Lesson 8: The Difference Between Theoretical Probabilities and Estimated Probabilities

Student Outcomes

- Given theoretical probabilities based on a chance experiment, students describe what they expect to see when they observe many outcomes of the experiment.
- Students distinguish between theoretical probabilities and estimated probabilities.
- Students understand that probabilities can be estimated based on observing outcomes of a chance experiment.

Did you ever watch the beginning of a Super Bowl game? After the traditional handshakes, a coin is tossed to determine which team gets to kick-off first. Whether or not you are a football fan, the toss of a fair coin is often used to make decisions between two groups.

Classwork

Example 1 (5 minutes): Why a Coin?

Example 1: Why a Coin?

Coins were discussed in previous lessons of this module. What is special about a coin? In most cases, a coin has two different sides: a head side ("heads") and a tail side ("tails"). The sample space for tossing a coin is {heads, tails}. If each outcome has an equal chance of occurring when the coin is tossed, then the probability of getting heads is – or . The probability of getting tails is also . Note that the sum of these probabilities is .

The probabilities formed using the sample space and what we know about coins are called the theoretical probabilities. Using observed relative frequencies is another method to estimate the probabilities of heads or tails. A relative frequency is the proportion derived from the number of the observed outcomes of an event divided by the total number of outcomes. Recall from earlier lessons that a relative frequency can be expressed as a fraction, a decimal, or a percent. Is the estimate of a probability from this method close to the theoretical probability? The following exercise investigates how relative frequencies can be used to estimate probabilities.

This lesson focuses on the chance experiment of tossing a coin. The outcomes are simple, and in most cases, students understand the theoretical probabilities of the outcomes. It is also a good example to build on their understanding of estimated probabilities. This example then sets up the situation of estimating these same probabilities using relative frequencies. The term *relative frequency* is introduced and defined in this example and the following exercise.

Have students read through the example. Then, use the following questions to guide the discussion:

- Are there other situations where a coin toss would be used?
  - *As a part of this discussion, you might indicate that in several state constitutions, if two candidates receive the same number of votes, the winning candidate is determined by a coin toss.*

MP.

- Is it possible to toss a fair coin and get heads in a row? How about heads?
  - *Make sure students understand that it is possible to get several heads or tails in a row, and that evaluating how likely it would be to get three, five, or even heads in a row, are examples of probability problems.*

Exercises 1–9 (15 minutes)

MP.

The following Exercises are designed to have students develop an estimate of the probability of getting heads by collecting data. In this example, students are provided with data from actual tosses of a fair coin. Students calculate the relative frequencies of getting heads from the data, and then use the relative frequencies to estimate the probability of getting a head.

Let students work in small groups to complete Exercises 1–9.

Exercises 1–9

Beth tosses a coin times and records her results. Here are the results from the tosses:

Toss										
Result	**H**	**H**	**T**	**H**	**H**	**H**	**T**	**T**	**T**	**H**

The total number of heads divided by the total number of tosses is the relative frequency of heads. It is the proportion of the time that heads occurred on these tosses. The total number of tails divided by the total number of tosses is the relative frequency of tails.

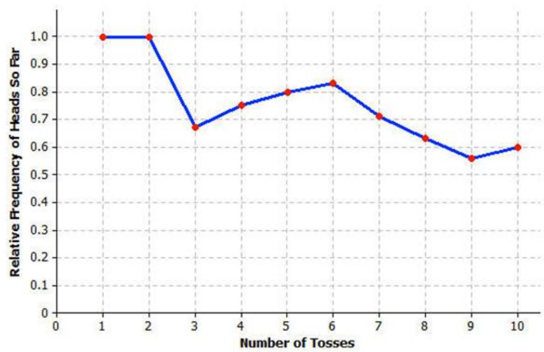
1. Beth started to complete the following table as a way to investigate the relative frequencies. For each outcome, the total number of tosses increased. The total number of heads or tails observed so far depends on the outcome of the current toss. Complete this table for the tosses recorded above.

