Applying Radical Functions

Math Background

Previously, you
- Found inverses of linear and quadratic functions
- Worked with composition of functions and used them to verify inverses
- Graphed linear and quadratic piecewise functions
- Created and solved linear, quadratic and polynomial functions

In this unit you will
- Find the inverse of square root and cubic functions
- Verify functions are inverses by composition
- Graph piecewise functions with radical portions with and without technology
- Create and solve problems involving radical equations and inequalities

You can use the skills in this unit to
- Determine if functions are inverses.
- Explain a situation of a real-life square root or cube root function.
- Create square root and cube root functions to model and solve real-world problems.

Vocabulary
- **Composite Function** – The result of composing two functions together so that the output of the first becomes the input of the second.
- **Cube Root Function** – A function whose value is the cube root of its argument. $f(x)=\sqrt[3]{x}$
- **Inverse functions** – Two functions are inverse functions if the domain of the original function matches the range of the second function.
- **One-to-One Functions** – A function whose inverse is a function. Both must pass the vertical and horizontal line tests.
- **Piecewise Function –** A function defined piecewise, that is $f(x)$ is given by different expressions on various intervals.
- **Square Root Function** – A function that maps the set of non-negative real numbers onto itself and when graphed is a half of a parabola with a vertical directrix.

Essential Questions

- How can we verify that two functions are inverses of each other? What is a composite function and why is it so important?
- What do the key features of the graphs of square root and cube roots tell you about the function?
- When do you use an equation versus an inequality? How do you determine which relationship to use for the model? How can we model applications and solve specific problems? How do the answers apply in context?

Overall Big Ideas

Composite functions are common representation of real life situations and are used whenever a change in one quantity produces a change in another, which in turn produces a third quantity.

The graph shows the solutions for the functions illustrating domain and range.

Variable equations or inequalities model real-life situations and generalize applications, building a foundation for solving equations and inequalities with more than one variable.

Skill

To find the inverse of a radical function and verify it by composition.

To graph piecewise functions including radical portions.

To create and solve radical equations and inequalities.

Related Standards

F.BF.B.4b
Verify by composition that one function is the inverse of another.

A.CED.A.4-2
Rearrange formulas of all types to highlight a quantity of interest, using the same reasoning as in solving equations. For example, rearrange Ohm's law V = IR to highlight resistance R. *(Modeling Standard)

F.IF.B.5-2
Relate the domain of any function to its graph and, where applicable, to the quantitative relationship it describes. For example, if the function *h(n)* gives the number of person-hours it takes to assemble *n* engines in a factory, then the positive integers would be an appropriate domain for the function. *(Modeling Standard)

F.BF.B.4a-2
Solve an equation of the form $f(x)=c$ for a simple radical, rational, power, or exponential function *f* that has an inverse and write an expression for the inverse. For example, $f(x)=2x^3$ or $f(x)=\frac{x+1}{x-1}$ for $x\neq 1$.

F.IF.C.7b-2
Graph square root and cube root functions. *(Modeling Standard)

A.CED.A.1-2
Create equations and inequalities in one variable and use them to solve problems. Include equations arising from all types of functions, including simple rational and radical functions. *(Modeling Standard)

Notes, Examples, and Exam Questions

Review Inverse relations and functions:

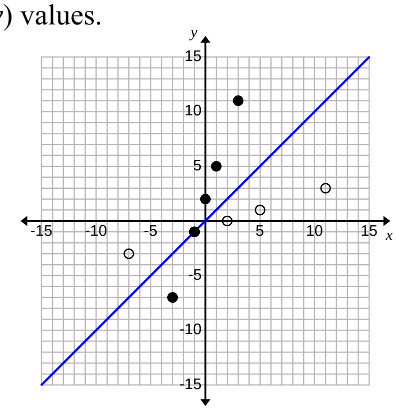
Inverse Relation: a mapping of the output values of a relation to its input values

Ex: Find the inverse relation of the relation.

x	-3	-1	0	1	3
y	-7	-1	2	5	11

To find the inverse relation, we will switch the input (x) values and the output (y) values.

x	-7	-1	2	5	11
y	-3	-1	0	1	3



Note: Looking at the graph of the relation (solid points) and its inverse relation (open points), we can see that the inverse relation includes all of the points reflected over the line $y = x$.

Finding the Equation of an Inverse Relation: recall that the inverse of a relation is its reflection over the line $y = x$. Therefore, to find the equation of an inverse relation, we will reverse the x and y variables and solve for y.

Note: If both the relation and its inverse relation are functions, then the two relations are called **inverse functions**.

Notation for Inverse Functions: The inverse of a function f is denoted f^{-1}.

