Rational Functions

Math Background

Previously, you

- Simplified linear, quadratic, radical and polynomial functions
- Performed arithmetic operations with linear, quadratic, radical and polynomial functions
- Identified the domain, range and x-intercepts of real-life functions
- Graphed linear, quadratic and polynomial functions
- Composed linear, quadratic and radical functions
- Created functions to represent a real life situation
- Solved linear, quadratic, radical and polynomial functions
- Transformed parent functions of linear, quadratic, radical and polynomial functions

In this unit you will

- Simplify rational expressions
- Perform arithmetic operations with rational expressions
- Transform rational functions
- Solve problems involving rational equations and inequalities
- Compose rational functions with other functions
- Create rational functions to represent real life situations

You can use the skills in this unit to

- Use the structure of an expression to identify ways to rewrite it.
- Use long division to perform partial fraction decomposition.
- Interpret the domain and its restrictions of a real-life function.
- Identify extraneous solutions of a rational equation.
- Describe how a rational function graph is related to its parent function.
- Model and solve real-world problems with rational functions
- Interpret the composition of functions as applied to real world problems.

Vocabulary

- **Composition of Functions –** The act of combining two mathematical functions.
- **Domain of a rational expression** – the set of all real numbers except the value(s) of the variable that result in division by zero when substituted into the expression.
- **Extraneous Solutions** – A root of a transformed equation that is not a root of the original equation because it was excluded from the domain of the original equation.
- **Horizontal Asymptote** – A horizontal line that the graph of a function approaches as x tends to plus or minus infinity. It describes the function's end behavior.
- **Partial Fraction Decomposition –** the operation that consists in expressing the fraction as a sum of a polynomial and one or several fractions with a simpler denominator.
- **Rational expression –** a fraction with a polynomial in the numerator and a nonzero polynomial in the denominator. Also known as an algebraic fraction.

- **Simplify a rational expression** – Write the fraction so there are no common factors other than 1 or -1.
- **Undefined** – An expression in mathematics which does not have meaning and so which is not assigned an interpretation. For example, division by zero is undefined in the field of real numbers.
- **Vertical asymptote –** A vertical line that the curve approaches more and more closely but never touches as the curve goes off to positive or negative infinity. The vertical lines correspond to the zeroes of the denominator of the rational function.

Essential Questions

- How do you find sums, differences, products, and quotients of rational expressions?
- What knowledge and skills are required to rewrite simple rational expressions?
- How do I solve a rational equation? How are extraneous solutions generated from a rational equation?
- How can the composition of two functions be used to represent real life applications?

Overall Big Ideas

The rules for addition, subtraction, multiplication and division of rational expressions are analogous to those for rational numbers.

Rational expressions can be rewritten using properties of fractions and elementary numerical algorithms.

We solve rational equations by transforming the equation into a simpler form to solve. However, this can produce solutions that do not exist in the original domain.

Real world applications can be modeled as a composition of two functions where the input is generally a function of time.

Skill

To transform the graph of $y=\frac{1}{x}$.

To simplify rational expressions.

To add, subtract, multiply, and divide rational expressions.

To solve rational equations and inequalities.

To compose a rational function with other functions.

To create and apply a rational function to a real life situation. (e.g. concentrations from chemistry)

Related Standards

F.IF.B.5-2

Relate the domain of any function to its graph and, where applicable, to the quantitative relationship it describes. For example, if the function *h(n)* gives the number of person-hours it takes to assemble *n* engines in a factory, then the positive integers would be an appropriate domain for the function. *(Modeling Standard)

F.BF.B.3-2

Identify the effect on the graph of replacing $f(x)$ by $f(x) + \text{k}$, $k\,f(x)$, $f(\text{k}x)$, and $f(x + \text{k})$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. Include recognizing even and odd functions from their graphs and algebraic expressions for them. Include simple radical, rational, and exponential functions, note the effect of multiple transformations on a single graph, and emphasize common effects of transformations across function types.

A.SSE.A.2-2

Use the structure of an expression, including polynomial and rational expressions, to identify ways to rewrite it. For example, see $x^4 - y^4$ as $\left(x^2\right)^2 - \left(y^2\right)^2$, thus recognizing it as a difference of squares that can be factored as $\left(x^2 - y^2\right)\left(x^2 + y^2\right)$.

A.APR.D.6

Rewrite simple rational expressions in different forms; write $\frac{a(x)}{b(x)}$ in the form $q(x) + \frac{r(x)}{b(x)}$, where $a(x), b(x), q(x),$ and $r(x)$ are polynomials with the degree of $r(x)$ less than the degree of $b(x)$, using inspection, long division, or, for the more complicated examples, a computer algebra system.

A.APR.D.7

Understand that rational expressions form a system analogous to the rational numbers, closed under addition, subtraction, multiplication, and division by a nonzero rational expression; add, subtract, multiply, and divide rational expressions.

A.REI.A.2

Solve simple rational and radical equations in one variable, and give examples showing how extraneous solutions may arise.

A.CED.A.1-2

Create equations and inequalities in one variable and use them to solve problems. Include equations arising from all types of functions, including simple rational and radical functions. *(Modeling Standard)

F.BF.A.1c

Compose functions. For example, if *T(y)* is the temperature in the atmosphere as a function of height, and *h(t)* is the height of a weather balloon as a function of time, then *T(h(t))* is the temperature at the location of the weather balloon as a function of time. *(Modeling Standard)

A.CED.A.2-2

Create equations in two or more variables to represent relationships between quantities and graph equations on coordinate axes with labels and scales. Use all types of equations. *(Modeling Standard)

A.CED.A.3-2

Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or non-viable options in a modeling context. Use all types of equations. For example, represent inequalities describing nutritional and cost constraints on combinations of different foods. *(Modeling Standard)

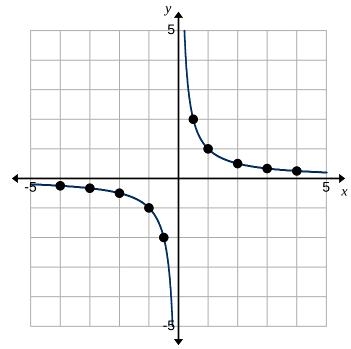
Notes, Examples, and Exam Questions

Unit 6.1: Graphing Rational Functions

Rational Function: a function of the form $f(x)=\dfrac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomial functions.

Ex 1: Use a table of values to graph the function $y=\dfrac{1}{x}$.

x	-4	-3	-2	-1	$-\frac{1}{2}$	$\frac{1}{2}$	1	2	3	4
y	$-\frac{1}{4}$	$-\frac{1}{3}$	$-\frac{1}{2}$	-1	-2	2	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$



Note: The graph of $y=\dfrac{1}{x}$ has two branches. The *x*-axis $(y=0)$ is a horizontal asymptote, and the *y*-axis $(x=0)$ is a vertical asymptote. Domain and Range: All real numbers not equal to zero.

A **horizontal asymptote** is a horizontal line that the graph of a function approaches as *x* tends to plus or minus infinity. As *x* grows very large or very small, the function approaches this value. A **vertical asymptote** is a vertical line near which the function grows to infinity or negative infinity. It is a place where the function is undefined.

Exploration: Graph each of the functions on the graphing calculator. Describe how the graph compares to the graph of $y=\dfrac{1}{x}$. Include horizontal and vertical asymptotes and domain and range in your answer.

