Graph, Identify, Solve, and Model Exponential Functions

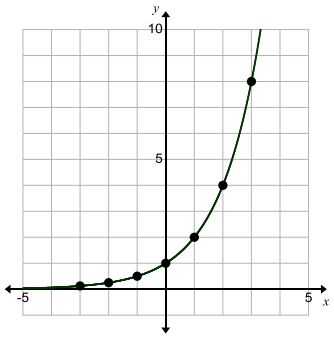
Skill: Graph <u>exponential</u> functions.

<u>Exponential Function</u>: $y = a \cdot b^x,\ b > 0, b \neq 1$

<u>Graph of an Exponential Function</u>

Ex 1: Create a table of values to graph the exponential function $y = 2^x$.

x	-3	-2	-1	0	1	2	3
y	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8



End Behavior: As $x \to +\infty,\ f(x) \to +\infty$.

As $x \to -\infty,\ f(x) \to 0$, therefore $y = 0$ is an **asymptote**.

Explore: Graph the following exponential functions and describe the change from the graph of $y = 2^x$.

1. $y = \frac{1}{3} \cdot 2^x$
2. $y = 3 \cdot 2^x$
3. $y = -\frac{1}{4} \cdot 2^x$
4. $y = -4 \cdot 2^x$
5. $y = 2^x + 3$
6. $y = 2^x - 4$
7. $y = 2^{x-1}$

General characteristics of $y = ab^{x-h} + k$:

Graph of $y = ab^x$ is shifted horizontally by *h* units.

Graph of $y = ab^x$ is shifted vertically by *k* units.

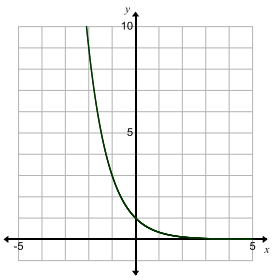
If $a > 0$ and $b > 1$, it is an **exponential growth function.**

Domain of an Exponential Growth Function: All Reals

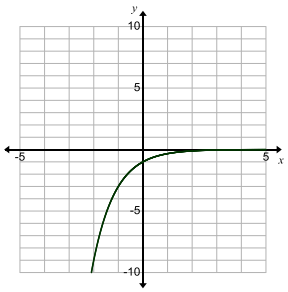
Range of an Exponential Growth Function: $y > k$

Ex 2: Sketch the graph of $y = -\left(\frac{1}{3}\right)^x + 4$

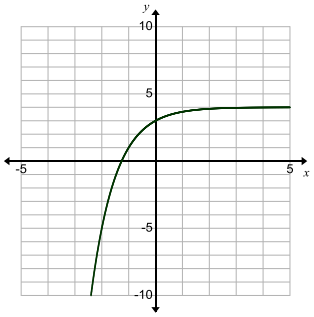
Step One: Sketch the graph of $y = \left(\frac{1}{3}\right)^x$



Step Two: Sketch the graph of $y = -\left(\frac{1}{3}\right)^x$ by reflecting over the *x*-axis.



Step Three: Translate the graph vertically 4 units up.



Exponential Growth Models: In a real-life situation, if a quantity increases by *r* percent each time period *t*, the situation can be modeled by the equation $y = a(1+r)^t$, where *a* is the initial amount. The quantity $(1+r)$ is called the **growth factor**.

Ex 3: In 1990 the cost of tuition at a state university was $4300. During the next 8 years, the tuition rose 4% each year. Write a model that gives the tuition *y* (in dollars) *t* years after 1990. Then estimate the cost of tuition in 1999.

Model: $y = 4300(1+0.04)^t \Rightarrow \boxed{y = 4300 \cdot 1.04^t}$

1999: $t = 9 \Rightarrow y = 4300 \cdot 1.04^9 \approx \boxed{\$6120.24}$

Compound Interest: $A = P\left(1+\frac{r}{n}\right)^{nt}$ *A* = amount in account after *t* years, *P* = principal (amount when *t* = 0, *r* = annual interest rate, *n* = number of times per year the interest is compounded

Ex 4: Jane deposits $1500 in an account that pays 6% annual interest. Find the balance after 3 years if the interest is compounded semiannually.