Caution: **This is not to be confused with the exponent -1!!**

Ex: Find the equation of the inverse function of $f(x) = 3x + 2$.

Step One: Switch the x and y variables. Note: $f(x) = y$.

$$x = 3y + 2$$

Step Two: Solve for y.

$$x - 2 = 3y$$

$$\frac{1}{3}x - \frac{2}{3} = y$$

Step Three: Write in inverse notation.

$$\boxed{f^{-1}(x) = \frac{1}{3}x - \frac{2}{3}}$$

We can verify our answer using the graph of the ordered pairs, as in our first example. However, a function and its inverse have another special relationship.

Verifying Inverse Functions: To verify that two functions are inverses, we must show that $f\left(f^{-1}(x)\right) = f^{-1}\left(f(x)\right) = x$. The function $y = x$ is the identity function, so the composition of a function and its inverse is the identity.

Ex: Show algebraically and graphically that the functions $f(x) = 3x + 2$ and $f^{-1}(x) = \frac{1}{3}x - \frac{2}{3}$ are inverses.

Step One: Show that $f\left(f^{-1}(x)\right) = x$.

$$f\left(f^{-1}(x)\right) = 3\left(\frac{1}{3}x - \frac{2}{3}\right) + 2$$
$$f\left(f^{-1}(x)\right) = x - 2 + 2$$
$$f\left(f^{-1}(x)\right) = x$$

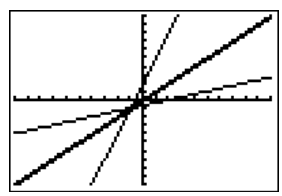
Step Two: Show that $f^{-1}\left(f(x)\right) = x$.

$$f^{-1}\left(f(x)\right) = \frac{1}{3}(3x + 2) - \frac{2}{3}$$
$$f^{-1}\left(f(x)\right) = x + \frac{2}{3} - \frac{2}{3}$$
$$f^{-1}\left(f(x)\right) = x$$

Step Three: Graph the functions and show that they are a reflection over the line $y = x$.

In this graph, the line $y = x$ is bold.

We have shown that these functions are inverses.



Inverses of Radical Expressions Unit 5.6

Ex 1: Find the inverse of the function $y = x^2$ for $x \ge 0$.

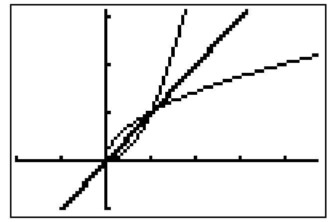
Step One: Switch the *x* and *y*. $x = y^2$

Step Two: Solve for *y*. $y = \pm\sqrt{x}$

Because $x \ge 0$, we only need the positive square root. $y = \sqrt{x}$

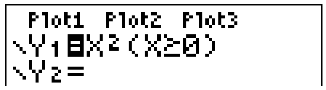
Step Three: Rewrite in inverse notation. $\boxed{y^{-1} = \sqrt{x}}$

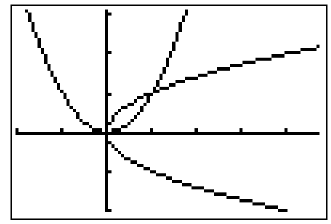
*****Note: The inverse of a quadratic function on the restricted domain of $x \ge 0$ is a square root function.**



Let's take a look at the graphs of these two functions. They are reflections of each other over the line $y = x$.

Calculator Note: To graph $y = x^2$ on its restricted domain, use parentheses after the function.



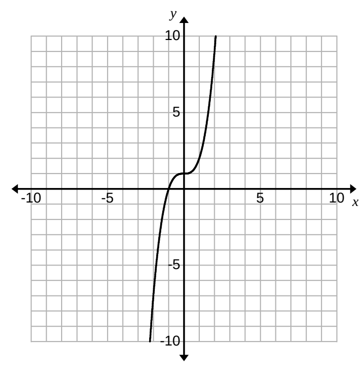


What if the domain of $y = x^2$ was not restricted? Let's take a look at the graphs. You can see that the inverse of $y = x^2$ is NOT a function.

By looking at a function's graph, we can see if it has an inverse function using the Horizontal Line Test.

Horizontal Line Test: If a horizontal line intersects the graph of a function *f* not more than once, then the inverse of *f* is a function.

Ex 2: Determine whether $y = x^3 + 1$ is a function. If it is, determine if it has an inverse function. If it does, find the inverse function and graph to verify.



Step One: Look at the graph of $y = x^3 + 1$ and use the vertical line test to verify it is a function.

It passes the vertical line test, so it is a function.

Step Two: Look at the graph of $y = x^3 + 1$ and use the horizontal line test to verify it has an inverse function.

It passes the horizontal line test, so it has an inverse function.