1. $y=\dfrac{2}{x}$
2. $y=-\dfrac{2}{x}$
3. $y=\dfrac{1}{x-3}$
4. $y=\dfrac{1}{x+5}$
5. $y=\dfrac{1}{x}-4$
6. $y=\dfrac{1}{x}+3$

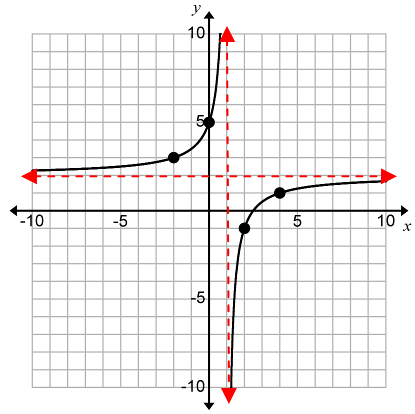
Hyperbola: the graph of the function: $y=\dfrac{a}{x-h}+k$

Ex: From the exploration above, describe the asymptotes, domain and range, and the effects of *a* on the general equation of a hyperbola $y=\dfrac{a}{x-h}+k$. Horizontal Asymptote: $y=k$ Vertical Asymptote: $x=h$

Domain: All real numbers not equal to *h*. Range: All real numbers not equal to *k*.

As $|a|$ gets bigger, the branches move farther away from the origin. If $a>0$, the branches are in the first and third quadrants. If $a<0$, the branches are in the second and fourth quadrants.

Ex 2: Describe the transformations of the graph of $y = -\dfrac{3}{x-1} + 2$



Horizontal Asymptote: $y = 2$ Vertical Asymptote: $x = 1$

The graph would be reflected over the x-axis ($a < 0$), shifted up 2

from its parent function $y = \dfrac{1}{x}$ and shifted right 1 unit.

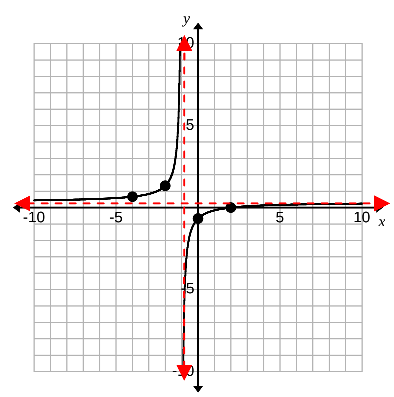
<u>The graph of the function is at the right. Note the asymptotes are the dashed red lines.</u>

<u>More Hyperbolas:</u> graphing in the form $y = \dfrac{ax+b}{cx+d}$

Horizontal Asymptote: $y = \dfrac{a}{c}$ Vertical Asymptote: $cx + d = 0$

$x = -\dfrac{d}{c}$

Ex 3: Describe the transformations of $y = \dfrac{x-2}{3x+3}$.



Horizontal Asymptote: $y = \dfrac{1}{3}$ Vertical Asymptote: $3x + 3 = 0$

$x = -1$

Using long division: $3x+3 \overline{)\, x-2}$ with quotient $\dfrac{1}{3}$; subtract $\underline{x+1}$; remainder -3

The quotient is: $\dfrac{1}{3} - \dfrac{3}{3x+3}$.

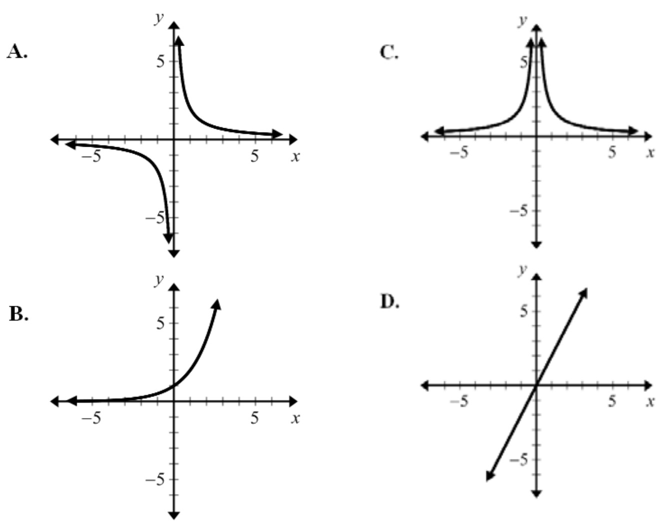
Rewriting this, we get: $-\dfrac{\cancel{3}}{\cancel{3}(x+1)} + \dfrac{1}{3} = \boxed{-\dfrac{1}{x+1} + \dfrac{1}{3}}$. Using transformations in this form, it tells us to reflect the graph, go up by $\dfrac{1}{3}$ and go left one unit. By dividing the rational function so that the numerator is one less degree than the denominator, we can use transformations to draw the graph of our rational function. The graph is shown above.

<u>Summarize the Graphs of Rational Functions:</u> $f(x) = \dfrac{p(x)}{q(x)}$

- x-intercepts: the zeros of $p(x)$
- Vertical Asymptotes: occur at the zeros of $q(x)$
- Horizontal Asymptote: describes the END BEHAVIOR of the graph (as $x \to -\infty$ and $x \to \infty$)
  - If the degree of $p(x)$ is less than the degree of $q(x)$, then $y = 0$ is a horizontal asymptote.
  - If the degree of $p(x)$ is equal to the degree of $q(x)$, then y = (the ratio of the leading coefficients) is a horizontal asymptote.
  - If the degree of $p(x)$ is greater than the degree of $q(x)$, then the graph has no horizontal asymptote. It will have a slant asymptote.

SAMPLE EXAM QUESTIONS

1. What is the graph of $f(x) = \dfrac{2}{x}$?



Ans: A

2. What are the asymptotes of the function: $f(x) = \dfrac{-x^2}{x^2 - 25}$?

A. $y = 0, x = -5$

B. $y = -1, x = -5, x = 5$

C. $y = -1, x = -5$

D. $y = 0, x = -5, x = 5$

Ans: B

3. Which set contains all the real numbers that are not part of the domain of $f(x)=\dfrac{x+4}{x^2+4x-32}$ **?**

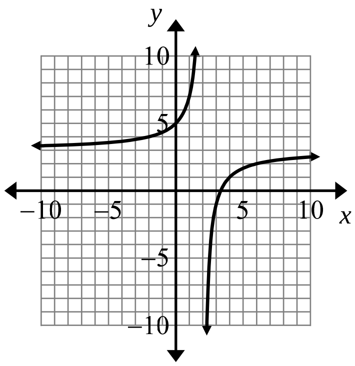
A. {8} **C.** {-4}

B. {-4, 8} **D.** {-8, 4}

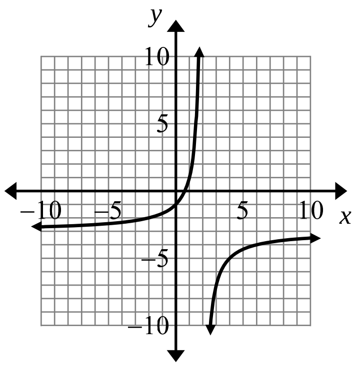
Ans: D

4. Which is the graph of $f(x)=\dfrac{-4}{x+2}-3$ **?**

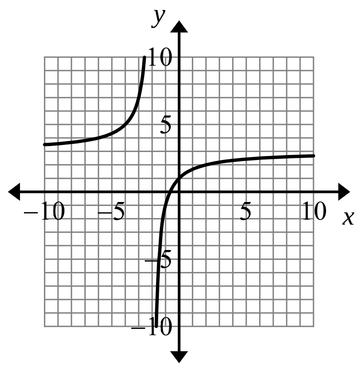
A.



