Exponential and Log Functions – Part Two

Math Background

Previously, you

- Simplified linear, quadratic, radical, polynomial and rational functions
- Performed arithmetic operations with linear, quadratic, radical, polynomial and rational functions
- Created functions to represent a real life situation
- Solved linear, quadratic, radical, polynomial and rational functions
- Transformed parent functions of linear, quadratic, radical, polynomial and rational functions

In this unit you will

- Simplify logarithmic expressions
- Solve exponential equations and inequalities
- Solve logarithmic functions
- Use exponential and logarithmic functions to model data

You can use the skills in this unit to

- Use the structure of an expression to identify ways to rewrite it.
- Interpret the domain and its restrictions of a real-life function.
- Model and solve real-world problems with exponential and logarithmic functions using graphs
- Model real-life data using exponential and logarithmic functions.

Vocabulary

- **Base –** The number *b* in the logarithm expression $\log_b C$
- **Common logarithm –** Logarithm to the base 10
- **Decay factor –** It is the percentage by which the original amount will decline.
- **Exponential Function –** A function whose exponent involves at least one variable.
- **Growth factor** – It is the percentage by which the original amount will increase.
- **Horizontal Asymptote** – A horizontal line that the graph of a function approaches as x tends to plus or minus infinity. It describes the function's end behavior.
- **Logarithm –** The exponent when expressing a number as the exponent of another number, usually to the base 10.
- **Logarithmic function –** A function in the form of $\log_b x$ in which *b* is a constant and *x* is a positive variable.
- **Natural logarithm –** The logarithm of a given number to the base *e*, where *e* is 2.71828….
- **Vertical asymptote –** A vertical line that the curve approaches more and more closely but never touches as the curve goes off to positive or negative infinity. The vertical lines correspond to the zeroes of the denominator of the rational function.

Essential Questions

- Why is it helpful to be able to change the forms of exponential expressions?
- Why are exponential and log functions important in real-life applications?

Overall Big Ideas

Changing the forms of an expression from one exponential form to another reveals important attributes in the function's meaning.

Many real world data can be modeled with exponential or logarithmic functions.

Skill

To use properties to simplify logarithmic expressions.

To solve exponential equations and inequalities.

To use base e to solve and graph exponential and logarithmic functions.

To use exponential and logarithmic functions to model data.

Related Standards

A.SSE.B.3c

Use the properties of exponents to transform expressions for exponential functions. For example, the expression 1.15*t* can be rewritten as $(1.151/12)12t \approx 1.01212t$ to reveal the approximate equivalent monthly interest rate if the annual rate is 15%. *(Modeling Standard)

F.LE.A.4

For exponential models, express as a logarithm the solution to $abct = d$ where *a*, *c*, and *d* are numbers and the base *b* is 2, 10, or *e*; evaluate the logarithm using technology. *(Modeling Standard)

A.CED.A.1-2

Create equations and inequalities in one variable and use them to solve problems. Include equations arising from all types of functions, including simple rational and radical functions. *(Modeling Standard)

A.REI.D.11-2

Explain why the x-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$ approximately, e.g., using technology to graph the functions, make table of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, radical, absolute value, exponential, and logarithmic functions. *(Modeling Standard)

F.IF.B.5-2

Relate the domain of any function to its graph and, where applicable, to the quantitative relationship it describes. For example, if the function h(n) gives the number of person-hours it takes to assemble *n* engines in a factory, then the positive integers would be an appropriate domain for the function *(Modeling Standard)

A.CED.A.2-2

Create equations in two or more variables to represent relationships between quantities and graph equations on coordinate axes with labels and scales. Use all types of equations. *(Modeling Standard)

A.CED.A.3-2

Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or non-viable options in a modeling context. Use all types of equations. For example, represent inequalities describing nutritional and cost constraints on combinations of different foods. *(Modeling Standard)

F.IF.C.7e-2

Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude. *(Modeling Standard)

S.ID.B.6a

Informally assess the fit of a function by plotting and analyzing residuals. *(Modeling Standard)

S.ID.B.6b

Fit a function to the data; use functions fitted to data to solve problems in the context of the data. Use given functions or choose a function suggested by the context. Emphasize linear, quadratic, and exponential models. *(Modeling Standard)

Notes, Examples, and Exam Questions

Unit 7.6 Use Properties to simplify logarithmic expressions.