Toss	Outcome	Total number of heads so far	Relative frequency of heads so far (to the nearest hundredth)	Total number of tails so far	Relative frequency of tails so far (to the nearest hundredth)
	H		− =		− =
	H		− =		− =
	T		− =		− =
	H		*− =*		*− =*
	H		*− =*		*− =*
	H		*− =*		*− =*
	T		*− =*		*− =*
	T		*− =*		*− =*
	T		*− =*		*− =*
	H		*— =*		*— =*

2. What is the sum of the relative frequency of heads and the relative frequency of tails for each row of the table?

   The sum of the relative frequency of heads and the relative frequency of tails for each row is .

3. Beth's results can also be displayed using a graph. Complete this graph using the values of relative frequency of heads so far from the table above:



4. Beth continued tossing the coin and recording results for a total of tosses. Here are the results of the next tosses:

Toss										
Result	T	H	T	H	T	H	H	T	H	T

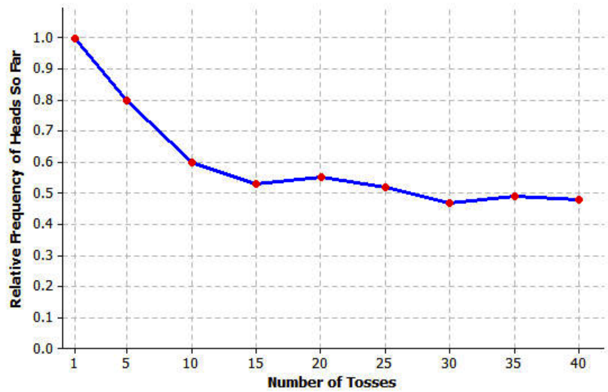
Toss										
Result	H	T	T	H	T	T	T	T	H	T

Toss										
Result	H	T	H	T	H	T	H	H	T	T

As the number of tosses increases, the relative frequency of heads changes. Complete the following table for the coin tosses:

Number of tosses	Total number of heads so far	Relative frequency of heads so far (to the nearest hundredth)

5. **Complete the graph below using the relative frequency of heads so far from the table above for total number of tosses of , , , , and :**



6. **What do you notice about the changes in the relative frequency of number of heads so far as the number of tosses increases?**

   The relative frequencies seem to change less as the number of tosses increases. The line drawn to connect the relative frequencies seems to be leveling off.

7. **If you tossed the coin times, what do you think the relative frequency of heads would be? Explain your answer.**

   Answers will vary. Anticipate most students will indicate 50 heads result in tosses, for a relative frequency of . This is a good time to indicate that the value of is where the graph of the relative frequencies seems to be approaching. However, the relative frequencies will vary. For example, if the relative frequency for tosses were (and it could be), what would the relative frequency for tosses be? Point out to students that no matter the outcome on the st toss, the relative frequency of heads would not be exactly .

8. **Based on the graph and the relative frequencies, what would you estimate the probability of getting heads to be? Explain your answer.**

   Answers will vary. Anticipate that students will estimate the probability to be , as that is what they determined in the opening discussion, and that is the value that the relative frequencies appear to be approaching. Some students may estimate the probability as , as that was the last relative frequency obtained after tosses. That estimate is also a good estimate of the probability.

9. **How close is your estimate in Exercise 8 to the theoretical probability of ? Would the estimate of this probability have been as good if Beth had only tossed the coin a few times instead of ?**

   In the beginning, the relative frequencies jump around. The estimated probabilities and the theoretical probabilities should be nearly the same as the number of observations increase. The estimated probabilities would likely not be as good after just a few coin tosses. Direct students to draw a horizontal line representing their estimate of the probability (or, in most cases,).

The value you gave in Exercise 8 is an estimate of the theoretical probability and is called an experimental or estimated probability.

