Pre-Algebra Notes:
Proportional Relationships & Linear Functions

Slope

The idea of slope is used quite often in our lives. However, outside of school, it goes by different names. People involved in home construction might talk about the pitch of a roof. If you were riding in your car, you might have seen a sign on the road indicating a grade of 6% up or down a hill. Both of those cases refer to what we call slope in mathematics.

Kids use slope on a regular basis without realizing it. Let's look at our drink example again. A student buys a cold drink for $0.50. If two cold drinks were purchased, the student would have to pay $1.00. I could describe that mathematically by using ordered pairs: (1, $0.50), (2, $1.00), (3, $1.50), and so on. The first element in the ordered pair represents the number of cold drinks; the second number represents the cost of those drinks. Easy enough, don't you think?

Now if I asked the student how much more was charged for each additional cold drink, hopefully the student would answer $0.50. So the difference in cost from one cold drink to adding another is $0.50. The cost would change by $0.50 for each additional cold drink. The change in price for each additional cold drink is $0.50. Another way to say that is the *rate of change* is $.50. We call the rate of change *slope*.

In math, the *rate of change* is called the slope and is often described by the ratio $\frac{rise}{run}$.

The rise represents the change (difference) in the vertical values (the *y*'s); the run represents the change in the horizontal values, (the *x*'s). Mathematically, we write

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Let's look at any two of those ordered pairs from buying cold drinks: (1, $0.50) and (3, $1.50). Find the slope. Substituting in the formula, we have:

$$\begin{aligned} m &= \frac{1.50 - 0.50}{3 - 1} \\ &= \frac{1.00}{2} \\ &= 0.50 \end{aligned}$$

We find the slope is $0.50. The rate of change per drink is $0.50.

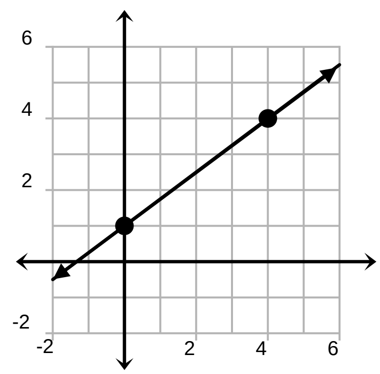
Example: *Find the slope of the line that connects the ordered pairs (3, 5) and (7, 12).*

To find the slope, I use $m = \dfrac{y_2 - y_1}{x_2 - x_1}$.

Subtract the *y* values and place that result over the difference in the *x* values.

$\dfrac{12-5}{7-3} = \dfrac{7}{4}$ The slope is $\dfrac{7}{4}$.

Example: *Find the slope of the line on the graph to the right.*



Pick two points that are easy to identify.

(0, 1) and (4, 4)

Find the slope:

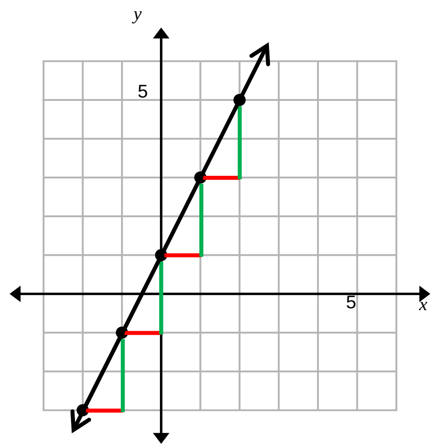
$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{4-1}{4-0}$$
$$= \frac{3}{4}$$

The slope of the graphed line is $\dfrac{3}{4}$.

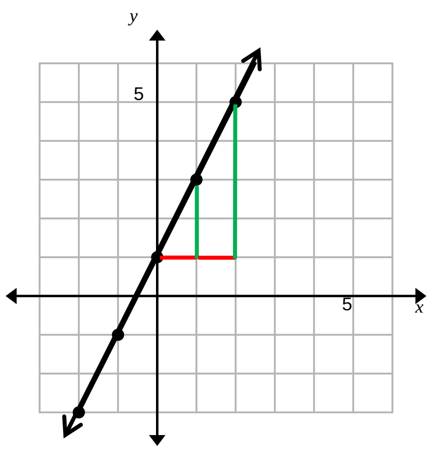
Using Similar Triangles to Explain Slope

NVACS 8.EE.6 (part): Use similar triangles to explain why the slope m is the same between any two distinct points on a non-vertical line in the coordinate plane.

Look at the line graphed. Let's choose several points with integer coordinates to help us determine the slope of the line. In each case, we will note the horizontal distance by a red segment and the vertical distance by a green segment.