Review: Properties of Exponents (Allow students to come up with these on their own.) We will now extend these properties for use with logarithms.

Let *a* and *b* be real numbers, and let *m* and *n* be integers.

Product of Powers Property $a^m \cdot a^n = a^{m+n}$

Quotient of Powers Property $\frac{a^m}{a^n} = a^{m-n}$ or $\frac{a^m}{a^n} = \frac{1}{a^{n-m}}, a \neq 0$

Power of a Power Property $\left(a^m\right)^n = a^{mn}$

Because of the relationship between logarithms and exponents, the properties of logarithms are similar.

Properties of Logarithms: Let *b*, *r*, and *v* be positive numbers with $b \neq 1$.

Product Property	$\log_b uv = \log_b u + \log_b v$	$\ln(ab) = \ln a + \ln b$
Quotient Property	$\log_b \frac{u}{v} = \log_b u - \log_b v$	$\ln\left(\frac{a}{b}\right) = \ln a - \ln b$
Power Property	$\log_b u^n = n \log_b u$	$\ln a^b = b \ln a$

These properties may also come in handy: $\log_b 1 = 0$ $\log_b b = 1$ $\log_b b^n = n$ $b^{\log_b x} = x$

$\ln 1 = 0$ $\ln e = 1$ $\ln\left(e^x\right) = x$ $e^{\ln x} = x$

Using Properties of Logarithms to Approximate the Value of a Logarithmic Expression

Ex 1: Use the approximations $\log_3 5 \approx 1.46$ and $\log_3 7 \approx 1.77$ to approximate the expressions.

a. $\log_3 35$ Rewrite as a product: $\log_3(5 \cdot 7)$

Use the product property of logarithms. $\log_3(5 \cdot 7) = \log_3 5 + \log_3 7$

Substitute the values of the logarithms. $\log_3 5 + \log_3 7 \approx 1.46 + 1.77 = \boxed{3.23}$

b. $\log_3 \frac{7}{5}$ Use the property of logarithms. $\log_3 \frac{7}{5} = \log_3 7 - \log_3 5$

Substitute the values of the logarithms. $\log_3 7 - \log_3 5 \approx 1.77 - 1.46 = \boxed{0.31}$

c. $\log_3 25$ Rewrite as a power: $\log_3 5^2$

Use the power property of logarithms. $2\log_3 5$

Substitute the values of the logarithms. $2\log_3 5 \approx 2(1.46) = \boxed{2.92}$

d. $\log_3 63$ Rewrite as a product and power: $\log_3\left(3^2 \cdot 7\right)$

Use the power property and product property of logarithms. $2\log_3 3 + \log_3 7$

Substitute the values of the logarithms. $2\log_3 3 = 2(1) + 1.77 = \boxed{3.77}$

Rewriting Logarithmic Expressions

Ex 2: Expand the expression $\log_5 \frac{x^4 y}{z}$. Assume all variables are positive.

Quotient Property: $\log_5 \frac{x^4 y}{z} = \log_5 x^4 y - \log_5 z$

Product Property: $= \log_5 x^4 + \log_5 y - \log_5 z$

Power Property: $= \boxed{4\log_5 x + \log_5 y - \log_5 z}$

Ex 3: Expand the expression $\ln\left(\sqrt{x^2+5}/3x\right)$. Assume all variables are positive.

Quotient Property: $\ln \frac{\sqrt{x^2+5}}{3x} = \ln\left(x^2+5\right)^{\frac{1}{2}} - \ln 3x$

Product Property: $\ln\left(x^2+5\right)^{\frac{1}{2}} - \ln 3 + \ln x$

Power Property: $\boxed{\frac{1}{2}\ln\left(x^2+5\right) - \ln 3 + \ln x}$

Ex 4: Write an equivalent form of the expression $\frac{1}{2}\ln y - 5\ln x - \ln z$. Assume all variables are positive.

Power Property: $\frac{1}{2}\ln y - 5\ln x - \ln z = \ln y^{\frac{1}{2}} - \ln x^5 - \ln z$

Quotient Property: $\ln y^{\frac{1}{2}} - \ln x^5 - \ln z = \ln \frac{y^{\frac{1}{2}}}{x^5 z} = \boxed{\ln \frac{\sqrt{y}}{x^5 z}}$

Ex 5: Write an equivalent form of the expression $\log_2 x^5 - 2\log_2(xy)$. Assume all variables are positive.