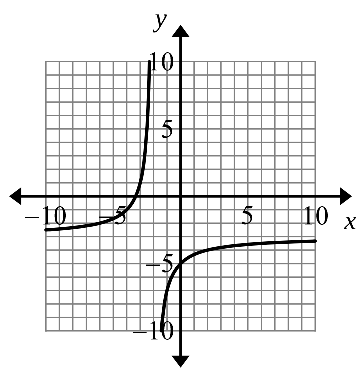
B.



C.



D.



Ans: D

5. **What are the asymptotes of the function** $f\left(x\right)=\dfrac{-x^2}{x^2+x-30}$ **?**

A. $x=-6,\ x=5,\ y=-1$

B. $x=-6,\ x=5,\ y=1$

C. $x=-5,\ x=6,\ y=-1$

D. $x=-5,\ x=6,\ y=1$

Ans: A

6. **Which intervals correctly define the domain of** $f(x)=\dfrac{1}{x+4}-2$

A. $(-\infty,4)$ and $(4,\infty)$

B. $(-\infty,-4)$ and $(4,\infty)$

C. $(-\infty,-4)$ and $(-4,\infty)$

D. $(-\infty,-4)$ and $(-2,\infty)$

Ans: B

7. **Which statement is true for the function** $f(x)=\dfrac{1}{x+4}$ **?**

A. 4 is not in the range of the function.

B. 4 is not in the domain of the function.

C. -4 is not in the range of the function.

D. -4 is not in the domain of the function.

Ans: D

Unit 6.2: Simplify Rational Expressions

Recall: When simplifying fractions, we divide out any common factors in the numerator and denominator

Ex: Simplify $\dfrac{16}{20}$.

The numerator and denominator have a common factor of 4. They can be rewritten. $\dfrac{4\cdot 4}{4\cdot 5}$

Now we can divide out the common factor of 4. The remaining numerator and denominator have no common factors (other than 1), so the fraction is now simplified. $\dfrac{\not{4}\cdot 4}{\not{4}\cdot 5}=\boxed{\dfrac{4}{5}}$

Simplified Form of a Rational Expression: a rational expression in which the numerator and denominator have no common factors other than 1

Simplifying a Rational Expression

1. Factor the numerator and denominator if you can
2. Divide out any common factors

Ex 4: Simplify the expression $\dfrac{x^2-5x-6}{x^2-1}$.

Factor. $$\frac{(x-6)(x+1)}{(x+1)(x-1)}$$

Divide out common factors. $$=\frac{(x-6)\cancel{(x+1)}}{\cancel{(x+1)}(x-1)}=\boxed{\frac{x-6}{x-1}}$$

Ex 5: Simplify the expression $\dfrac{x^2-5x-6}{x^2-1}$.

Factor. $$\frac{(x-6)(x+1)}{(x+1)(x-1)}$$

Divide out common factors. $$=\frac{(x-6)\cancel{(x+1)}}{\cancel{(x+1)}(x-1)}=\boxed{\frac{x-6}{x-1}}$$

Ex 6: Simplify the expression $\dfrac{9x^2+81x}{x^3+8x^2-9x}$.

Factor. $$\frac{9x(x+9)}{x(x+9)(x-1)}$$

Divide out common factors. $$=\frac{9\cancel{x}\cancel{(x+9)}}{\cancel{x}\cancel{(x+9)}(x-1)}=\boxed{\frac{9}{x-1}}$$

Unit 6.3: Multiply and Divide Rational Expressions

Recall: When multiplying fractions, simplify any common factors in the numerators and denominators, then multiply the numerators and multiply the denominators.

Ex: Multiply $\frac{25}{6}\cdot\frac{42}{50}$. Divide out common factors. $\frac{25}{7}\cdot\frac{42}{50}=\frac{\cancel{25}}{\cancel{7}}\cdot\frac{\cancel{2}\cdot 3\cdot\cancel{7}}{\cancel{25}\cdot\cancel{2}}=\boxed{3}$

Multiplying Rational Expressions

1. Factor numerators and denominators (if necessary).
2. Divide out common factors.
3. Multiply numerators and denominators.

Ex 7: Multiply $\frac{3x-27x^3}{3x^2-2x-1}\cdot\frac{3x^2-4x+1}{3x}$.

Factor. $\frac{3x\left(1-9x^2\right)}{(3x+1)(x-1)}\cdot\frac{(3x-1)(x-1)}{3x}=\frac{3x(1-3x)(1+3x)}{(3x+1)(x-1)}\cdot\frac{(3x-1)(x-1)}{3x}$

Divide out common factors. $=\frac{\cancel{3x}(1-3x)\cancel{(1+3x)}}{\cancel{(3x+1)}\cancel{(x-1)}}\cdot\frac{(3x-1)\cancel{(x-1)}}{\cancel{3x}}$

Multiply. $=(1-3x)(3x-1)=-(3x-1)(3x-1)=\boxed{-(3x-1)^2}$

Ex 8: Find the product. $\frac{x+2}{27x^3+8}\cdot\left(9x^2-6x+4\right)$

Factor. $\frac{(x+2)}{(3x+2)\left(9x^2-6x+4\right)}\cdot\frac{\left(9x^2-6x+4\right)}{1}$

Divide out common factors. $=\frac{(x+2)}{(3x+2)\cancel{\left(9x^2-6x+4\right)}}\cdot\frac{\cancel{\left(9x^2-6x+4\right)}}{1}=\boxed{\frac{x+2}{3x+2}}$

Multiplying Rational Expressions with Monomials

Use the properties of exponents to multiply numerators and denominators, then divide.

Ex 9: Multiply $\frac{6x^2y^3}{2x^2y^2}\cdot\frac{10x^3y^4}{18y^2}$.

Use the properties of exponents and simplify. $\frac{6x^2y^3}{2x^2y^2}\cdot\frac{10x^3y^4}{18y^2}=\frac{60x^5y^7}{36x^2y^4}=\boxed{\frac{5x^3y^3}{3}}$

Recall: When dividing fractions, multiply by the reciprocal.

Ex: Find the quotient. $\frac{5}{36}\div\frac{10}{20}$ $=\frac{5}{36}\cdot\frac{20}{10}=\frac{5}{36}\cdot\frac{2}{1}=\boxed{\frac{5}{18}}$

Dividing Rational Expressions

Multiply the first expression by the reciprocal of the second expression and simplify.

Ex 10: Divide. $\frac{3}{4x-8}\div\frac{x^2+3x}{x^2+x-6}$

Multiply by the reciprocal. $\frac{3}{4x-8}\cdot\frac{x^2+x-6}{x^2+3x}$

Factor and simplify. $=\frac{3}{4(x-2)}\cdot\frac{(x+3)(x-2)}{x(x+3)}=\boxed{\frac{3}{4x}}$

Ex 11: Find the quotient of $\frac{8x^2+10x-3}{4x^2}$ and $4x^2-x$.

Multiply by the reciprocal. $\frac{8x^2+10x-3}{4x^2}\cdot\frac{1}{4x^2-x}$

Factor and simplify. $\frac{(4x-1)(2x+3)}{4x^2}\cdot\frac{1}{x(4x-1)}=\boxed{\frac{2x+3}{4x^3}}$

Ex 12: Simplify. $\frac{x}{x-2}\cdot(2x+3)\div\frac{4x^2-9}{x-2}$

Multiply by the reciprocal. $\frac{x(2x+3)}{x-2}\cdot\frac{x-2}{4x^2-9}$

Factor and simplify. $\frac{x(2x+3)}{x-2}\cdot\frac{x-2}{(2x+3)(2x-3)}=\boxed{\frac{x}{2x-3}}$

QOD: What is the factoring pattern for a sum/difference of two cubes?