When we compare the change in the *y*-value (2) to the change in the *x*-value (1) for each "slope triangle", we have the ratio $\frac{2}{1}$, which is the slope for the line.



Using the same line, look at a different pair of slope triangles.

The ratio of the larger slope triangle is $\frac{4}{2}$.

The ratio of the smaller slope triangle is $\frac{2}{1}$.

The ratios are equivalent! $\frac{4}{2}=\frac{2}{1}=2$

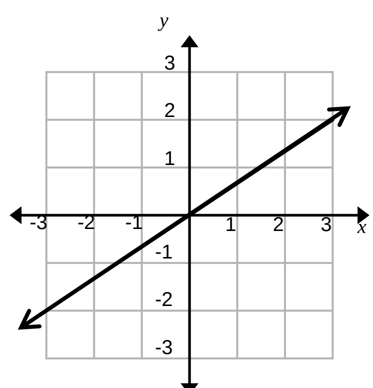
These slope triangles are similar by angle-angle (the right angle and the common angle). When we have similar triangles, we know that the ratios of the corresponding sides must be equal. That is the reason that the slope is the same for both slope triangles.

Example problems:

1. Determine the slope for the line graphed to the right.

   Change in the y-value is 2; change in the x-value is 3.

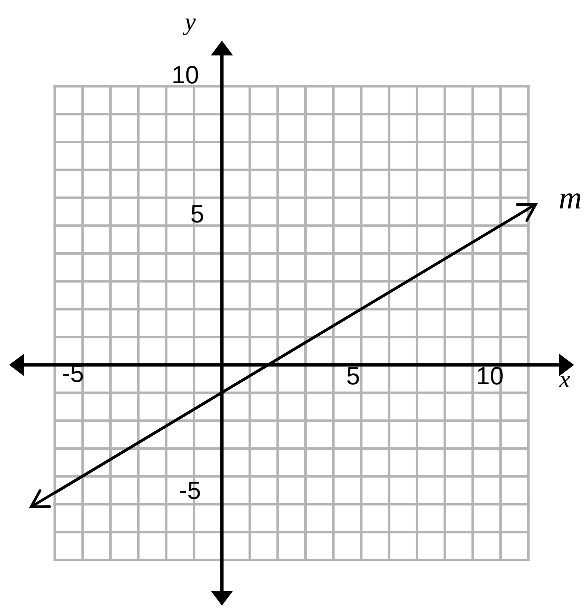
   Therefore, the slope is $\frac{2}{3}$.



2. Start at (0, 0). Move right 6 units. From the location defined by completing these steps, how many units up is the line?

   Using the slope, move 3 more units to the right and up 2 units, identifies another point on the line (6, 4). Therefore, the answer is 4 units.

Show another way to show that slope of a line is constant by using similar triangles:



1. On the graph to the right, plot four points with integer coordinates on the line *m*. Label the points *P*, *Q*, *R*, and *S*.
2. Draw the slope triangle using points *P* and *Q*. Label the right angle vertex *T*.
3. Draw the slope triangle using points *R* and *S*. Label the right angle vertex *V*.
4. Extend $\overleftrightarrow{PT}$ and $\overleftrightarrow{RV}$ to create horizontal (and parallel) lines across the coordinate plane.
5. Since line *m* is a transversal intersecting the parallel lines you just created, you know that $\angle QPT$ is a corresponding angle to $\angle$______. Corresponding angles are _____________.
6. The two right angles $\angle$____ and $\angle$_____ are ________.
7. Therefore, Δ______ is similar to Δ______ by angle-angle similarity.
8. When we have similar triangles, we know that the ratios of the corresponding sides must be equal. Therefore, the slope of the line is constant. What is the slope of this line? ________
9. Start at (0, 0). Move right 15 units. From the location defined by completing these steps, how many units up is line *m*? _____

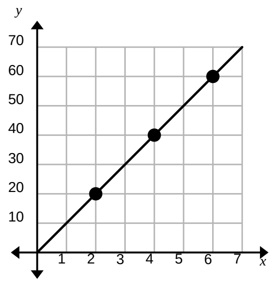
Graphing Proportional Relationships

NVACS 8.EE.5: Graph proportional relationships, interpreting the unit rate as the slope of the graph. Compare two different proportional relationships represented in different ways.

Direct Variation: Constant of Proportionality (Variation), Slope & Unit Rate

Consider the table. Note that the ratio of the two quantities is constant ($\frac{20}{2}=\frac{40}{4}=\frac{60}{6}$), indicating a proportional relationship. This relationship is called a **direct variation**. This constant ratio is called the **constant of proportionality** or **constant of variation.**

Babysitting (hours), ***x***	*Money Earned ($),* ***y***
2	20
4	40
6	60



Consider the graph of a line containing these points.