Power Property: $\log_2 x^5 - \log_2 (xy)^2$

Quotient Property: $\log_2 \frac{x^5}{x^2 y^2} = \log_2 \frac{\cancel{x^5}}{\cancel{x^2} y^2} = \boxed{\log_2 \frac{x^3}{y^2}}$

Evaluating Logarithms of Base *b*

Start with $\log_b u = x$.

⚠ Properties: $\log_b x = y$ if and only if $b^y = x$

$\ln x = y$ if and only if $e^y = x$

Rewrite in exponential form: $b^x = u$

Take the "log" of both sides (Any log will do!): $\log b^x = \log u$

We will use either the common (base 10) or natural logarithm (base *e*) because these are the logs that the calculator can evaluate.

Use the power property: $x \log b = \log u$ Solve for *x*: $x = \frac{\log u}{\log b}$

Change of Base Formula: $\log_b u = \frac{\log_c u}{\log_c b}$

Ex 6a: Use the change of base formula to approximate the value of $\log_3 35$.

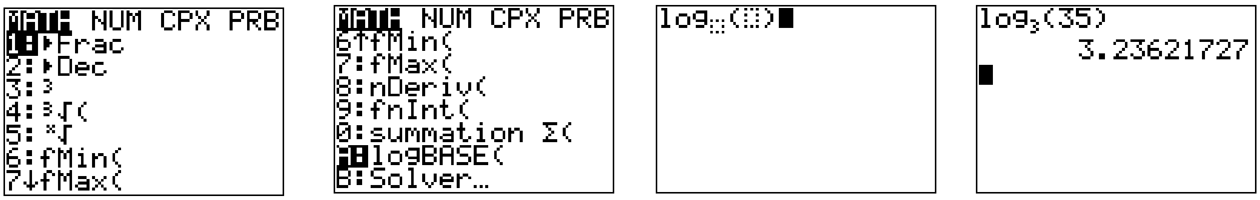
Using common logarithms: $\log_3 35 = \frac{\log 35}{\log 3} \approx \boxed{3.236}$

Now try the same example using natural logarithms: $\log_3 35 = \frac{\ln 35}{\ln 3} \approx \boxed{3.236}$

⚠ Note: We approximated the value of this logarithm previously in the notes – Example 1a. Compare!

Ex 6b: Use the calculator to approximate the value of $\log_3 35$.

The latest software update for the TI-84 calculator has a log solver. To evaluate $\log_3 35$, go to the MATH menu and find the logBASE app. Plug in the values.

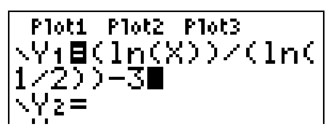


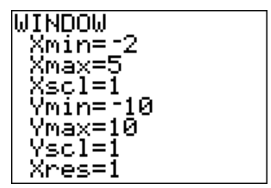
Graphing Logarithmic Functions: We can now use the change of base formula to graph logarithmic functions of any base on the graphing calculator.

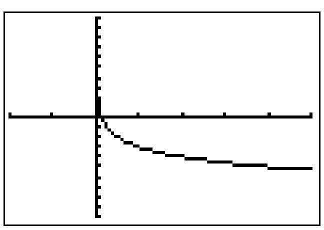
Ex 7: Graph the function $f(x) = \log_{1/2} x - 3$.

Rewrite the function using the change of base formula. (We will use the natural logarithm.)

$$f(x) = \log_{1/2} x - 3 = \frac{\ln x}{\ln\left(\frac{1}{2}\right)} - 3$$







Note: We graphed this function by hand earlier in earlier notes. (Notes Section 7.1 – 7.5) Compare!

QOD: Explain how you can rewrite the expression $\log x - \log y$ without using the quotient property of logarithms. (Hint: Use the power and product properties.)