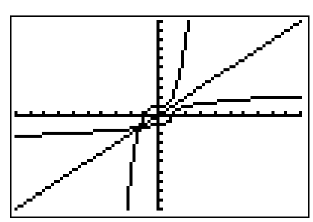
Step Three: Find the inverse function.

$$x = y^3 + 1$$
$$x - 1 = y^3$$
$$\sqrt[3]{x-1} = y$$

$$y^{-1} = \sqrt[3]{x-1}$$
$$\boxed{f^{-1}(x) = \sqrt[3]{x-1}}$$

Step Four: Graph the two functions.

The inverse is a function, and it is the reflection of the original function over the line $y = x$.



*****Note: The inverse of a cubic function is a cube root function.**

Ex 3: Find the inverse of the function $f(x)=\sqrt[3]{x-1}+5$.

Step One: Switch the *x* and *y* variables. Note: $f\left(x\right)=y$.

$$x=\sqrt[3]{y-1}+5$$

Step Two: Solve for *y*.

$$x-5=\sqrt[3]{y-1}$$
$$\left(x-5\right)^3=\left(\sqrt[3]{y-1}\right)^3$$
$$\left(x-5\right)^3=y-1$$
$$\left(x-5\right)^3+1=y$$

Step Three: Write in inverse notation.

$$\boxed{f^{-1}\left(x\right)=(x-5)^3+1}$$

Ex 4: Find the inverse of the function $f(x)=2\sqrt{x-2}+3$ and verify it by composition.

Step One: Switch the *x* and *y* variables. Note: $f\left(x\right)=y$.

$$x=2\sqrt{y-2}+3$$

Step Two: Solve for *y*.

$$x-3=2\sqrt{y-2}$$
$$\frac{x-3}{2}=\sqrt{y-2}$$
$$\left(\frac{x-3}{2}\right)^2=y-2$$
$$\left(\frac{x-3}{2}\right)^2+2=y$$

Step Three: Write in inverse notation.

$$\boxed{f^{-1}\left(x\right)=\left(\frac{x-3}{2}\right)^2+2}$$

Step Four: Verify by showing that $f\left(f^{-1}\left(x\right)\right)=x$.

$$f\left(f^{-1}\left(x\right)\right)=2\sqrt{\left(\frac{x-3}{2}\right)^2+2-2}+3$$
$$f\left(f^{-1}\left(x\right)\right)=2\left(\frac{x-3}{2}\right)+3$$
$$f\left(f^{-1}\left(x\right)\right)=x-3+3$$
$$f\left(f^{-1}\left(x\right)\right)=x$$

Step Five: Verify by showing that $f^{-1}\left(f\left(x\right)\right)=x$.

$$f^{-1}\left(f\left(x\right)\right)=\left(\frac{2\sqrt{x-2}+3-3}{2}\right)^2+2$$

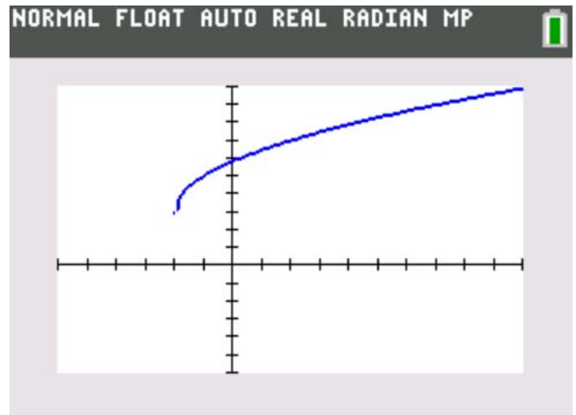
$$f^{-1}\left(f\left(x\right)\right)=\frac{4(x-2)}{4}+2$$

$$f^{-1}\left(f\left(x\right)\right)=x-2+2$$

$$f^{-1}\left(f\left(x\right)\right)=x$$

Graphing Inverses on the Graphing Calculator

Your calculator cannot find the inverse of a function, but it can draw the inverse.