If time permits, you might point out some history on people who wanted to observe long-run relative frequencies. Share with students how, for each of these cases, the relative frequencies were close to . Students may also find it interesting that the relative frequencies were not exactly . Ask students, "If they were closer to than in our example, why do you think that was the case?"

- The French naturalist Count Buffon (1707–1788) tossed a coin times.
  Result: heads, or proportion for heads.

- Around 1900, the English statistician, Karl Pearson, tossed a coin times.
  Result: heads, a proportion of .

- While imprisoned by the Germans during World War II, the South African mathematician, John Kerrich, tossed a coin times.
  Result: heads, a proportion of .

Example 2 (5 minutes): More Pennies!

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Beth received nine more pennies. She securely taped them together to form a small stack. The top penny of her stack showed heads, and the bottom penny showed tails. If Beth tosses the stack, what outcomes could she observe?

This example moves the discussion to a chance experiment in which the theoretical probability is not known. Prepare several stacks of the pennies as described in this example. Make sure the pennies are stacked with one end showing heads and the other end tails. It is suggested you use scotch tape to wrap the entire stack. Because constructing the stacks might result in pennies flying around, it is suggested you prepare the stacks before this exercise is started.

Introduce students to the following exercise by tossing the stack a few times (and testing that it did not fall apart!). Then ask:

- What are the possible outcomes?
  Head, Tail, and on the Side. These three outcomes represent the sample space.

Exercises 10–17 (15 minutes)

Let students continue to work in small groups.

Exercises 10–17

10. **Beth wanted to determine the probability of getting heads when she tosses the stack. Do you think this probability is the same as the probability of getting heads with just one coin? Explain your answer.**

    The outcomes when tossing this stack would be: {Head, Tail, Side}. This changes the probability of getting heads, as there are three outcomes.

11. Make a sturdy stack of pennies in which one end of the stack has a penny showing heads and the other end tails. Make sure the pennies are taped securely, or you may have a mess when you toss the stack. Toss the stack to observe possible outcomes. What is the sample space for tossing a stack of pennies taped together? Do you think the probability of each outcome of the sample space is equal? Explain your answer.

The sample space is {Head, Tail, Side}. A couple of tosses should clearly indicate to students that the stack lands often on its side. As a result, the probabilities of heads, tails, and on the side do not appear to be the same.

12. Record the results of tosses. Complete the following table of the relative frequencies of heads for your tosses:

Answers will vary; the results of an actual toss are shown below.

Toss										
Result	*Head*	*Head*	*Side*	*Side*	*Side*	*Tail*	*Side*	*Side*	*Tail*	*Side*
Relative frequency of heads so far										

13. Based on the value of the relative frequencies of heads so far, what would you estimate the probability of getting heads to be?

If students had a sample similar to the above, they would estimate the probability of tossing a head as (or something close to that last relative frequency).

14. Toss the stack of pennies another times. Complete the following table:

Answers will vary; student data will be different.

Toss										
Result	*Head*	*Head*	*Tail*	*Side*	*Side*	*Tail*	*Tail*	*Side*	*Tail*	*Side*

Toss										
Result	*Side*	*Head*	*Side*	*Side*	*Head*	*Tail*	*Tail*	*Head*	*Head*	*Side*

15. Summarize the relative frequencies of heads so far by completing the following table:

Sample table is provided using data from Exercise 14.

Number of tosses	Total number of heads so far	Relative frequency of heads so far (to the nearest hundredth)

16. **Based on the relative frequencies for the tosses, what is your estimate of the probability of getting heads? Can you compare this estimate to a theoretical probability like you did in the first example? Explain your answer.**

Answers will vary. Students are anticipated to indicate an estimated probability equal or close to the last value in the relative frequency column. For this example, that would be . An estimate of for this sample would have also been a good estimate. Students would indicate that they could not compare this to a theoretical probability, as the theoretical probability is not known for this example. Allow for a range of estimated probabilities. Factors that might affect the results for the long-run frequencies include how much tape is used to create the stack and how sturdy the stack is. Discussing these points with students is a good summary of this lesson.