Simplifying Logarithmic expressions using the inverse property:

We can use the inverse logarithm properties to simplify expressions. $\log_b b^x = x$ $\quad b^{\log_b x} = x$

Ex 8: Simplify the following log expressions

a. $3^{\log_3 5x} = \boxed{5x}$

b. $\log_4 16^x = \log_4 \left(4^2\right)^x = \log_4 (4)^{2x} = \boxed{2x}$

c. $\log_2 16 = \log_2\left(2^4\right) = \boxed{4}$

SAMPLE EXAM QUESTIONS

1. **Expand the expression $\log\left(3n^5\right)$.**

   A. $5\log 3 + 5\log n$

   B. $\log 3 + 5\log n$

   C. $\left(\log 3\right)\left(\log n^5\right)$

   D. $\left(\log 3\right)\left(5\log n\right)$

Ans: B

2. **Find the value of** $\log_2 32$.

   A. 5　　C. 1024

   B. 16　　D. 4

Ans: A

3. **Which is the same function as $f(x) = \ln\frac{x}{3}$?**

   A. $g(x) = \ln x - \ln 3$

   B. $g(x) = \frac{\ln x}{\ln 3}$

   C. $g(x) = \ln 3 - \ln x$

   D. $g(x) = \ln x + \ln 3$

Ans: A

4. **Rewrite $\log_9 9^{2x+3} = y$ in exponential form.**

   A. $9y = 9(2x+3)$

   B. $9^y = 9^{2x+3}$

   C. $y = 2x+3$

   D. $9y = 18x + 27$

Ans: B

Unit 7.7 To solve exponential equations and inequalities.

Solving Exponential Equations

Method 1: Rewrite both sides of the equation so that they have the same base.

Note: If $b^x = b^y$, then $x = y$.

Ex 9: Solve the equation $8^x = 4^{x-1}$.

Rewrite both sides with a base of 2. (Note: $8 = 2^3$ and $4 = 2^2$) $\left(2^3\right)^x = \left(2^2\right)^{x-1}$

Use the power of a power property. $2^{3x} = 2^{2x-2}$

Equate the exponents and solve for *x*. $3x = 2x - 2$

$\boxed{x = -2}$

Check the solution by substituting into the original equation. $(8)^{-2} = (4)^{-2-1}$

$\frac{1}{64} = 4^{-3}$ True

Ex 10: Solve the equation $20\left(\frac{1}{2}\right)^{\frac{x}{3}} = 5$.

Divide both sides by 20 to isolate the base. $\left(\frac{1}{2}\right)^{\frac{x}{3}} = \frac{1}{4}$

Rewrite both sides with a base of $\frac{1}{2}$ Note: $\frac{1}{4} = \left(\frac{1}{2}\right)^2$ $\left(\frac{1}{2}\right)^{\frac{x}{3}} = \left(\frac{1}{2}\right)^2$

Equate the exponents and solve for *x*. $\frac{x}{3} = 2$

$\boxed{x = 6}$

Check the solution by substituting into the original equation. $20\left(\frac{1}{2}\right)^{\frac{6}{3}} = 5 \Rightarrow 20\left(\frac{1}{4}\right) = 5$ True

Method 2: Taking a logarithm of both sides.

Ex 11: Solve the equation $3^x = 8$.

Take the log base 3 of both sides. $\log_3 3^x = \log_3 8$

Simplify using inverses. $x = \log_3 8$

Use the change of base formula to evaluate. $x = \boxed{\frac{\log 8}{\log 3} \approx 1.89}$

Alternate Method:

Take the natural log of both sides. $\ln 3^x = \ln 8$

Use the power property of logs. $x \ln 3 = \ln 8$

Solve for *x*. $x = \boxed{\frac{\ln 8}{\ln 3} \approx 1.89}$ Check your answer on the calculator.

Ex 12: Solve the equation $8 + 2^{5x+4} = 35$.

Isolate the exponential term. $2^{5x+4} = 27$

Take the common logarithm of both sides. $\log 2^{5x+4} = \log 27$

Use the power property of logs. $(5x+4)\log 2 = \log 27$

$$5x + 4 = \frac{\log 27}{\log 2}$$

Solve for *x*. $5x = \frac{\log 27}{\log 2} - 4$

$$x = \boxed{\frac{1}{5}\left(\frac{\log 27}{\log 2} - 4\right) \approx -0.151}$$

Check the solution.

Because of the complexity of the solution, a good way to check would be to graph both sides of the original equation on the graphing calculator and find the point of intersection.

Solving Exponential Inequalities:

To solve exponential inequalities, use the same methods for solving exponential equations.