Determine the slope: $\frac{change\,in\,y}{change\,in\,x}=\frac{20}{2}=10$

What does this mean? *$10 is earned per hour babysitting*

Recall, this is also called the *unit rate* (a rate with 1 in the denominator).

Therefore, note that the **constant of proportionality (variation), the slope, and the unit rate all have the same value.**

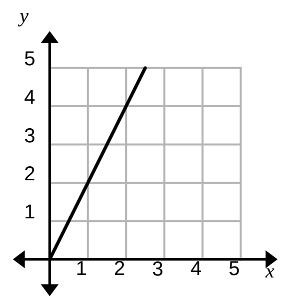
Example problems: Determine the unit rates:

1. Bamboo that grows 5 inches in 2.5 hours.

2.

Cyclist Ride	
Hours	Miles
3	24
6	48

3.



Representing Proportional Relationships and Slope

A **proportional relationship** between two quantities exists if they have a constant ratio and a constant rate of change. This relationship is also called a **direct variation**. The equations of such relationships are always in the form ***y* = *mx***. When graphed, they produce a line that passes through the origin. In this equation, ***m*** is the **slope** of the line; it is also called the **unit rate**, the rate of change, or the **constant of proportionality** of the function.

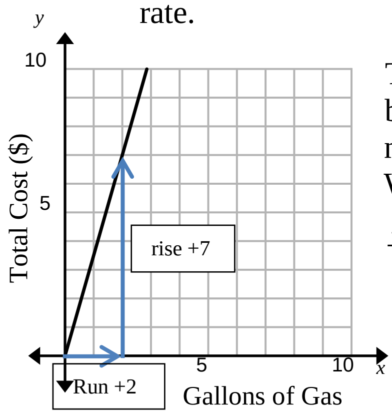
Example: Graph the proportional relationship between the two quantities, write the equation representing the relationship, and describe how the unit rate or slope is represented on the graph.

Gasoline cost $3.50 per gallon

We can start by creating a table to show how these two quantities, gallons of gas and cost, vary. Two things show us that this is definitely a proportional relationship. First, it contains the origin, (0, 0), and this makes sense: if we buy zero gallons of gas it will cost zero dollars. Second, if the number of gallons is doubled, the cost is doubled; if it is tripled, the cost is tripled.

Gas (gal)	Cost ($)
0	0
1	3.50
2	7
3	10.50

The equation that will represent this data is $y = 3.50x$, where x is the number of gallons of gasoline and y is the total cost ($y = mx$). Slope is 3.50 as indicated in the table as the unit rate.



The graph is shown. (Note: The equation does extend into the third quadrant because this region does not make sense for the situation. We will not buy negative quantities of gasoline, nor pay for it with negative dollars!) We can find the slope by creating a "slope triangle" which represents $\frac{rise}{run} = \frac{7}{2} = 3.5$, which confirms the slope we show in the equation.

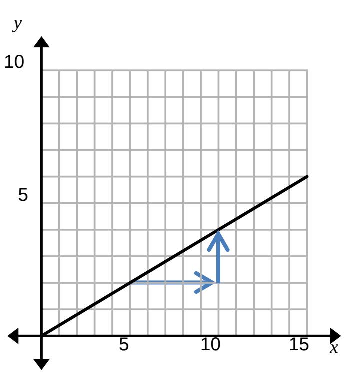
Either way, the constant of proportionality is the slope, which is 3.5.

Example: Graph the proportional relationship between the two quantities, write the equation representing the relationship, and describe how the unit rate or slope is represented on the graph.

Five Fuji Apples cost $2

Again, we can begin by creating a table relating the number of apples to their cost. We can use this table to plot the points and determine the slope of the line.

# of apples	0	5	10	15
cost ($)	0	2	4	6



Using the slope triangle, we can see that $\frac{rise}{run} = \frac{2}{5}$.

Using $y = mx$, the equation for the line is $y = \frac{2}{5}x$.

For unit rate: if five apples cost $2.00, then one apple costs $\frac{2.00}{5} = .40$ or 40 cents per apple. (It is also represented on the graph: for one apple, the graph rises .40.)

Comparing Proportional Relationships in Different Formats

You can use table, graphs, words or equations to represent and compare proportional relationships. Different cyclists rates are represented below.

Words Cyclist *A*

A cyclist can ride 24 miles in 2 hours.