17. **Create another stack of pennies. Consider creating a stack using pennies, pennies, or pennies taped together in the same way you taped the pennies to form a stack of pennies. Again, make sure the pennies are taped securely, or you might have a mess!**

Toss the stack you made times. Record the outcome for each toss:

Toss										
Result										

Toss										
Result										

Toss										
Result										

The problem set involves another example of obtaining results from a stack of pennies. Suggestions include stacks of , , and (or a number of your choice). The problem set includes questions based on the results from tossing one of these stacks. Provide students in small groups one of these stacks. Each group should collect data for tosses to use for the problem set.

Closing (5 minutes)

When students finish collecting data for the problem set, ask the following:

- When you toss the stack and calculate a relative frequency, are you getting an estimated probability or a theoretical probability?
  - *You are getting an estimated probability.*
- Is there an exact number of times you should toss the stack to estimate the probability of getting heads?
  - *There is no exact number of times you should toss the stack; however, the larger the number of tosses, the closer the estimated probability will approach to the probability of the event.*

Lesson Summary

- **Observing the long-run relative frequency of an event from a chance experiment (or the proportion of an event derived from a long sequence of observations) approximates the theoretical probability of the event.**
- **After a long sequence of observations, the observed relative frequencies get close to the probability of the event occurring.**
- **When it is not possible to compute the theoretical probabilities of chance experiments, then the long-run relative frequencies (or the proportion of events derived from a long sequence of observations) can be used as estimated probabilities of events.**

Exit Ticket (5–8 minutes)

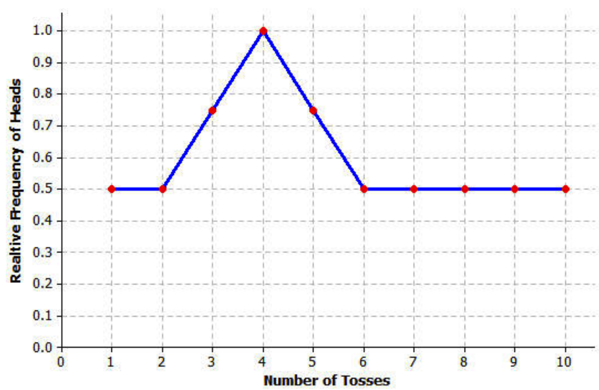
Name ____ Date____________________

Lesson 8: The Difference Between Theoretical Probabilities and Estimated Probabilities

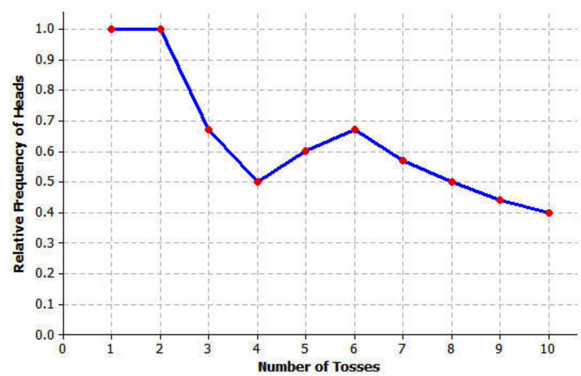
Exit Ticket

1. Which of the following graphs would NOT represent the relative frequencies of heads when tossing penny? Explain your answer.

Graph A



Graph B



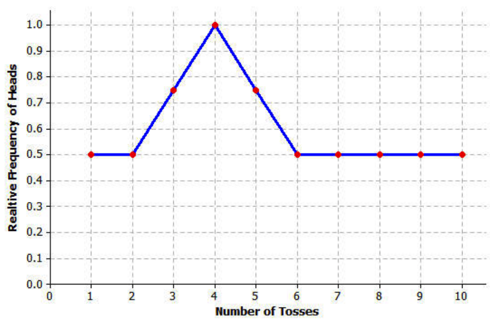
2. Jerry indicated that after tossing a penny times, the relative frequency of heads was (to the nearest hundredth). He indicated that after times, the relative frequency of heads was . Are Jerry's summaries correct? Why or why not?