Ex 13: Solve the inequality $5^{2x+3} \le 125$

Rewrite both sides with a base of 5. (Note: $5^3 = 125$) $5^{2x+3} \le 5^3$

Equate the exponents and solve for *x*. $2x + 3 \le 3$

$\boxed{x \le 0}$

Ex 14: Solve the inequality $e^{3x} > 32$

Take the natural log of both sides. $\ln e^{3x} > \ln 32$

Use the inverse property of logs. $3x > \ln 32$

Solve for *x*. $\frac{3x}{3} > \frac{\ln 32}{3}$

$\boxed{x > \frac{\ln 32}{3}}$

SAMPLE EXAM QUESTIONS

1. Which equation is equivalent to $3^4 = 81$?

A. $\log_4 3 = 81$

B. $\log_3 81 = 4$

C. $\frac{\log_{10} 4}{\log_{10} 3} = 81$

D. $\log_{81} 3 = 4$

Ans: B

2. Solve the equation $e^n = 10$ for *n*.

A. $n = \ln \frac{10}{e}$

B. $n = \ln 10$

C. $n = \ln(10 - e)$

D. $n = \frac{\ln 10}{e}$

Ans: B

3. What is the value of *x* if $3^{2x} = 9^{3x-1}$?

A. $\frac{1}{4}$

B. $\frac{1}{2}$

C. 1

D. 3

Ans: B

Unit 7.8 To use base e to solve and graph exponential and logarithmic functions.

Solving a Logarithmic Equation

Method 1: If the logarithms on both sides have the same base, use the property that states: $\log_b x = \log_b y$ if and only if $x = y$.

Ex 15: Solve the equation $\log_4(x+3) = \log_4(8x+17)$.

Both sides have a logarithm with base 4, so we can equate the arguments. $x+3 = 8x+17$

Solve for *x*. $-7x = 14$

$\boxed{x = -2}$

Check the solution in the original equation. $\log_4(-2+3) = \log_4(8(-2)+17)$

$\log_4 1 = \log_4 1$

Ex 16: Solve the equation $\ln(3x-2) + \ln(x-1) = 2\ln x$.

Use the product and power properties $\ln[(3x-2)(x-1)] = \ln x^2$

Equate the arguments as they both have base e $(3x-2)(x-1) = x^2$

Isolate *x*. $3x^2 - 5x + 2 = x^2 \Rightarrow 2x^2 - 5x + 2 = 0$

Factor x. $(2x-1)(x-2) = 0$

Solve for x. $\boxed{x = 2 \text{ or } x = \frac{1}{2}}$

Check the solution in the original equation.

$\ln\left(3(2)-2\right)+\ln(2-1)=2\ln 2$
$\ln(4)+\ln(1)=2\ln 2$ True
$\ln(4\cdot 1)=\ln 2^2 \Rightarrow \ln 4=\ln 4$

Since the domain of the natural logarithm must be positive real numbers, if we substitute $\frac{1}{2}$ into the original equation, we will get $\ln\left(-\frac{1}{2}\right)$, making our second solution an extraneous solution. The only solution is: $\boxed{x=2}$

Method 2: Rewriting the equation in exponential form.

Ex 17: Solve the equation $\log_5\left(x-4\right)=2$.

Rewrite in exponential form. $5^2=x-4$

Solve for *x*.
$25=x-4$
$\boxed{x=29}$

Check the solution in the original equation.
$\log_5\left(29-4\right)=2$
$\log_5 25=2$ True

<u>Simplifying Before Solving in a Logarithmic Equation</u>

Ex 18: Solve the equation $\log_6\left(x+5\right)+\log_6 x=2$.

Rewrite the left side of the equation using the product property of logs. $\log_6\left(x^2+5x\right)=2$

Write the equation in exponential form. $6^2=x^2+5x$

Solve for *x*.
$36=x^2+5x$
$x^2+5x-36=0$
$\left(x+9\right)\left(x-4\right)=0$
$x=-9,4$

Check the solutions in the original equation.

$x=-9$: $\log_6\left(-9+5\right)+\log_6\left(-9\right)\neq 2$ Not possible to take the log of a negative number!

$x=4$: $\log_6\left(4+5\right)+\log_6 4=\log_6 9+\log_6 4=\log_6 36=2$ True

Solution: $\boxed{x=4}$ (Note: -9 is an extraneous solution)

Ex 19: Solve the equation $3\ln(x-3)+4=5$.

