The Binomial Theorem Notes

Patterns: Many important mathematical discoveries have begun with the study of patterns. Note the pattern below.

$$(a+b)^0 = 1$$
$$(a+b)^1 = a+b$$
$$(a+b)^2 = a^2+2ab+b^2$$
$$(a+b)^3 = a^3+3a^2b+3ab^2+b^3$$
$$(a+b)^4 = a^4+4a^3b+6a^2b^2+4ab^3+b^4$$
$$(a+b)^5 = a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5$$

Can you predict what the expansion of $(a+b)^6$ will look like?

$$\binom{0}{0}$$
$$\binom{1}{0} \quad \binom{1}{1}$$
$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$
$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$
$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$
$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$

This is where the coefficients come from. Now you should be able to find the sixth row if you did not see the pattern! This leads us to the definition of the Binomial Coefficient.

Binomial Coefficient:

The binomial coefficients that appear in the expansion of $(a+b)^n$ are the values of ${}_nC_r$ for $r = 0, 1, 2, 3, \ldots, n$.

A classical notation for ${}_nC_r$, especially in the context of binomial coefficients, is $\binom{n}{r}$. Both notations are read "n choose r."

Table Trick:

Recursion Formula For Coefficients:

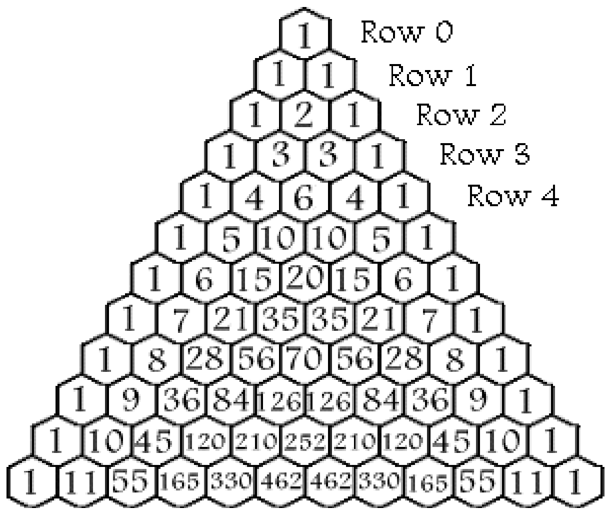
$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r} \text{ or, equivalently, } {}_nC_r = {}_{n-1}C_{r-1} + {}_{n-1}C_r$$

Pascal's Triangle: If we eliminate the variables and the signs and just look at the binomial coefficients, we get Pascal's Triangle.

The Binomial Theorem can be stated as:

$$(a + b)^n = a^n + na^{n-1}b^1 + \frac{n(n-1)}{2}a^{n-2}b^2 + \ldots\ldots + b^n$$

The co-efficients generated by expanding binomials of the form $(a + b)^n$ **can be shown in the form of a symmetrical triangle:**



The Binomial Theorem:

For any positive integer n,

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \cdots + \binom{n}{r}a^{n-r}b^r + \cdots + \binom{n}{n}b^n,$$

where

$$\binom{n}{r} = {}_nC_r = \frac{n!}{r!(n-r)!}.$$

Another way of writing the binomial theorem in summation form is:

$$(a+b)^n = \sum_{r=0}^{n}\binom{n}{r}a^{n-r}b^r$$

Basic Factorial Identities:

For any integer $n \geq 1$, $n! = n(n-1)!$

For any integer $n \geq 0$, $(n+1)! = (n+1)n!$

Example 1: Expand $(p+q)^8$.

Example 2: Find the coefficient of the x^4 term. $(x-2)^{12}$

Example 3: Expand $(2y-3x)^5$.

Example 4: Find the fifth term of $(x+3y)^8$.

Example 5: Prove $\binom{n}{1} = \binom{n}{n-1} = n$ for $n \geq 1$

Exploring Pascal's Triangle: By looking at patterns in Pascal's triangle, guess the answer to the following questions:

a) What number appears the least number of times?

b) What number appears the greatest number of times?

c) Is there any positive integer that does NOT appear in Pascal's triangle?

d) If you go along any row alternately adding and subtracting the numbers, what is the result?

e) If p is a prime number, what do all the interior numbers along the *pth* row have in common?

f) Which rows have all even interior numbers?

g) Which rows have all odd numbers?