P = 1500, *r* = 0.06, *t* = 3, *n* = 2 $A = 1500\left(1+\frac{0.06}{2}\right)^{2(3)} \approx \boxed{\$1791.08}$

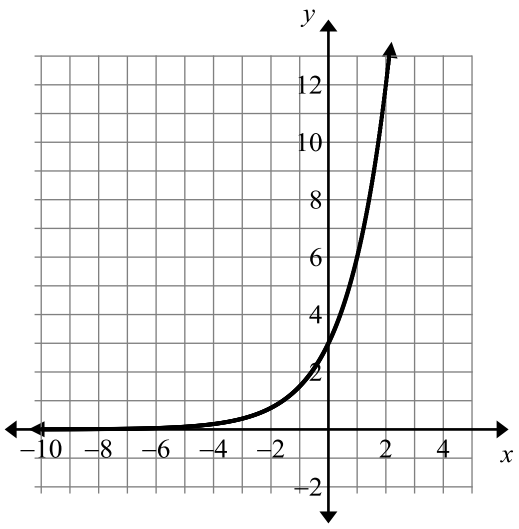
You Try: 1. Sketch the graph of the exponential function. $y = 2 \cdot 3^{x-2}$

2. Use the example above and find out the balance of Jane's account if the interest is compounded quarterly.

QOD: What is the difference between percent increase and growth factor?

Sample CCSD Common Exam Practice Question(s):

What function describes the graph below?



A. $y = x^2 + 3$

B. $y = 3x^3 + 3$

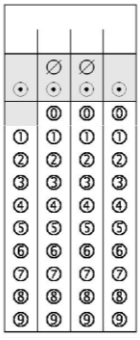
C. $y = 3 \cdot 2^x$

D. $y = 6^x$

Sample SAT Question(s): Taken from College Board online practice problems.

1. If *x* is a positive integer, what is one possible value of the units digit of 103^{2x} after it has been multiplied out?

   Grid-In



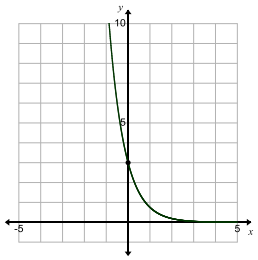
Skill: Graph growth and decay <u>exponential</u> functions.

<u>Exponential Decay Function</u>: $f(x) = ab^x$, where $a > 0$ and $0 < b < 1$

Domain: All Real Numbers; Range: $y > 0$

Ex 5: Graph the function $y = 3\left(\frac{1}{4}\right)^x$ and state its domain and range.

y – intercept: $(0,3)$ Asymptote: $y = 0$



Domain: All real numbers

Range: $y > 0$

Ex 6: State whether $f(x)$ is an exponential growth or decay function.

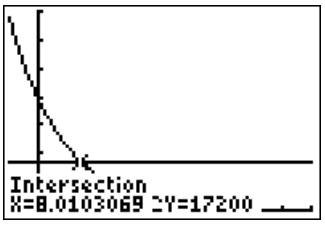
1. $f(x) = 4\left(\frac{3}{4}\right)^x$ DECAY (because $a > 0$ and $b < 1$)
2. $f(x) = 3\left(\frac{5}{2}\right)^x$ GROWTH (because $a > 0$ and $b > 1$)
3. $f(x) = 6(4)^{-x}$ DECAY (Can be rewritten as $f(x) = 6\left(\frac{1}{4}\right)^x$.)
4. $f(x) = -8\left(\frac{2}{3}\right)^x$ GROWTH (because $a < 0$ and $b < 1$)
5. $f(x) = -2 \cdot 3^x$ DECAY (because $a < 0$ and $b > 1$)

<u>Exponential Decay Models:</u> In a real-life situation, if a quantity decreases by r percent each time period t, the situation can be modeled by the equation $y = a(1-r)^t$, where a is the initial amount. The quantity $(1-r)$ is called the **decay factor**.