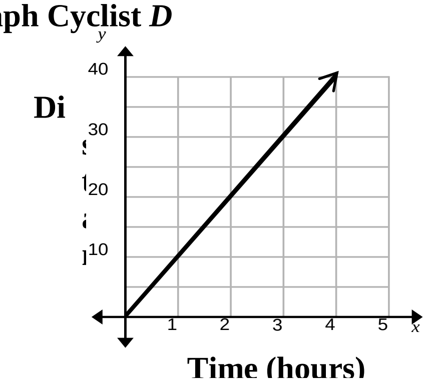
Equation Cyclist *C*

$y = 9x$

Table Cyclist *B*

Time Hours	Distance (miles)
0	0
1	5
2	10
3	15

Graph Cyclist *D*



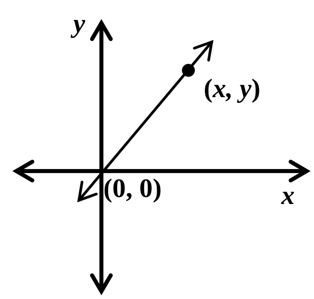
Which cyclist is faster? **A** Slower? **B** Explain your reasoning. *Cyclist A's rate is* $\frac{24\,miles}{2\,hours} = \frac{12\,miles}{1\,hour}$ *, cyclist B's rate is 5mph, cyclist C's rate is 9mph and Cyclist D's rate is 10mph.*

Deriving *y* = *mx* and *y* = *mx* + *b*

NVACS 8.EE.6 (part): Derive the equation y=mx for a line through the origin and the equation y=mx+b for a line intercepting the vertical axis at b.

Deriving $y = mx$

We know that the graphs for direct variation always go through the origin (0, 0). Knowing that, let's derive the equation for direct variation.



$\frac{y_2 - y_1}{x_2 - x_1} = m$ slope formula

$\frac{y - 0}{x - 0} = m$ $(x_1, y_1) = (0, 0)$ and $(x_2, y_2) = (x, y)$

$\frac{y}{x} = m$ simplify

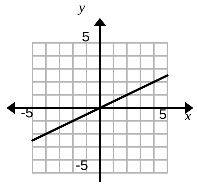
$y = mx$ Multiplication Property of Equality

So, in a **direct variation equation,** $y = mx$**,** the *m* represents the constant of proportionality (variation), the slope and the unit rate.

Example: Which functions show a proportional relationship? How do you know?

x	0	1	2
y	0	3	6

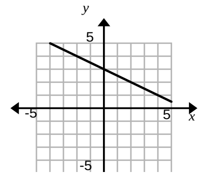
Yes, passes through (0,0)



Yes, passes through (0,0)

$y = -2x$

Yes, $y=mx$



No, does not pass through (0,0)

$y = 2x - 3$

No, not $y=mx$

x	3	6	9
y	1	2	3

Yes, does pass through (0,0)

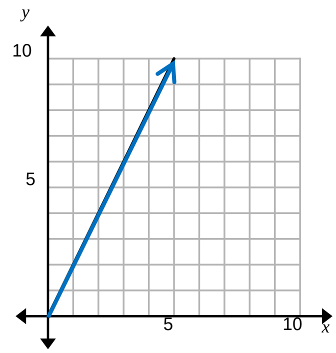
Interpreting the *y*-intercept

Create a chart and graph for the scenarios below. Identify the slope.

(A) *Each week Marlow puts 2 dollars away per week to save up some money to buy a new video game.*

Slope: $\dfrac{2\,dollars}{1\,week}$

Week	Total Money Saved ($)
0	0
1	2
2	4
3	6

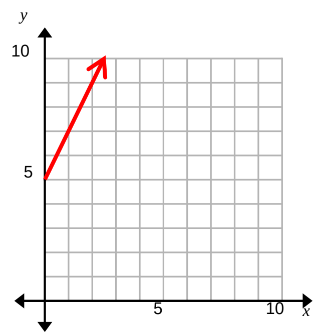


(B) *Each week a friend Jayden puts 2 dollars away per week to save up some money to buy a new video game—the same as Marlow. However, she already had 5 dollars when they started.*

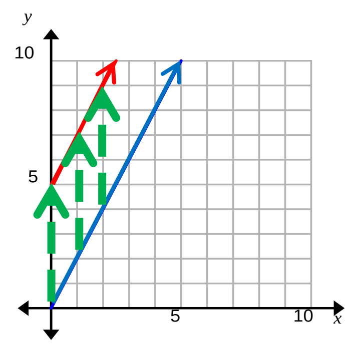
Slope: $\dfrac{2\,dollars}{1\,week}$

Week	Total Money Saved ($)
0	5
1	5+2=7
2	5+4=9
3	5+6=11

+1 +2



Note the slope of the lines. They are the same! Each step horizontally is 1 week. Each step vertically is 2 dollars. Putting the graphs on the same grid should make it clear that Jayden's line is a vertical translation of Marlow's line by 5 units.