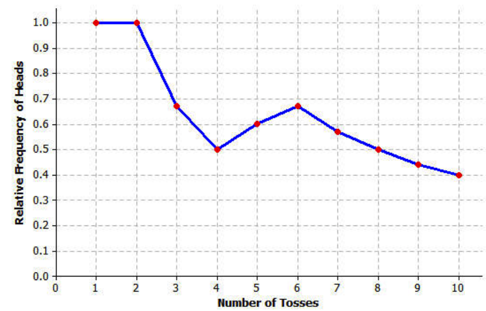
3. Jerry observed heads in tosses of his coin. Do you think this was a fair coin? Why or why not?

Exit Ticket Sample

1. **Which of the following graphs would NOT represent the relative frequencies of heads when tossing penny? Explain your answer.**

Graph A **Graph B**





Graph A would not represent a possible graph of the relative frequencies. Students could point out a couple of reasons. The first problem is the way Graph A starts. After the first toss, the probability would either be a or a . Also, it seems to settle exactly to the theoretical probability without showing the slight changes from toss to toss.

2. **Jerry indicated that after tossing a penny times, the relative frequency of heads was (to the nearest hundredth). He indicated that after times, the relative frequency of heads was . Are Jerry's summaries correct? Why or why not?**

Something is wrong with Jerry's information. If he tossed the penny 30 times, and the relative frequency of heads was , then he had heads. If his next toss were heads, then the relative frequency would be — or (to the nearest hundredth). If his next toss were tails, then the relative frequency would be — or (to the nearest hundredth).

3. **Jerry observed heads in tosses of his coin? Do you think this was a fair coin? Why or why not?**

Students should indicate Jerry's coin is probably not a fair coin. The relative frequency of heads for a rather large number of tosses should be close to the theoretical probability. For this problem, the relative frequency of is quite different from , and probably indicates that the coin is not fair.

Problem Set Sample Solutions

1. If you created a stack of pennies taped together, do you think the probability of getting a head on a toss of the stack would be different than for a stack of pennies? Explain your answer.

   The estimated probability of getting a head for a stack of pennies would be different than for a stack of pennies. A few tosses indicate that it is very unlikely that the outcome of heads or tails would result as the stack almost always lands on its side. (The possibility of a head or a tail is noted, but it has a small probability of being observed.)

2. If you created a stack of pennies taped together, what do you think the probability of getting a head on a toss of the stack would be? Explain your answer.

   The estimated probability of getting a head for a stack of pennies is very small. The toss of a stack of this number of pennies almost always lands on its side. Students might indicate there is a possibility but with this example, the observed outcomes are almost all on their side.

   Note: If students selected a stack of 5 coins, the outcomes are nearly the same as if it was only coin. The probability of landing on its side is small (close to). As more pennies are added, the probability of the stack landing on its side increases, until it is nearly (or).

3. Based on your work in this lesson, complete the following table of the relative frequencies of heads for the stack you created:

   Answers will vary based on the outcomes of tossing the stack. As previously stated, as more pennies are added to the stack, the probability that the stack will land on its side increases. Anticipate results of for a stack of pennies. Samples involving pennies will have a very small probability of showing heads.

Number of tosses	Total number of heads so far	Relative frequency of heads so far (to the nearest hundredth)

4. What is your estimate of the probability that your stack of pennies will land heads up when tossed? Explain your answer.

   Answers will vary based on the relative frequencies.

5. Is there a theoretical probability you could use to compare to the estimated probability? Explain you answer.

   There is no theoretical probability that could be calculated to compare to the estimated probability.