.   **Use the data to estimate the probability that a mouse finds food. Show your calculation.**

        Using the given   simulated paths:

	A(LL)	B(LS)	C(LR,RL)	D(RR)
Simulated	—	—	——	—
Theoretical				

    c.   **Paige suggests that it is equally likely that a mouse gets to any of the four terminal stations. What does your simulation suggest about whether her equally likely model is believable? If it is not believable, what do your data suggest is a more believable model?**

        Paige is incorrect. The mouse still makes direction decisions that are equally likely, but there is one decision point that has three rather than two emanating paths. Based on the simulation, the mouse is more likely to end up at terminal C.

    d.   **Does your simulation support the following theoretical probability model? Explain.**

        The probability a mouse finds terminal point A is   .

        The probability a mouse finds terminal point B is   .

        The probability a mouse finds terminal point C is   .

        The probability a mouse finds terminal point D is   .

        Yes, the simulation appears to support the given probability model as the estimates are reasonably close.

	A(LL)	B(LS)	C(LR,RL)	D(RR)
Simulated	—	—	——	—
Theoretical				