SAMPLE EXAM QUESTIONS

1. Simplify the expression. $\dfrac{3x^3-27x}{x^3-4x^2+3x}$

A. $\dfrac{3(x+3)}{x-1}$ **C.** $\dfrac{3(x+3)(x-3)}{x^2-4x+3}$

B. $\dfrac{3(x-3)}{x+1}$ **D.** $\dfrac{x^2+3x+9}{x(x-1)}$

Ans: A

2. Simplify the expression: $\dfrac{\mathbf{x^2-3x-4}}{\mathbf{x^2-16}}\cdot\dfrac{\mathbf{x^2+4x}}{\mathbf{x^2-x-2}}$

A. 1 **C.** $\dfrac{1}{x-4}$

B. $\dfrac{x-2}{x+4}$ **D.** $\dfrac{x}{x-2}$

Ans: D

3. Multiply. Simplify your answer.

$\dfrac{8x^4y^2}{3z^3}\bullet\dfrac{9xy^2z^6}{4y^4}$

A. $6x^5y^8z^9$ **C.** $6x^5z^3$

B. $6x^4yz^2$ **D.** $\dfrac{3}{2}x^3y^2z$

Ans: C

4. Which expression represents the quotient? $\dfrac{8x^6z^4+4x^4z^2}{4x^2z}$

A. $2x^4z^3+x^2z$ **C.** $2x^3z^4+x^2z^2$

B. $4x^4z^3+3x^2z$ **D.** $4x^3z^4+3x^2z^2$

Ans: A

5. Which expression is equivalent to $\dfrac{y^{\frac{1}{2}}}{8x^{\frac{4}{3}}} \div \dfrac{x^{\frac{1}{3}}y^{\frac{5}{2}}}{6}$ **for all** $x, y \neq 0$**?**

A. $\dfrac{1}{2x^{\frac{5}{6}}y^2}$

B. $\dfrac{3y^2}{4x^{\frac{4}{9}}}$

C. $\dfrac{3}{4x^{\frac{5}{3}}y^2}$

D. $\dfrac{y^3}{48x^4}$

Ans: C

Unit 6.3: Add and Subtract Rational Expressions

Recall: To add or subtract fractions with like denominators, add or subtract the numerators and keep the common denominator.

Ex: Find the difference. $\dfrac{2}{15} - \dfrac{8}{15}$ $\quad = \dfrac{2-8}{15} = \dfrac{-6}{15} = \boxed{-\dfrac{2}{5}}$

Adding and Subtracting Rational Expressions with Like Denominators

Add or subtract the numerators. Keep the common denominator. Simplify the sum or difference.

Ex 13: Subtract. $\dfrac{3}{2x} - \dfrac{7}{2x}$ $\quad = \dfrac{3-7}{2x} = \dfrac{-4}{2x} = \boxed{-\dfrac{2}{x}}$

Ex 14: Add. $\dfrac{3x}{x+2} + \dfrac{6}{x+2}$ $\quad = \dfrac{3x+6}{x+2} = \dfrac{3(x+2)}{(x+2)} = \boxed{3}$

Recall: To add or subtract fractions with unlike denominators, find the least common denominator (LCD) and rewrite each fraction with the common denominator. Then add or subtract the numerators.

Ex: Add. $\dfrac{2}{15} + \dfrac{3}{20}$

Note: To find the LCD, it is helpful to write the denominators in factored form. $15 = 3 \cdot 5,\ 20 = 2 \cdot 2 \cdot 5$

$\text{LCD} = 2 \cdot 2 \cdot 3 \cdot 5 = 60$

Adding and Subtracting Rational Expressions with Unlike Denominators

1. Find the least common denominator (in factored form).
2. Rewrite each fraction with the common denominator.
3. Add or subtract the numerators, keep the common denominator, and simplify.

Ex 15: Find the sum. $\dfrac{4}{3x^3}+\dfrac{x}{6x^3+3x^2}$

Factor and find the LCD. $\dfrac{4}{3x^3}+\dfrac{x}{3x^2\left(2x+1\right)}$ LCD = $3x^3\left(2x+1\right)$

Rewrite each fraction with the LCD. $\dfrac{4}{3x^3}\cdot\dfrac{\left(2x+1\right)}{\left(2x+1\right)}+\dfrac{x}{3x^2\left(2x+1\right)}\cdot\dfrac{x}{x}=\dfrac{8x+4}{3x^3\left(2x+1\right)}+\dfrac{x^2}{3x^3\left(2x+1\right)}$

Add the fractions. $=\boxed{\dfrac{x^2+8x+4}{3x^3\left(2x+1\right)}}$ Note: Our answer cannot be simplified because the numerator cannot be factored. We can leave the denominator in factored form.

Ex 16: Subtract: $\dfrac{x+1}{x^2+6x+9}-\dfrac{1}{x^2-9}$

Factor and find the LCD. $\dfrac{x+1}{\left(x+3\right)^2}-\dfrac{1}{\left(x+3\right)\left(x-3\right)}$ LCD = $\left(x+3\right)^2\left(x-3\right)$

Rewrite each fraction with the LCD.

$$\frac{\left(x+1\right)}{\left(x+3\right)^2}\cdot\frac{\left(x-3\right)}{\left(x-3\right)}-\frac{1}{\left(x+3\right)\left(x-3\right)}\cdot\frac{\left(x+3\right)}{\left(x+3\right)}=\frac{x^2-2x-3}{\left(x+3\right)^2\left(x-3\right)}-\frac{\left(x+3\right)}{\left(x+3\right)^2\left(x-3\right)}$$

Subtract the fractions. $=\dfrac{x^2-2x-3-x-3}{\left(x+3\right)^2\left(x-3\right)}=\boxed{\dfrac{x^2-3x-6}{\left(x+3\right)^2\left(x-3\right)}}$

Ex 17: Perform the indicated operations and simplify. $\dfrac{4}{x+1}+\dfrac{5}{x-1}-\dfrac{3}{x}$

Find the LCD. LCD = $x(x+1)(x-1)$

Rewrite each fraction with the LCD.

$$\frac{4}{x+1}\cdot\frac{x(x-1)}{x(x-1)}+\frac{5}{x-1}\cdot\frac{x(x+1)}{x(x+1)}-\frac{3}{x}\cdot\frac{x^2-1}{x^2-1}$$

Add and subtract the fractions.

$$\frac{4x^2-4}{x(x-1)(x+1)}+\frac{5x^2+5}{x(x-1)(x+1)}-\frac{3x^2-3}{x(x-1)(x+1)}$$

$$=\frac{4x^2+5x^2-3x^2-4+5+3}{x(x-1)(x+1)}=\boxed{\frac{6x^2+4}{x(x-1)(x+1)}}$$

Note: Denominator can be left in factored form.

QOD: Explain if the following is a true statement. The LCD of two rational expressions is the product of the denominators.

SAMPLE EXAM QUESTIONS

1. **A board of length $\frac{5}{x+2}$ cm was cut into two pieces. If one piece is $\frac{7}{x-2}$ cm, express the length of the other board as a rational expression.**

A. $\frac{-2x-24}{(x+2)(x-2)}$

B. $\frac{-2x-24}{(x+2)^2}$

C. $\frac{-6x-24}{(x+2)^2}$

D. $\frac{-6x-24}{(x+2)(x-2)}$

Ans: C

Unit 6.4 Solve rational equations

Recall: When solving equations involving fractions, **we can eliminate** the fractions by multiplying every term in the equation by the LCD.