Simplify $\ln(x-3)=\frac{1}{3}$

Rewrite in exponential form $e^{\frac{1}{3}}=x-3$

Solve for *x*. $e^{\frac{1}{3}}+3=x$

Evaluate x. $1.3956+3\approx x$

$\boxed{x\approx 4.3956}$

Check the solution in the original equation. $3\ln\left(e^{\frac{1}{3}}+3-3\right)+4=5 \Rightarrow 3\ln\left(e^{\frac{1}{3}}\right)=1$

$\ln\left(e^{\frac{1}{3}}\right)^3=1 \Rightarrow \ln e=1$ True

QOD: Explain why logarithmic equations can have extraneous solutions.

SAMPLE EXAM QUESTIONS

1. What is the solution of the equation $\log_2(2k-3)=5$?

A. $k=5$

B. $k=\frac{13}{2}$

C. $k=14$

D. $k=\frac{35}{2}$

Ans: D

2. **A biologist studying the relationship between the brain weight and body weight in mammals uses the formula:**

$$\ln(w_{body}) = \ln(w_{brain}) - 669$$

Where w_{body} **=body weight in grams and** w_{brain} **=brain weight in grams. What is the formula for the body weight?**

A. $w_{body} = (w_{brain})(e^{-669})$

B. $w_{body} = (w_{brain}) - (e^{-669})$

C. $w_{body} = e^{(w_{brain})(e^{-669})}$

D. $w_{body} = -669(w_{brain})$

Ans: B

3. **If** $\log_4\{\log_2[\log_3(3x)]\} = \frac{1}{2}$**, then what is** x **?**

A. 81
B. 48
C. 27
D. 9

Ans: C

4. **Which equation has the same solution as** $\log_4(x+7) = 5$ **?**

A. $4^{x+7} = 5$

B. $5^{x+7} = 5$

C. $5^4 = x+7$

D. $4^5 = x+7$

Ans: D

Unit 7.10 To use exponential and logarithmic functions to model data.

Writing an Exponential Function: two points determine a unique exponential function $y = ab^x$

Ex 20: Write an exponential function $y = ab^x$ whose graph passes through $(1,7)$ and $(3,63)$.

Substitute each ordered pair in for *x* and *y* in the equation $y = ab^x$: $7 = ab^1$ $63 = ab^3$

To eliminate *a*, divide the two equations. Put the highest power of *b* on top: $\frac{63 = ab^3}{7 = ab^1} \Rightarrow 9 = b^2$

Solve for *b*: (Note: In an exponential function, *b* cannot be negative.) $b = 3$

Substitute this value of b into one of the original equations and solve for *a*: $7 = a(3)^1 \Rightarrow a = \frac{7}{3}$

Exponential Function: $y = \frac{7}{3} \cdot 3^x$

Ex 21: A certain culture of bacteria will grow from 250 to 2000 bacteria in 1.5 hours. Find the constant *k* for the growth formula. Use $y = ae^{kt}$.

Substitute the values into the exponential growth formula: $2000 = 250e^{k(1.5)}$

Divide each side of the equation by 250: $8 = e^{k(1.5)}$

Use the property of equality: $\ln 8 = \ln e^{k(1.5)}$

Use the inverse property of natural logarithms: $\ln 8 = 1.5k$

Solve for k: $2.0794 = 1.5k \Rightarrow k \approx 1.3863$

Ex 22: The population of a city 10 years ago was 150,000. Since then, the population has increased at a steady rate each year. If the population is currently 185,000. Write an exponential function that could be used to model the population and find the population in 25 years.