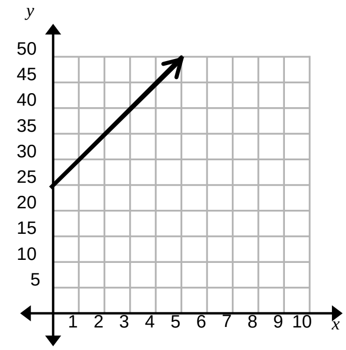


Note that the relationship represented by Marlow's savings is proportional, so we know the equation for Marlow's line is $y = 2x$. But the relationship represented by Jayden's is not proportional, so what is the equation for her line? The rate of change is the same for both graphs, so we know the slope is 2. Every point on Jaydens' line is a vertical translation of Marlow's by 5. Therefore,

$y = 2x + 5$. The slope m (value of 2 in this problem) represents the rate of change, and the initial value of 5 is labeled ***b*** and called the ***y*-intercept. Nonproportional linear relationship can be written in the form *y* = *mx* + *b* called the slope-intercept form; *m* is the slope and *b* is the y-intercept.**

Let's look at a similar problem:

The senior class is selling t-shirts for homecoming week. It costs $25 for the original design and then $5 to print each shirt. Show the graph for this scenario and write an equation.



The graph intersects at 25 (initial cost), so b =25. Slope (m) is $\frac{5}{1} = 5$.

Using the slope intercept form $y = mx + b$

$y = 5x + 25$

Deriving $y = mx + b$

Complete the steps to derive the equation for a nonproportional linear relationship by using the slope formula.

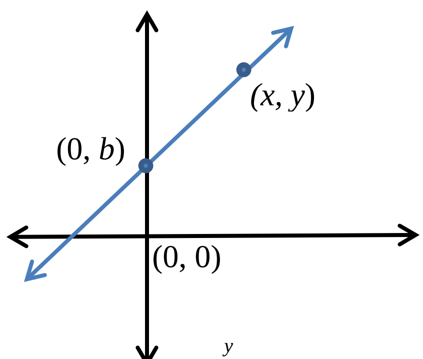
$\frac{y_2 - y_1}{x_2 - x_1} = m$ slope formula

$\frac{y - b}{x - 0} = m$ $(x_1, y_1) = (0, b)$ *and* $(x_2, y_2) = (x, y)$

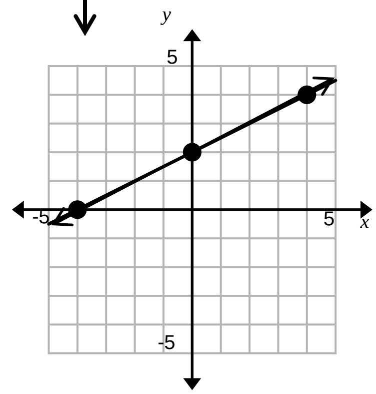
$\frac{y - b}{x} = m$ simplify

$y - b = m \cdot x$ Multiplication Property of Equality

$y = mx + b$



Problem: Write an equation in the slope-intercept form for the graph shown.



The y-intercept is 2. From (0, 2), you can move up 2 units and to the right 4 units to reach another point on the line. That makes the slope $\frac{2}{4} = \frac{1}{2}$.

$y = mx + b$

$y = \frac{1}{2}x + 2$

Graphing Linear Equations

In order to plot the graph of a linear equation, we solve the equation for *y* in terms of *x*. Then we assign values for *x* and find the value of *y* that corresponds to that *x*. Each *x* and *y*, called an ordered pair (*x*, *y*), represents the coordinate of a point on the graph.

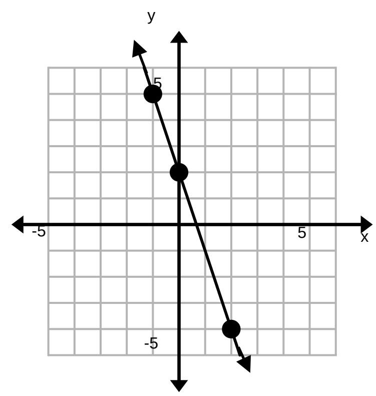
Example: *Graph* $3x + y = 2$.

Solving for *y*, I subtract 3*x* from both sides.

$$y = 2 - 3x \text{ or } y = -3x + 2$$

When I assign values for *x*, I get these *y* values:

x	*y*
0	2
2	–4
–1	5



Rewriting as ordered pairs, I get (0, 2), (2, -4), (-1, 5) which I plot on the graph. When I connect those three points, I get a straight line called a LINEAR equation.