Ex 7: There are 40,000 homes in a certain city. Each year 10% of the homes are expected to disconnect from septic systems and connect to the sewer system. Write an exponential decay model for the number of homes that still use septic systems. Use the graph of the model to estimate when about 17,200 homes will still not be connected to the sewer system.

Model: $y = 40000(1-0.1)^t \Rightarrow \boxed{y = 40000(0.9)^t}$

Graph: On the graphing calculator, graph $y = 40000(0.9)^x$ and $y = 17200$. Find the point of intersection.



There will be 17,200 homes not connected to the sewer system after about 8 years.

You Try: A new car costs $23,000. The value decreases by 15% each year. Write an exponential decay model for the car's value. Use the model to estimate the value of the car after 3 years.

QOD: Describe the end behaviors of an exponential decay function.

Skill: Simplify an expression involving real-number exponents using laws of exponents.

Review: Properties of Exponents (Allow students to come up with these on their own.) We will now extend these properties for use with logarithms.

Let *a* and *b* be real numbers, and let *m* and *n* be integers.

Product of Powers Property $a^m \cdot a^n = a^{m+n}$

Quotient of Powers Property $\frac{a^m}{a^n} = a^{m-n}$ or $\frac{a^m}{a^n} = \frac{1}{a^{n-m}}, a \neq 0$

Power of a Power Property $\left(a^m\right)^n = a^{mn}$

Skill: Solve exponential equations including.

Solving Exponential Equations

Method 1: Rewrite both sides of the equation so that they have the same base.

Note: If $b^x = b^y$, then $x = y$.

Ex 8: Solve the equation $8^x = 4^{x-1}$.

Rewrite both sides with a base of 2. (Note: $8 = 2^3$ and $4 = 2^2$) $\left(2^3\right)^x = \left(2^2\right)^{x-1}$

Use the power of a power property. $2^{3x} = 2^{2x-2}$

Equate the exponents and solve for *x*. $3x = 2x - 2$

$\boxed{x = -2}$

Check the solution by substituting into the original equation.

$$(8)^{-2} = (4)^{-2-1}$$

$$\frac{1}{64} = 4^{-3}$$

True

Method 2: Taking a logarithm of both sides.

Ex 9: Solve the equation $2^x = 8$.

Rewrite each side using a common base. $2^x = 2^3$

If the bases are equal, the exponents must be equal $x = 3$

Ex 10: Solve the equation $3^{5x+6} = 9^x$.

Rewrite each side using a common base..

$$3^{5x+6} = (3^2)^x$$

$$3^{5x+6} = 3^{2x}$$

If the bases are equal, the exponents must be equal.

$$5x + 6 = 2x$$

$$3x = -6$$

$$x = -2$$

Check the solution.

Skill: Develop mathematical models using exponential or logarithmic equations to solve real world problems.

Writing an Exponential Function: two points determine a unique exponential function $y = ab^x$

Ex 11: Write an exponential function $y = ab^x$ whose graph passes through $(1,7)$ and $(3,63)$.

Substitute each ordered pair in for *x* and *y* in the equation $y = ab^x$: $7 = ab^1$ $63 = ab^3$

To eliminate *a*, divide the two equations. Put the highest power of *b* on top: $\dfrac{63 = ab^3}{7 = ab^1} \Rightarrow 9 = b^2$

Solve for *b*: (Note: In an exponential function, *b* cannot be negative.) $b = 3$

Substitute this value of b into one of the original equations and solve for *a*: $7 = a(3)^1 \Rightarrow a = \dfrac{7}{3}$

Exponential Function: $\boxed{y = \dfrac{7}{3} \cdot 3^x}$

Writing a Power Function: two points determine a unique power function $y = ax^b$

Ex 12: Write a power function $y = ax^b$ whose graph passes through $(3,8)$ and $(9,12)$.

