Understanding Math

Southern Nevada
Regional Professional Development Program

Graphing Functions with Asymptotic Lines – Rational Functions

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When students learn how to graph rational functions, *asymptotes* are certain to be a point of discussion. Different types of asymptotes are explored along with their relevant characteristics. Unfortunately, there are some aspects of asymptotes that our students fail to grasp. We will examine those details in the paragraphs below.

Any discussion must begin with the definition of asymptote: *An* ***asymptote*** *is a line related to a given curve such that the distance between the line and a point on the curve approaches zero as the distance of the point from the origin increases without bound.* More informally, as we "travel" along the curve, the curve gets closer and closer to the line but does not touch it. We then classify asymptotes as vertical, horizontal, or slant/oblique, with each having its own characteristics.

The first thing that students don't always get is that slant asymptotes are very similar to horizontal asymptotes. With horizontal asymptotes we examine the end behavior of the function; what it does as *x* gets "very large" in either direction, that is, $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$. For example, if $f(x)=\frac{2x+1}{x}$, then $\lim_{x\to\infty}\frac{2x+1}{x}=2$ and $\lim_{x\to-\infty}\frac{2x+1}{x}=2$, thus $f(x)=\frac{2x+1}{x}$ has a horizontal asymptote of $y=2$. On the other hand, if $f(x)=\frac{2x^2+1}{x}$, then $\lim_{x\to\infty}\frac{2x^2+1}{x}=\infty$ and $\lim_{x\to-\infty}\frac{2x^2+1}{x}=\infty$, thus $f(x)=\frac{2x^2+1}{x}$ has no horizontal asymptotes. However, it *does* have a *slant* asymptote. As *x* gets "very large", $f(x)$ is just a tiny bit more than 2*x*, with that "tiny bit" decreasing as *x* increases.. (See Figure 1.)

$f(x)=\frac{2x^2+1}{x}$ therefore has a slant asymptote of $y=2x$.

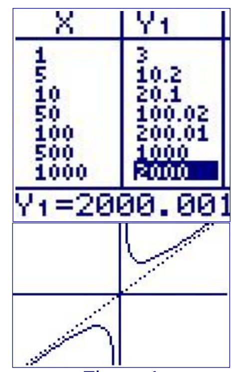

Figure 1

The next detail that students miss is that a function may have different horizontal asymptotes for positive and negative values of *x*. Consider the function $f(x)=\frac{|2x+1|}{x}$: $\lim_{x\to\infty}\frac{|2x+1|}{x}=2$ and $\lim_{x\to-\infty}\frac{|2x+1|}{x}=-2$. Thus, $f(x)$ has a horizontal asymptote of $y=2$ when $x > 0$ and $y=-2$ when $x < 0$. (See Figure 2.)

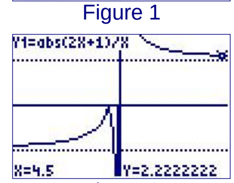

Figure 2

The last point students don't get is the misconception that a graph can't cross an asymptote. For vertical asymptotes, this is true, as the function is undefined where they occur. However, a function can definitely cross a horizontal asymptote. Take a look at Figures 3 and 4, of $f(x)=\frac{2x^2+2x+1}{x^2-1}$ along with its horizontal asymptote $y=2$. We can see that function crosses the horizontal asymptote at $x = -1.5$. How can this be? The definition of asymptote does not allow the curve to touch the asymptote, not to mention crossing it!


Figure 3

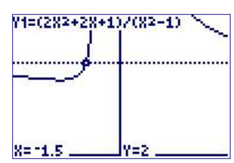

Figure 4

There is no contradiction here. Horizontal asymptotes are important as we consider values of *x* that are "very large," again, $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$. We really don't care about the horizontal asymptotes when *x* is "small." This characteristic of curves crossing horizontal asymptotes also applies to slant asymptotes, that is, the curve may "break through" the asymptote from one side and approach it from the other as *x* increases without bound.

One final note is about how we ask students to identify special points on curves—intercepts, extrema, etc. In the example above, there is a local minimum at $\left(\frac{-3-\sqrt{5}}{2},\frac{1+\sqrt{5}}{2}\right)$ that students should identify. (See Figure 5.)

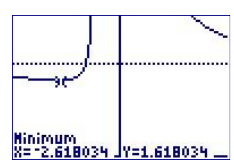

Figure 5