Use the exponential growth model: $y = A(1+r)^t$

Substitute the known values: $185000 = 150000(1+r)^{10}$

Divide by 150000: $1.23333 = (1+r)^{10}$

Solve for r: $\left(1.23333\right)^{\frac{1}{10}} = \left[(1+r)^{10}\right]^{\frac{1}{10}} \Rightarrow 1.0212 \approx 1 + r$

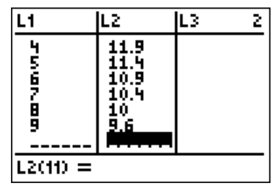
Find for *time* = 25 years: $y = 150000(1.0212)^{25} \Rightarrow \boxed{\approx 253{,}432}$

Using Exponential Models

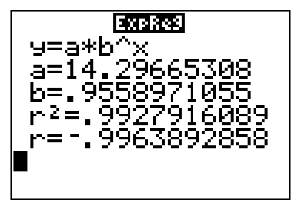
Ex 23: Find an exponential model to fit the data. Use the model to estimate *y* when *x* is 15.

x	0	1	2	3	4	5	6	7	8	9
y	14.7	13.5	12.9	12.4	11.9	11.4	10.9	10.4	10.0	9.6

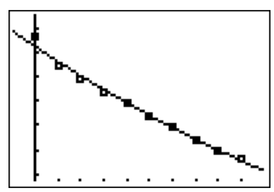
STAT, EDIT: Enter the *x*-values into L1 and the *y*-values into L2.



On the Home screen, go to STAT ,CALC and choose option 0, ExpReg. Then type Y1 (found in the VARS menu), and press Enter. This will calculate the exponential regression model and store it in Y1.



Graph the scatter plot along with the exponential regression equation to see if the model fits the data. Use ZoomStat.



Exponential Model: $\boxed{y = 14.297 \cdot 0.956^{x}}$

Interpret the coefficient of determination (r^2) that was given when the model was created and determine if the predictions will be accurate. The coefficient of determination, $r^2 \approx 0.99279$, tells us that the model is going to be very accurate when making predictions (the closer to 1, the better the estimator). 99.279% of the variation in the y-variable can be explained by the exponential correlation in the x-variable.

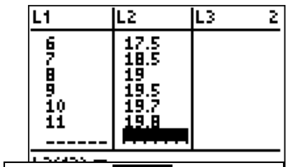
When *x* is 15, $y = 14.297 \cdot 0.956^{15} \approx \boxed{7.28}$

Using Logarithmic Models

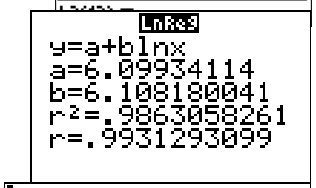
Ex 24: The data below show the average growth rates of 12 Weeping Higan cherry trees planted in Washington, D.C. At the time of planting, the trees were one year old and were all 6 feet in height.

Age of Tree (in years)	1	2	3	4	5	6	7	8	9	10	11
Height (in feet)	6	9.5	13	15	16.5	17.5	18.5	19	19.5	19.7	19.8

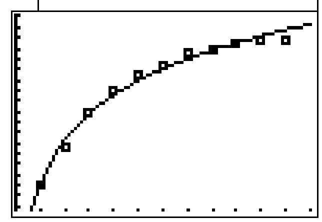
Enter the ages into L1 and the heights into L2 (Go to STAT – Edit)



On the Home screen, go to STAT – CALC and choose option 9, LnrReg. Then type Y1 (found in the VARS menu), and press Enter. This will calculate the natural logarithmic regression model and store it in Y1.



Graph the scatter plot along with the logarithmic regression equation to see if the model fits the data. Use ZoomStat.



Logarithmic Model: $\boxed{y = 6.099 + 6.108\ln x}$

Is the model a good fit?

On the scatterplot, the model seems to fit the data points very well. Also, the coefficient of determination, r^2 is 0.9863 which means that 98.6% of the total variation in height can be explained by the relationship between age and height. Yes, it is a very good fit.

Interpolate: What was the average height of the trees at one and one-half years of age?

$$y = 6.099 + 6.108\ln(1.5) \approx \boxed{8.576 \text{ ft}}$$

Extrapolate: What is the predicted average height of the trees at 20 years of age? Is this prediction realistic?

$y = 6.099 + 6.108\ln(20) \approx \boxed{24.397 \text{ ft}}$ Extrapolations far from the stated data are often inaccurate and unreliable. Nine years away from the data set is a large span of time and the reading of 24.4 ft may be "high" based upon the observed leveling nature of the graph, which appears to be levelling off around 20 feet.

QOD: Which values are constant in an exponential function?