I could have chosen any values for *x* and found the corresponding *y*. However, it is easier to choose a convenient number, like zero, one, or two. Choosing a number like 100 would make my graph a lot larger; or I could have chosen a fraction, but that can be messy.

You can also graph a line quickly by choosing the points where the line crosses the axes.

To find the ***x*-intercept** (*x*-coordinate of the point where the line crosses the *x*-axis), substitute 0 for *y* in the line's equation and solve for *x*.

To find the ***y*-intercept** (*y*-coordinate of the point where the line crosses the *y*-axis), substitute 0 for *x* in the line's equation and solve for *y*.

Example: *Find the intercepts of the graph* $3x - 2y = 6$. Graph.

To find the *x*-intercept, we will let $y = 0$ and solve for *x*.

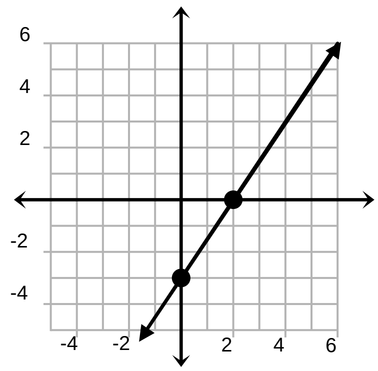
$$3x - 2y = 6$$
$$3x - 2(0) = 6$$
$$3x = 6$$
$$x = 2$$

To find the *y*-intercept, we will let $x = 0$ and solve for *y*.

$$3x - 2y = 6$$
$$3(0) - 2y = 6$$
$$-2y = 6$$
$$y = -3$$

Answer: The *x*-intercept is 2 and the *y*-intercept is -3.
I can rewrite these as the ordered pairs (2, 0) and $(0, -3)$.

Graphing these 2 intercepts I can now graph the line.



Example: *Graph* $x - 3y = 8$**.**

Solve for *y* in terms of *x*.

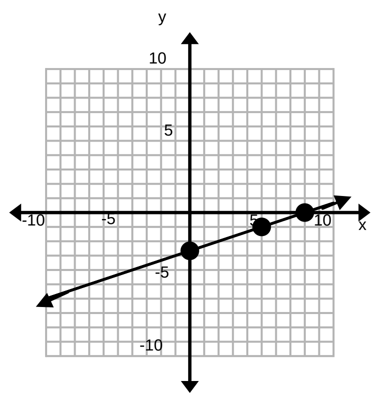
$$x - 3y = 8$$

$$-3y = 8 - x \quad \text{(subtract } x \text{ from both sides)}$$

$$y = \frac{1}{3}x - \frac{8}{3} \quad \text{(divide both sides by } -3\text{)}$$

Assign values for *x*, and find the corresponding *y*'s. Choose numbers that will make the values for *y* easy to work with.

x	***y***
0	$-\frac{8}{3}$
8	0
5	-1



The ordered pairs $(0, -\frac{8}{3})$, (8, 0), and $(5, -1)$ represent the points on the graph.

If we did enough of these problems, we would see a quicker way of graphing linear equations. First, all linear equations are graphs of lines; therefore, all we need do is graph two points. Second, we would see that the value of *x* when the graph crosses the *y*-axis is always zero. Look at the last two examples. What's the value of *y* when the graph crosses the *x*-axis? Look at the graphs we've already plotted. When the graph crosses the *x*-axis, the value of *y* is zero.

Let's look at another example. First we'll plot the graph as we did in the previous two examples; then we'll look at a short cut.

Example: *Graph* $3x - 4y = 12$.

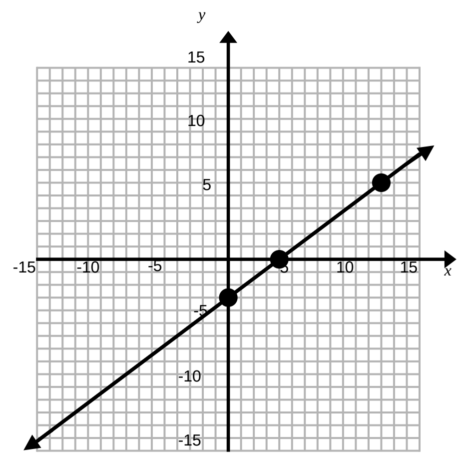
Solve for *y:* $3x - 4y = 12$

$-4y = -3x + 12$ (subtract $3x$ from both sides)

$y = \frac{3}{4}x - 3$ (divide both sides by -4)

Now assign values for *x* and find the corresponding *y*'s. Graph each ordered pair.

x	***y***
0	–3
4	0
12	6



Now for the shortcut….