Substitute each ordered pair in for *x* and *y* in the equation $y = ax^b$: $8 = a(3)^b$ $12 = a(9)^b$

Solve one of the equations for *a*: $8 = a(3)^b \Rightarrow a = \dfrac{8}{3^b}$

Substitute the expression into the other equation for *a*: $12 = \left(\dfrac{8}{3^b}\right)9^b$

Use the power of a quotient property to simplify: $12 = 8 \cdot \left(\dfrac{9}{3}\right)^b \Rightarrow 12 = 8 \cdot 3^b$

Solve for *b* using logarithms: $1.5 = 3^b \Rightarrow b = \log_3 1.5 = \dfrac{\log 1.5}{\log 3} \approx 0.369$

Substitute this value in for *b* to solve for *a*: $a = \dfrac{8}{3^b} \approx \dfrac{8}{3^{0.369}} \approx 5.33$

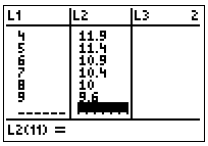
Power Function: $\boxed{y = 5.33x^{0.369}}$

Using Exponential and Power Models

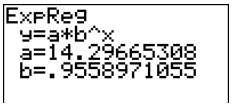
Ex 13: Find an exponential model to fit the data. Use the model to estimate *y* when *x* is 15.

x	0	1	2	3	4	5	6	7	8	9
y	14.7	13.5	12.9	12.4	11.9	11.4	10.9	10.4	10.0	9.6

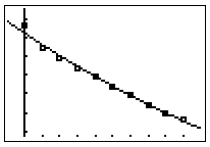
Enter the *x*-values into L1 and the *y*-values into L2 (Go to STAT – Edit)



On the Home screen, go to STAT – CALC and choose option 0, ExpReg. Then type Y1 (found in the VARS menu), and press Enter. This will calculate the exponential regression model and store it in Y1.



Graph the scatter plot along with the exponential regression equation to see if the model fits the data. Use ZoomStat.

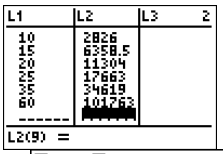


Exponential Model: $\boxed{y = 14.297 \cdot 0.956^{x}}$ When *x* is 15, $y = 14.297 \cdot 0.956^{15} \approx \boxed{7.28}$

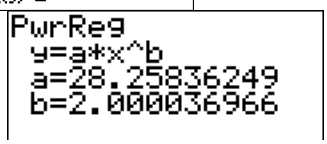
Ex 14: The ordered pairs (t, r) describe the circular area *r* (square feet) that oil from a leaking oil tanker covers *t* minutes after it begins leaking. Find a power model for the data. Use the model to estimate the area that will be covered by the leaking oil after 2 hours.

t	1	5	10	15	20	25	35	60
r	28.26	706.5	2826	6358.5	11304	17663	34618.5	101736

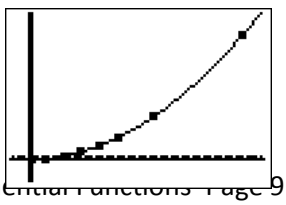
Enter the *t*-values into L1 and the r-values into L2 (Go to STAT – Edit)



On the Home screen, go to STAT – CALC and choose option A, PwrReg. Then type Y1 (found in the VARS menu), and press Enter. This will calculate the power regression model and store it in Y1.



Graph the scatter plot along with the exponential regression equation to see if the model fits the data. Use ZoomStat.



Power Model: $\boxed{y = 28.258x^2}$

After 2 hours, *t* is 120 minutes, $y = 28.258(120)^2 \approx \boxed{406915.2 \text{ ft}^2}$

You Try:

1. Write an exponential function of the form $y = ab^x$ that passes through the points $(2,6)$ and $(3,8)$. Check your answer using the exponential regression function on the graphing calculator.

2. Write a power function of the form $y = ax^b$ that passes through the points $(0.5,1)$ and $(10,150)$. Check your answer using the power regression function on the graphing calculator.

QOD: Which values are constant in an exponential function? In a power function?