Ex 18: Solve the equation. $\frac{2x}{3}-4x=\frac{x^2}{9}$

Multiply every term by the LCD = 9.

$$9\cdot\frac{2x}{3}-9\cdot4x=9\cdot\frac{x^2}{9}$$
$$6x-36x=x^2$$

Solve the equation.

$$-30x=x^2$$
$$x^2+30x=0$$
$$x(x+30)=0$$
$$\boxed{x=0,-30}$$

Rational Equation: an equation that involves rational expression

To solve a rational equation, multiply every term by the LCD. Then check your solution(s) in the original equation.

Ex 19: Solve the equation $\frac{3}{x}-\frac{1}{2}=\frac{12}{x}$

Multiply every term by the LCD = 2*x*

$$2\cancel{x}\cdot\frac{3}{\cancel{x}}-\cancel{2}x\cdot\frac{1}{\cancel{2}}=2\cancel{x}\cdot\frac{12}{\cancel{x}}$$
$$6-x=24$$

Solve the equation.

$$-x=18$$
$$x=-18$$

Check. $\frac{3}{-18}-\frac{1}{2}=\frac{12}{-18}\Rightarrow-\frac{1}{6}-\frac{1}{2}=-\frac{2}{3}$ true , so $\boxed{x=-18}$

Ex 20: Solve the equation $\frac{5x}{x+1}=4-\frac{5}{x+1}$

Multiply every term by the LCD = *x* + 1

$$\cancel{(x+1)}\cdot\frac{5x}{\cancel{(x+1)}}=(x+1)\cdot4-\cancel{(x+1)}\cdot\frac{5}{(x+1)}$$
$$5x=4x+4-5$$

Solve the equation. $x=-1$

Check. $\frac{5(-1)}{(-1)+1}=4-\frac{5}{(-1)+1}$

This solution leads to division by zero in the original equation. Therefore, it is an *extraneous solution*. This equation has no solution.

Ex 21: Solve the equation $\frac{3x-2}{x-2}=\frac{6}{x^2-4}+1$

Multiply every term by the LCD = $(x-2)(x+2)$

$$\cancel{(x-2)}(x+2)\cdot\frac{(3x-2)}{\cancel{(x-2)}}=\cancel{(x-2)(x+2)}\cdot\frac{6}{\cancel{(x-2)(x+2)}}+(x-2)(x+2)\cdot1$$
$$3x^2+4x-4=6+x^2-4$$

Solve the equation.

$$2x^2+4x-6=0 \qquad (x+3)(x-1)=0$$
$$x^2+2x-3=0 \qquad x=-3,\ 1$$

Check.

$$\frac{3(-3)-2}{(-3)-2}=\frac{6}{(-3)^2-4}+1 \qquad \frac{3(1)-2}{(1)-2}=\frac{6}{(1)^2-4}+1$$
$$\frac{11}{5}=\frac{6}{5}+1 \text{ true} \qquad \frac{1}{-1}=\frac{6}{-3}+1 \text{ true}$$

Solutions: $\boxed{x=-3,\ 1}$

Recall: When solving a proportion, cross multiply the two ratios.

Ex: Solve the proportion $\frac{x}{5}=\frac{3}{20}$.

$$20x=15$$
$$x=\frac{15}{20}=\boxed{\frac{3}{4}}$$

Solving a Rational Equation by Cross Multiplying: this can be used when each side of the equation is a single rational expression

Ex 22: Solve the equation $\frac{3}{x^2+4x}=\frac{1}{x+4}$.

Cross multiply.

$$3(x+4)=1(x^2+4x)$$
$$3x+12=x^2+4x$$

Solve.

$$0=x^2+x-12$$
$$0=(x+4)(x-3)$$

$$x=-4,\ 3$$

Check.

$$\frac{3}{(-4)^2+4(-4)}=\frac{1}{(-4)+4}$$
$$\frac{3}{0}=\frac{1}{0} \text{ division by zero!}$$

$$\frac{3}{(3)^2+4(3)}=\frac{1}{(3)+4}$$
$$\frac{3}{21}=\frac{1}{7} \text{ true}$$

Solution: $\boxed{x=3}$ (Note: −4 is an extraneous solution.)

Ex 23: Solve the rational equation. $\frac{7x+1}{2x+5}+1=\frac{10x-3}{3x}$

This is not a proportion. We cannot cross multiply. We must find the LCD and multiply.

Multiply every term by the LCD = $3x(2x+5)$

$$3x\,\cancel{(2x+5)}\cdot\frac{7x+1}{\cancel{2x+5}}+3x(2x+5)\cdot 1=\cancel{3x}(2x+5)\cdot\frac{10x-3}{\cancel{3x}}$$
$$3x(7x+1)+3x(2x+5)=(2x+5)(10x-3)$$

Solve the equation.

$$21x^2+3x+6x^2+15x=20x^2+44x-15$$
$$7x^2-26x+15=0 \Rightarrow (7x-5)(x-3)=0$$
$$\boxed{x=3,\frac{5}{7}}$$

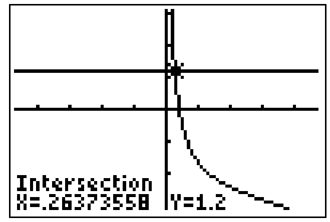
Check the solution:

$\frac{21+1}{6+5}+1=\frac{30-3}{9}$ is true. 3 is a solution.

$\frac{5+1}{\frac{10}{7}+5}+1=\frac{\frac{50}{7}-3}{\frac{15}{7}}$ is true. $\frac{5}{7}$ is also a solution.

Ex 24: use the graph of the rational model $y = \dfrac{50x-20}{x^2-18x-1}$ to find the value of x when $y = 1.2$ on the graphing calculator.

Put $\dfrac{50x-20}{x^2-18x-1}$ into Y_1 and put 1.2 in Y_2. Go to CALC 5 to find the point of intersection.



$\boxed{x = 0.2637}$

QOD: When is cross-multiplying an appropriate method for solving a rational equation?

Unit 6.4 Solve rational inequalities

To solve rational inequalities, which are inequalities that contain one or more rational expressions follow these steps:

1. State the excluded values. These are the values for which the denominator is 0.
2. Solve the related equation.
3. Use the values determined from the previous steps to divide a number line into intervals.
4. Test a value in each interval to determine which intervals contain values that satisfy the inequality.

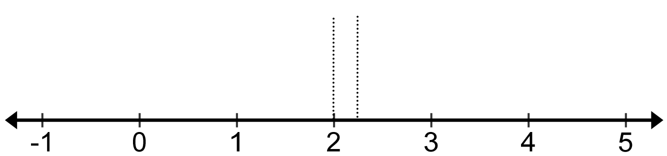
Ex 25: Solve $\dfrac{x}{x-2} > 9$

Step 1: The excluded value for this inequality is 2.

Step 2: Solve

$$\frac{x}{x-2} = 9 \quad \text{Cross multiply}$$
$$9x - 18 = x$$
$$8x = 18 \Rightarrow x = \frac{9}{4}$$

Step 3: Separate the number line into intervals at the solutions and the excluded value.



Step 4: Test a sample in each interval. Use x = 0, x = 17/8 and x = 4.