Slope-Intercept Form of a Linear Equation

NVACS 8.F.3 (part): Interpret the equation $y = mx + b$ ***as defining a linear function, whose graph is a straight line.***

We have $3x - 4y = 12$. Solving for *y*, we have $y = \frac{3}{4}x - 3$. We graphed that and found the graph to cross the *y*-axis at $(0, -3)$. We can determine the slope by using two of the points:

$$m = \frac{-3 - 0}{0 - 4} \rightarrow \frac{-3}{-4} \rightarrow \frac{3}{4}.$$

I know the graph will cross the *y*-axis at $(0, -3)$ and has a slope of $\frac{3}{4}$. That means I can locate the point the graph crosses the *y*-axis, called the ***y-intercept***; from there, go up 3 units and over 4 units to locate another point on the line. We can now draw the line.

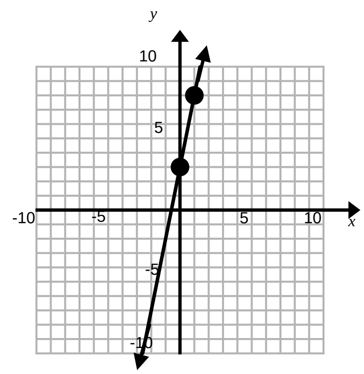
If I did enough problems like that, I would begin to notice the coefficient of the *x* is the *slope* and the constant is the *y-intercept (where the graph crosses the y-axis).* That leads us to the

*Slope-Intercept Form of an Equation of a Line***:**

y = mx + b*,*

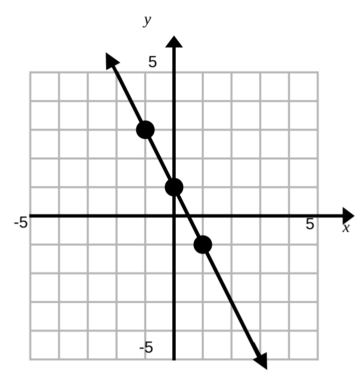
where ***m*** *is the* ***slope*** *and* ***b*** *is* ***the y-intercept****.*

Let's look at another problem, $y = 5x + 3$.



To graph that, I would find the graph crosses the *y*-axis at (0, 3) and has slope 5. That means, I would locate the point the graph crosses the *y*-axis, called *the y-intercept (0, 3),* and from there go up 5 spaces and over one *(recognizing slope as* $\frac{rise}{run} = \frac{5}{1}$*).* The graph always crosses the *y*-axis when $x = 0$.

Example: *Find the y-intercept and slope of* $y = -2x + 1$.
Graph.



Without plugging in *x*'s and finding *y*'s, I can graph this by inspection using the slope-intercept form of a line.

The equation $y = -2x + 1$ is in the $y = mx + b$ form. In our case, the *y*-intercept is 1, written (0, 1), and the slope is -2. This slope can be written as

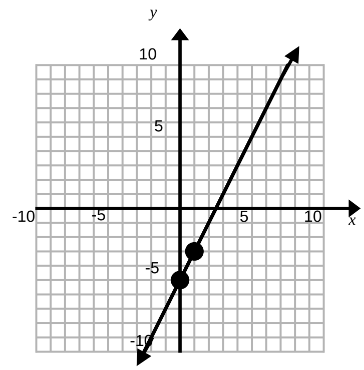
$$\frac{-2}{1} \text{ or } \frac{2}{-1}.$$

To graph, I begin by plotting a point at the *y*-intercept (0, 1).Next, the slope is -2. So from the *y*-intercept, I go down two and to the right one and place a point. (Or I could have gone up two and to the left one.) Connect those points, and I have my line.

Example: *Find the y-intercept and the slope of* $y = 2x - 5$. *Graph.*

The graph of that line would cross the y-axis at the *y-intercept* (0, –5) and has *slope* 2.

To graph that, I would go to the y-intercept (0, –5) and from there, go up two spaces and to the right one space.



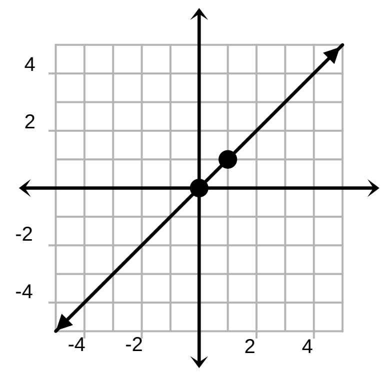
The question comes down to: would you rather learn a shortcut for graphing or do it the long way by solving for y and plugging in values for x?