SAMPLE EXAM QUESTIONS

1. **In 1950, the city of San Jose had a population of 95,000. Since then, on average, it grows 4% per year. What is the best formula to model San Jose's growth?**

   A. $95{,}000(1.04)^t$

   B. $95{,}000(0.96)^t$

   C. $-.04t + 95{,}000$

   D. $.04t + 95{,}000$

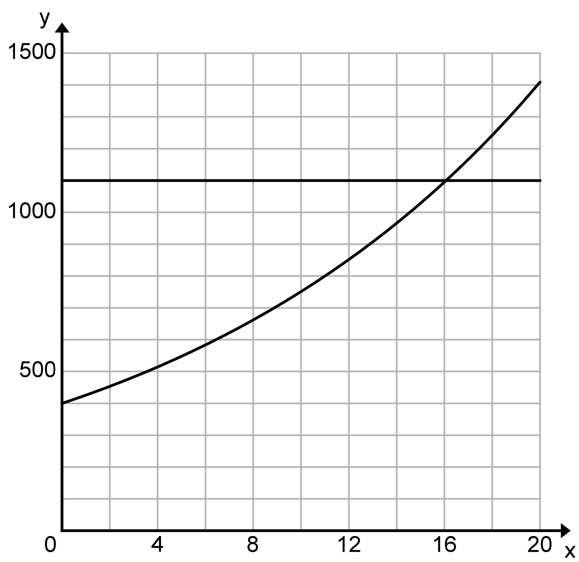
Ans: A

2. **Sarai bought $400 of Las Vegas Cellular stock in January 2005. The value of the stock is expected to increase by 6.5% per year.**

   a) Write a model to describe Sarai's investment.

   $A(x) = P(x)(1+r)^t = 400(1.065)^t$, ***t* is the number of years**

   b) Use the graph to show when Sarai's investment will reach $1100?



Sarai's investment will reach $1100 in just over 16 years.

3. **The loudness of sound is measured on a logarithmic scale according to the formula $L = 10\log(\frac{I}{I_0})$, where L is the loudness of sound in decibels (db), I is the intensity of sound, and I_0 is the intensity of the softest audible sound.**

a) Find the loudness in decibels of each sound listed in the table.

Sound	Intensity
Jet taking off	$10^{15} I_0$
Jackhammer	$10^{12} I_0$
Hairdryer	$10^{7} I_0$
Whisper	$10^{3} I_0$
Leaves rustling	$10^{2} I_0$
Softest audible sound	I_0

According to the logarithmic scale formula $L = 10\log\left(\frac{I}{I_0}\right)$ by replacing the intensity *I* by their formula in terms of I_0 we obtain:

Sound	Intensity
Jet taking off	150
Jackhammer	120
Hairdryer	70
Whisper	30
Leaves rustling	20
Softest audible sound	0

b) The sound at a rock concert is found to have a loudness of 110 decibels. Where should this sound be placed in the table in order to keep the sound intensities in order from least to greatest?

It should be placed between the jackhammer and the hairdryer.

Here is the explanation: $110 = 10\log\left(\frac{I}{I_0}\right)$, so $\frac{I}{I_0} = 10^{11}$

c) A decibel is $\frac{1}{10}$ of a *bel.* Is a jet plane louder than a sound that measures 20 *bels*? Explain.

The loudness of a jet plane is 150 db=15 bels, so the jet is not louder that a sound that measures 20 bels.

4. Aaron invested \$4000 in an account that paid an interest rate r compounded continuously. After 10 years he has \$5809.81. The compound interest formula is $A = Pe^{rt}$, where P is the principal (the initial investment), A is the total amount of money (principal plus interest), r is the annual interest rate, and t is the time in years.

a) **Divide both sides of the formula by P and then use logarithms to rewrite the formula without an exponent. Show your work.**

$$A = Pe^{rt}$$
$$\frac{A}{P} = e^{rt}$$
$$\ln\frac{A}{P} = \ln e^{rt}$$
$$\ln\frac{A}{P} = rt$$

b) **Using your answer for part (a) as a starting point, solve the compound interest formula for the interest rate r.**

$$r = \frac{1}{t}\ln\frac{A}{P}$$

c) **Use your equation from part (a) to determine the interest rate.**

$$r = \frac{1}{t}\ln\frac{A}{P} = \frac{1}{10}\ln\frac{5809.81}{4000} = \frac{1}{10}\cdot 0.37325350697 \approx 0.037$$

Which gives us an interest rate of 3.7%