$x=0$

$\dfrac{0}{0-2}>9$

$0>9$ FALSE

$x=\dfrac{17}{8}$

$\dfrac{\frac{17}{8}}{\frac{17}{8}-2}>9$

$17>9$ TRUE

$x=4$

$\dfrac{4}{4-2}>9$

$2>9$ FALSE

The statement is true for 17/8, so the solution is: $\boxed{2<x<\dfrac{9}{4}}$.

Ex 26: Solve $\dfrac{4}{3x}+\dfrac{7}{x}<\dfrac{5}{9}$

Step 1: The excluded value for this inequality is 0.

Step 2: Solve

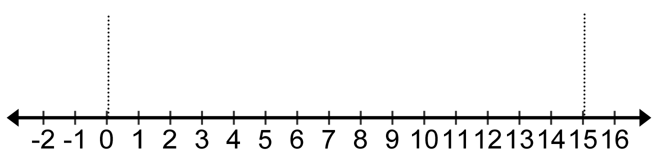
$$\frac{4}{3x}+\frac{7}{x}=\frac{5}{9} \qquad \text{LCD} = 9x$$

$$9x\cdot\frac{4}{3x}+9x\cdot\frac{7}{x}=9x\cdot\frac{5}{9}$$

$$12+63=5x \;\Rightarrow\; 75=5x$$

$$x=15$$

Step 3: Separate the number line into intervals at the solutions and the excluded value.



Step 4: Test a sample in each interval. Use x = -1, x = 1 and x = 16.

$x=-1$

$\dfrac{4}{3(-1)}+\dfrac{7}{-1}<\dfrac{5}{9}$

$-\dfrac{4}{3}-\dfrac{7}{1}<\dfrac{5}{9}$

$\dfrac{-25}{3}<\dfrac{5}{9}$ TRUE

$x=1$

$\dfrac{4}{3(1)}+\dfrac{7}{1}<\dfrac{5}{9}$

$\dfrac{4}{3}+\dfrac{7}{1}<\dfrac{5}{9}$

$\dfrac{25}{3}<\dfrac{5}{9}$ FALSE

$x=16$

$\dfrac{4}{3(16)}+\dfrac{7}{16}<\dfrac{5}{9}$

$\dfrac{4}{48}+\dfrac{7}{16}<\dfrac{5}{9}$

$\dfrac{25}{48}<\dfrac{5}{9}$ TRUE

The statement is true for x = -1 and x = 16. Therefore, the solution is $\boxed{x<0 \text{ or } x>15}$.

Ex 27: Solve $\frac{x}{3}-\frac{1}{x-2}<\frac{x+1}{4}$

Step 1: The excluded value for this inequality is 2.

Step 2: Solve

$$\frac{x}{3}-\frac{1}{x-2}=\frac{x+1}{4} \qquad \text{LCD} = 12(x-2)$$

$$12(x-2)\cdot\frac{x}{3}-12(x-2)\cdot\frac{1}{x-2}=12(x-2)\cdot\frac{x+1}{4}$$

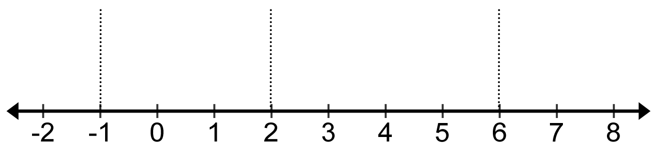
$$(4x-8)x-12=(3x-6)(x+1)$$

$$4x^2-3x-12=3x^2-3x-6$$

$$x^2-5x-6=0 \Rightarrow (x-6)(x+1)=0$$

$$x=6 \text{ or } -1$$

Step 3: Separate the number line into intervals at the solutions and the excluded value.



Step 4: Test a sample in each interval. Use x = -2, x = 0, x = 4 and x = 8.

$x=-2$

$$\frac{-2}{3}-\frac{1}{-2-2}<\frac{-2+1}{4}$$

$$\frac{-2}{3}-\frac{1}{-4}<\frac{-1}{4}$$

$$\frac{-5}{12}<\frac{-1}{4} \quad \text{TRUE}$$

$x=0$

$$\frac{0}{3}-\frac{1}{0-2}<\frac{0+1}{4}$$

$$\frac{1}{2}<\frac{1}{4} \quad \text{FALSE}$$

$x=4$

$$\frac{4}{3}-\frac{1}{4-2}<\frac{4+1}{4}$$

$$\frac{4}{3}-\frac{1}{2}<\frac{5}{4}$$

$$\frac{5}{6}<\frac{5}{4} \quad \text{TRUE}$$

$x=8$

$$\frac{8}{3}-\frac{1}{8-2}<\frac{8+1}{4}$$

$$\frac{32}{12}-\frac{2}{12}<\frac{9}{4}$$

$$\frac{30}{12}<\frac{27}{12} \quad \text{FALSE}$$

The statement is true for x = -3 and x = 4. Therefore, the solution is $\boxed{x<-1 \text{ or } 2<x<6}$.

SAMPLE EXAM QUESTIONS

1. Solve $7=\frac{1}{3n}+\frac{1}{4n}$.

A. $n=\frac{1}{12}$

B. $n=\frac{1}{42}$

C. $n=\frac{2}{49}$

D. $n=\frac{1}{84}$

Ans: A

2. Solve the rational equation for *y*:

$$\frac{5}{y-5}-\frac{5y-10}{y^2-8y+15}=\frac{5}{y-3}$$

A. $y=-2$

B. $y=2$

C. $y=0$

D. $y=4$

Ans: C

3. Solve the equation $\frac{x}{x^2-49}+\frac{7}{x-7}=\frac{1}{x+7}$.

A. -8

B. 8

C. 7

D. No solution

Ans: A

4. Solve: $\frac{1}{x^2+4x}=\frac{5}{x+4}$

A. $x=\frac{1}{5}$

B. $x=-4,\ x=\frac{1}{5}$

C. $x=5$

D. $x=-4,\ x=5$

Ans: A

5. Use the following expressions to answer the questions.

$A=\frac{x+5}{x^2+4x+3}$ $\quad B=\frac{2}{x^2+x}$ $\quad C=\frac{x-3}{2x+2}$ $\quad D=\frac{7}{2x}$

(1) Explain how to add two rational expressions. Simplify $A+B$ as an example. Be sure to give reasons for each step. Simplify completely.

$$A+B=\frac{x+5}{x^2+4x+3}+\frac{2}{x^2+x}=\frac{x+5}{(x+3)(x+1)}+\frac{2}{x(x+1)}=$$

$$\frac{(x+5)}{(x+3)(x+1)}\bullet\frac{x}{x}+\frac{2}{x(x+1)}\bullet\frac{(x+3)}{(x+3)}=\frac{x^2+5x+2x+6}{x(x+1)(x+3)}=$$

$$=\frac{x^2+7x+6}{x(x+1)(x+3)}$$

(2) Explain how subtracting (such as simplifying $A-B$) would be different from adding.

$$A-B=\frac{x+5}{x^2+4x+3}-\frac{2}{x^2+x}=\frac{x+5}{(x+3)(x+1)}-\frac{2}{x(x+1)}=$$
$$\frac{(x+5)}{(x+3)(x+1)}\bullet\frac{x}{x}-\frac{2}{x(x+1)}\bullet\frac{(x+3)}{(x+3)}=\frac{x^2+5x-2x-6}{x(x+1)(x+3)}=$$
$$=\frac{x^2+3x-6}{x(x+1)(x+3)}$$

(3) Explain how to multiply rational expressions, simplifying $B\cdot C$ as an example.

$$B\cdot C=\frac{2}{x^2+x}\cdot\frac{x-3}{2x+2}=\frac{2}{x(x+1)}\cdot\frac{x-3}{2(x+1)}=\frac{x-3}{x(x+1)^2}$$

(4) Show how to divide B by C.

$$B\div C=\frac{2}{x^2+x}\div\frac{x-3}{2x+2}=\frac{2}{x(x+1)}\cdot\frac{2(x+1)}{x-3}=\frac{4}{x(x-3)}$$

(5) Explain how you would solve a rational equation like $C=D$. Then describe, in detail, the strategy for solving an equation like $B+C=D$. Do not completely solve the equations. Instead, concentrate on the first two or three steps in solving; show what to do and explain why.