Let's try one more.

Example: *Graph* $y = x$.

The graph of that line would cross the y-axis at (0, 0) and has slope 1.

To graph that, I would go to the y-intercept (0, 0) and from there, go up one space and over one.

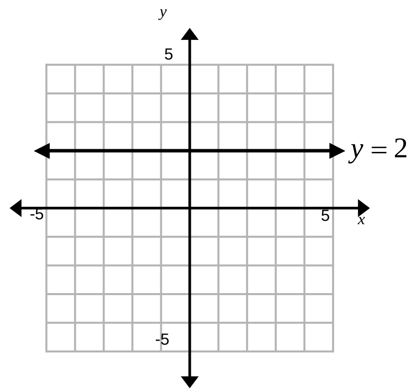


Horizontal and Vertical Lines

Don't forget to address graphing *horizontal* and *vertical* lines (students can have problems with these). The graph of the equation $y = b$ is the horizontal line through (0, b). The graph of the equation $x = a$ is the vertical line through (a, 0). Of course, remind students to simply set up a table of values if they are confused. ***If the line is horizontal, the slope is zero. If the line is vertical, the slope is undefined.***

Example: *Graph* $y = 2$.

The graph of the equation $y = 2$ is the horizontal line through (0, 2).



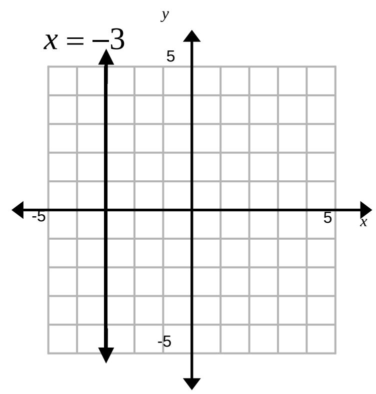
Or a quick table would give us points to plot, and then we could draw the line.

x	***y***
–2	2
0	2
2	2

$$m = \frac{2-2}{2-0} = \frac{0}{2} = 0$$

Example: *Graph* $x = -3$.

The graph of the equation $x = -3$ is the horizontal line through (0, −3).



Or a quick table would give us points to plot and then draw the line.

x	***y***
–3	–2
–3	0
–3	2

$$m = \frac{0-(-2)}{-3-(-3)} = \frac{2}{0}$$ *undefined*

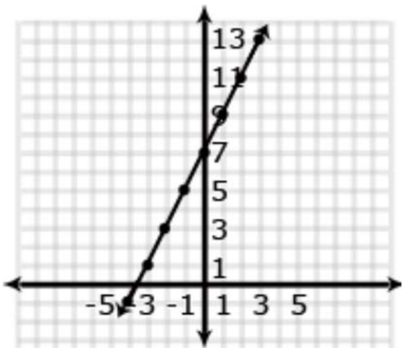
Comparing Properties of Two Functions

NVACS 8.F.2: Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions. For example, given a linear function represented by a table of values and a linear function represented by an algebraic expression, determine which function has the greater rate of change.

The following is adapted from the Arizona Academic content Standards, 2010.

Example: Compare the two linear functions listed below and determine which function represents a greater rate of change.

Function 1:



Function 2:

The function whose input *x* and output *y* are related by

$$y = 3x + 7$$

Solution:

Function 1 has a slope (rate of change) of $\frac{2}{1}$ *or* **2**. Function 2 has a slope of 3. The greater the slope, the greater the rate of change, so Function 2 has the greater rate of change. (Steeper slope indicates greater rate of change.)

Example: *Compare the two linear functions listed below and determine which has a negative slope.*

<u>Function 1: Gift Card</u>
Samantha starts with $20 on a gift card for the book store. She spends $3.50 per week to buy a magazine. Let *y* be the amount remaining as a function of the number of weeks, *x*.

x	*y*
0	20
1	16.50
2	13.00
3	9.50
4	6.00

<u>Function 2:</u>
The school bookstore rents graphing calculators for $5 per month. It also collects a non-refundable fee of $10.00 for the school year. Write the rule for the total cost (c) of renting a calculator as a function of the number of months (*m*).

Solution:
Function 1 is an example of a function whose graph has negative slope. Samantha starts with $20 and spends money each week. The amount of money left on the gift card decreases each week. The graph has a negative slope of -3.5, which is the amount the gift card balance decreases with Samantha's weekly magazine purchase. Function 2 is an example of a function whose graph has positive slope. Students pay a yearly nonrefundable fee for renting the calculator and pay $5 for each month they rent the calculator. This function has a positive slope of 5 which is the amount of the monthly rental fee. An equation for Example 2 could be $c = 5m + 10$.