First, I need to check the denominators: they tell me that x cannot equal zero or -1 (since these values would cause division by zero). Re-check at the end, to make sure any solutions are "valid".

$$C=\frac{x-3}{2x+2} \qquad D=\frac{7}{2x}$$

$$C=D\Leftrightarrow\frac{x-3}{2x+2}=\frac{7}{2X}\Leftrightarrow\frac{x-3}{x+1}=\frac{7}{x}$$

Cross product $x^2-3x=7x+7$ so $x^2-10x-7=0$

In the end, using the quadratic formula the solutions are: $x=5\pm4\sqrt{2}$ "valid" solutions.

Now, $B+C=D$

When you were adding and subtracting rational expressions, you had to find a common denominator. Now that you have equations (with an "equals" sign in the middle), you are allowed to multiply through by the LCD (because you have two sides to multiply on) and get rid of the denominators entirely. In other words, you still need to find the common denominator, but you don't necessarily need to use it in the same way.

$$\frac{2}{x(x+1)}+\frac{x-3}{2(x+1)}=\frac{7}{2x}$$

Now let's multiply through by LCD $=2x(x+1)$

So we get $\frac{2}{x(x+1)}\cdot 2x(x+1)+\frac{x-3}{2(x+1)}\cdot 2x(x+1)=\frac{7}{2x}\cdot 2x(x+1)$

Which is equivalent to: $4+(x-3)x=7(x+1)$ This is a quadratic equation with the following real solutions: $x_{1,2}=5\pm 2\sqrt{7}$

Unit 6.5 Compose a rational function with other functions

Recall that the composition of functions refers to the combining of functions in a manner where the output from one function becomes the input for the next function. In math terms, the range (the y-value answers) of one function becomes the domain (the x-values) of the next function.

Ex 28: Let $f(x)=2x-3$ and $g(x)=\frac{1}{x+1}$. Find $(f\circ g)(x)$ and $(g\circ f)(x)$.

$$(f\circ g)(x)=(f(g(x))=2\left(\frac{1}{x+1}\right)-3$$

$$=\frac{2}{x+1}-3 \Rightarrow \frac{2}{x+1}-\frac{3(x+1)}{x+1}$$

$$=\boxed{\frac{-3x-1}{x+1}}$$

$$(g\circ f)(x)=(g(f(x))=\frac{1}{(2x-3)+1}$$

$$=\boxed{\frac{1}{2x-2}}$$

Ex 29: Let $f(x)=3x^2-4$ and $g(x)=\frac{2x}{x-3}$. Find $(f\circ g)(x)$ and $(g\circ f)(x)$.

$$(f\circ g)(x)=(f(g(x))=3\left(\frac{2x}{x-3}\right)^2-4$$

$$=3\frac{4x^2}{(x-3)^2}-4 \Rightarrow \frac{12x^2}{(x-3)^2}-\frac{4(x-3)^2}{(x-3)^2}$$

$$=\frac{12x^2}{(x-3)^2}-\frac{4x^2-6x+9}{(x-3)^2} \Rightarrow \boxed{\frac{8x^2-6x+9}{(x-3)^2}}$$

$$(g\circ f)(x)=(g(f(x))=\frac{2x}{(3x^2-4)-3}$$

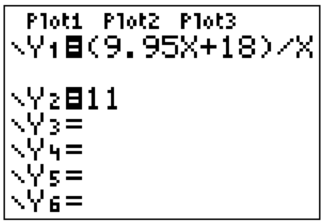
$$=\boxed{\frac{2x}{3x^2-7}}$$

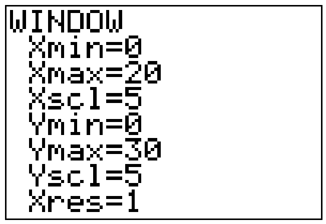
Unit 6.6 Create and Apply Rational Functions

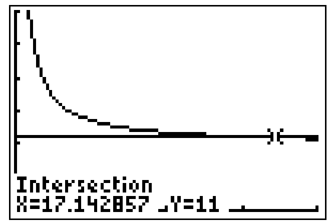
Application Problems with Rational Functions

Ex 30: The senior class is sponsoring a dinner. The cost of catering the dinner is \$9.95 per person plus an \$18 delivery charge. Write a model that gives the average cost per person. Graph the model and use it to estimate the number of people needed to lower the cost to \$11 per person. What happens to the average cost per person as the number increases?

Model: Average cost = (Total Cost) / (Number of People) $A = \dfrac{9.95x + 18}{x}$







They need at least 17 people to lower the cost to \$11 per person.

The average cost approaches \$9.95 as the number of people increases.

Application Problems with Local Extrema

Ex 31: A closed silo is to be built in the shape of a cylinder with a volume of 100,000 cubic feet. Find the dimensions of the silo that use the least amount of material.

Volume of a Cylinder: $V = \pi r^2 h$

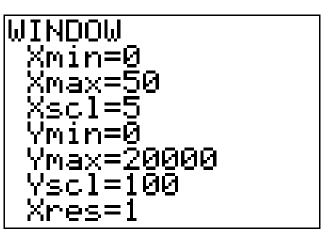
$$100,000 = \pi r^2 h$$
$$\frac{100,000}{\pi r^2} = h$$

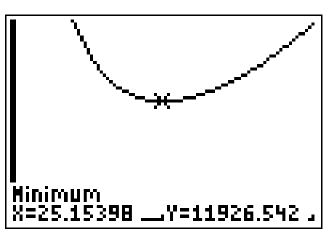
Using the least amount of material is finding the minimum surface area, *S*, of the cylinder.

Surface Area of a Cylinder: $S = 2\pi r^2 + 2\pi rh$

Substitute *h* from above: $S = 2\pi r^2 + 2\pi r\left(\dfrac{100,000}{\pi r^2}\right) = 2\pi r^2 + \dfrac{200,000}{r}$

Graph the function for surface area and find the minimum value.





The minimum surface area occurs when the radius is 25.15 ft. The height is

$$h=\frac{100,000}{\pi r^2}=\frac{100,000}{\pi(25.15)^2}\approx \boxed{50.32\text{ ft}}.$$

Ex 32: Every year, the junior and senior classes at Hillcrest High School build a house for the community. If it takes the senior class 24 days to complete a house and 72 days for the junior class to complete a house, how long would it take if they worked together?

Find the rate for each of the classes: The senior class can complete 1 house in 24 days, so their rate is $\frac{1}{24}$ of a house per day. The rate for the junior class is $\frac{1}{72}$ of a house per day. We will be combining the two rates.

Solve:

$$\frac{1}{24}+\frac{1}{72}=\frac{1}{c}\quad \text{Multiply by the LCD: } 72c$$

$$72c\cdot\frac{1}{24}+72c\cdot\frac{1}{72}=72c\cdot\frac{1}{c}$$

$$3c+c=72 \Rightarrow 4c=72$$

$$\boxed{c=18}$$

Check:

$$\frac{1}{24}+\frac{1}{72}=\frac{1}{18}$$

$$\frac{3}{72}+\frac{1}{72}=\frac{4}{72}$$

so $\boxed{c=18\text{ days}}$

Ex 33: Sandra is rowing a canoe on Stanhope Lake. Her rate in still water is 6 miles per hour. It takes Sandra 3 hours to travel 10 miles round trip. Assuming that Sandra rowed at a constant rate of speed, determine the rate of the current.

Use the formula $d=rt$, or $t=\frac{d}{r}$.

Time with the current	Time against the current	Total time
$\frac{5}{6+r}$	$\frac{5}{6-r}$	3 hours

Solve:

$$\frac{5}{6+r}+\frac{5}{6-r}=3 \quad \text{LCD}=(6+r)(6-r)$$

$$(6+r)(6-r)\cdot\frac{5}{6+r}+(6+r)(6-r)\cdot\frac{5}{6-r}=3(6+r)(6-r)$$

$$5(6-r)+5(6+r)=3\left(36-r^2\right)$$

$$30-5r+30+5r=108-3r^2 \Rightarrow 60=108-3r^2$$

$$48=3r^2 \Rightarrow \boxed{r=\pm 4}$$

Since speed cannot be negative, the speed of the current is 4 miles per hour.

Ex 34: Mia adds a 70% acid solution to 12 milliliters of a solution that is 15% acid. How much of the 70% acid solution should be added to create a solution that is 60% acid?

Each solution has a certain percentage that is acid. The percentage of acid in the final solution must equal the amount of acid divided by the total solution.

$$\text{Percentage of acid in solution}=\frac{\text{amount of acid}}{\text{total solution}}$$

	Original	Added	New
Amount of Acid	0.15(12)	0.7(x)	0.15(12)+0.7(x)
Total Solution	12	x	12+x

$$\frac{\text{percent}}{100}=\frac{\text{amt. of acid}}{\text{total solution}}$$

$$\frac{60}{100}=\frac{(0.15)(12)+0.7x}{12+x}\quad \text{Cross Multiply}$$

Solve:

$$60(12+x)=100(1.8+0.7x)$$

$$720+60x=180+70x \Rightarrow 540=10x$$

$$\boxed{x=54}$$

Check:

$$\frac{60}{100}=\frac{(0.15)(12)+0.7(54)}{12+54}$$

$$0.6=\frac{37.8}{66}$$

$$0.6=0.6$$

Mia needs to add 54 milliliters of the 70% acid solution.

QOD: Describe how to find the horizontal, vertical, and slant asymptotes of a rational function.

QOD: Write a rational function whose graph is a hyperbola that has a vertical asymptote at $x=2$ and a horizontal asymptote at $y=1$. Can you write more than one function with the same asymptotes?

QOD: In what line(s) is the graph of $y=\frac{1}{x}$ symmetric? What does this symmetry tell you about the inverse of this function?

SAMPLE EXAM QUESTIONS

1. **The equivalent resistance R of two resistors in parallel, with resistances r_1 and r_2, is given by the formula:**

$$\frac{1}{R}=\frac{1}{r_1}+\frac{1}{r_2}$$

If r_1 has twice the resistance of r_2, what is the value of r_2 if the equivalent resistance is 30 ohms?

A. 15 ohms **C.** 30 ohms

B. 45 ohms **D.** 90 ohms

Ans: C

2. **Last week, Wendy jogged for a total of 10 miles and biked for a total of 10 miles. She biked at a rate that was twice as fast as her jogging rate.**

(1) Suppose Wendy jogs at a rate of r miles per hour. Write an expression that represents the amount of time she jogged last week and an expression that represents the amount to time she biked last week. (*hint:* distance = rate•time)

$$t_{jogged}=\frac{d}{r}=\frac{10}{r}$$

$$t_{biked}=\frac{d}{2r}=\frac{10}{2r}=\frac{5}{r}$$

(2) Write and simplify an expression for the total amount of time Wendy jogged and biked last week.

$$t_{total} = \frac{10}{r} + \frac{5}{r} = \frac{15}{r}$$

(3) Wendy jogged at a rate of 5 miles per hour. What was the total amount of time Wendy jogged and biked last week?

$$t_{total} = \frac{15}{r} = \frac{15}{5} = 3(hours)$$

3. The rate of heat loss from a metal object is proportional to the ratio of its surface area to its volume.

(1) What is the ratio of a steel sphere's surface area to volume?

Here are the formulas for sphere's surface area and volume, where r is radius:

$$S.A._{sphere} = 4\pi r^2 \qquad V_{sphere} = \frac{4}{3}\pi r^3$$

So the ratio of sphere's surface area to volume is:

$$\frac{4\pi r^2}{\frac{4}{3}\pi r^3} = \frac{3}{r}$$

(2) Compare the rate of heat loss for two steel spheres of radius 2 meters and 3 meters, respectively.

The smaller sphere (the one with the radius of 2) will lose heat at a rate of 3/2=1.5 and the bigger one at a rate of 1. Since time is inverse proportion to heat rate we could also say that if it takes the big sphere an hour to cool down, is going to take the smaller one 2/3 of that time which is 40 minutes.

4. A sight-seeing boat travels at an average speed of 20 miles per hour in the clam water of a large lake. The same boat is also used for sight-seeing in a nearby river. In the river, the boat travels 2.9 miles downstream (with the current) in the same amount of time it takes to travel 1.8 miles upstream (against the current). Find the current of the river.

$$t_{downstream} = t_{upstream}$$
$$\frac{2.9}{20+r} = \frac{1.8}{20-r}$$
$$2.9(20-r) = 1.8(20+r)$$
$$1.8r + 2.9r = 2.9 \cdot 20 - 1.8 \cdot 20$$
$$4.7r = 58 - 36$$
$$4.7r = 22$$
$$r = 4.68mi/h$$

5. A baseball player's batting average is found by dividing the number of hits the player has by the number of at-bats the player has. Suppose a baseball player has 45 hits and 130 at-bats. Write and solve an equation to model the number of consecutive hits the player needs in order to raise his batting average to 0.400. Explain now you found your answer.

Let x be the number of consecutive hits the player needs. In that case the number of hits will become (45+x) and the number of at-bats will be (130+x), so the equation representing his batting average will be:

$$\frac{45+x}{130+x}=0.400$$
$$x+45=0.4x+52$$
$$0.6x=7$$
$$x=11.67$$

cross product, followed by solving the linear equation gives us:

$$x+45=0.4x+52$$
$$0.6x=7$$
$$x=11.67$$

and since x is a whole number we need to round it up to 12. So the number of consecutive hits the player needs in order to raise his batting average to 0.400 is 12 hits.

6. <u>Sample SAT Question(s):</u> Taken from College Board online practice problems.

The projected sales volume of a video game cartridge is given by the function $s(p)=\frac{3000}{2p+a}$, where *s* is the number of cartridges sold, in thousands; *p* is the price per cartridge, in dollars; and *a* is a constant. If according to the projections, 100,000 cartridges are sold at \$10 per cartridge, how many cartridges will be sold at \$20 per cartridge?

(A) 20,000
(B) 50,000
(C) 60,000
(D) 150,000
(E) 200,000

Ans: C