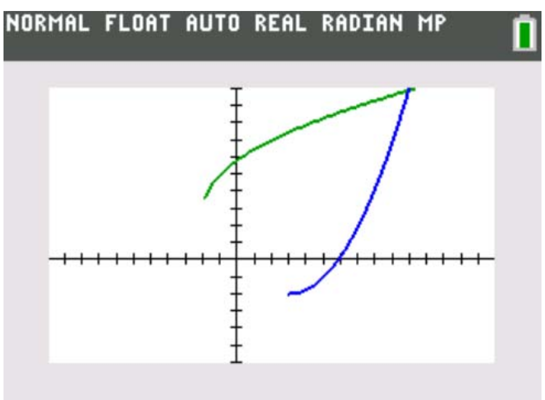


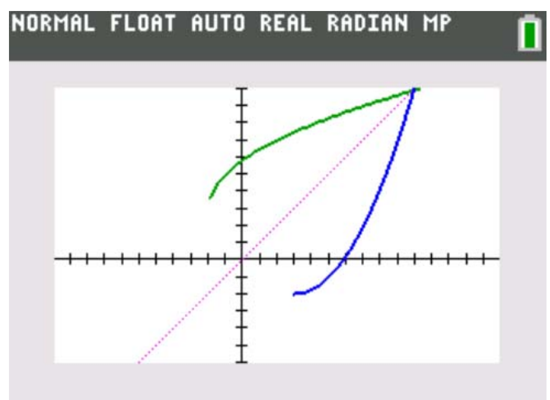
Ex 5: Use a graphing calculator to graph the inverse of $y=2\sqrt{x+2}+3$.

Step One: Enter the function into Y1.

Step Two: On the home screen choose 8:DrawInv from the Draw menu, then choose Y1 from the VARS menu. Press Enter, and the calculator will automatically draw the inverse.

Is the inverse a function? Did you know before the calculator graphed it?





**The dotted line is the graph of $y=x$. Recall that inverse of a function is reflected over the line $y=x$.

Ex 6: Find the inverse of the function $f\left(x\right)=2-2x^2, x\le 0$. Verify that these are inverses algebraically. Is the inverse a function?

Step 1: Switch the *x* and *y* variables. $x=2-2y^2$

Step 2: Solve for *y*.

$$x-2=-2y^2$$
$$\frac{x-2}{-2}=y^2$$
$$\pm\sqrt{\frac{-x+2}{2}}=y$$

Step 3: Write in inverse notation.

Because the domain is restricted to $x \le 0$, we choose the negative option. $\boxed{f^{-1}(x)=-\sqrt{\frac{-x+2}{2}}}$

Step 4: Verify $f\left(f^{-1}(x)\right)=x$.

$$f\left(f^{-1}(x)\right)=2-2\left(-\sqrt{\frac{-x+2}{2}}\right)^2$$
$$f\left(f^{-1}(x)\right)=2-2\left(\frac{-x+2}{2}\right)$$
$$f\left(f^{-1}(x)\right)=2-(-x+2)$$
$$f\left(f^{-1}(x)\right)=2+x-2=x$$

Step 5: Verify $f^{-1}\left(f(x)\right)=x$.

$$f^{-1}\left(f(x)\right)=-\sqrt{\frac{-\left(2-2x^2\right)+2}{2}}$$
$$f^{-1}\left(f(x)\right)=-\sqrt{\frac{-2+2x^2+2}{2}}$$
$$f^{-1}\left(f(x)\right)=-\sqrt{\frac{2x^2}{2}}=-\sqrt{x^2}$$
$$f^{-1}\left(f(x)\right)=x$$

Since the domain of *f(x)* is restricted, the inverse is a function.

QOD: What is the difference between the vertical and horizontal line tests? Explain why each test is used in each case.

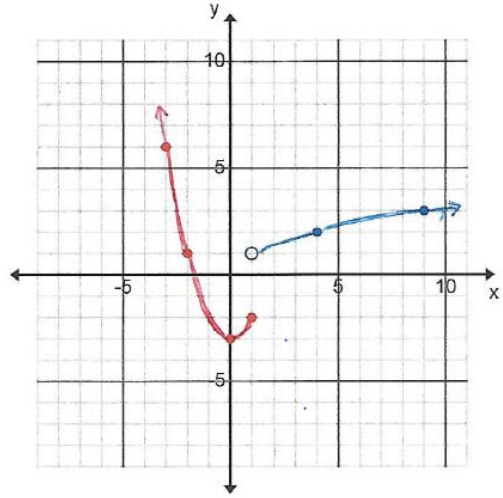
Graphing Piecewise Functions with Radical Portions – Unit 5.7

Piecewise functions are just what they are named: pieces of different functions all on one graph. The easiest way to think of them is if you drew more than one function on a graph, and you just erased parts of the functions where they aren't supposed to be (along the *x*'s); they are defined differently for different intervals of *x*. So *y* is defined differently for different values of *x*; we use the *x* to look up what interval it's in, so we can find out what the *y* is supposed to be. The piecewise functions in this unit will include radical portions.

Ex 7: Graph the function: $f(x)=\begin{cases}\sqrt{x} & x>1\\ x^2-3 & x\le 1\end{cases}$

There are two portions. We have a quadratic (parabola) and a square root. To get a nice graph, a T-table approach would be a good way of approaching the problem. Use x-values that are appropriate for each portion. Find "nice" values when dealing with the radical portion.

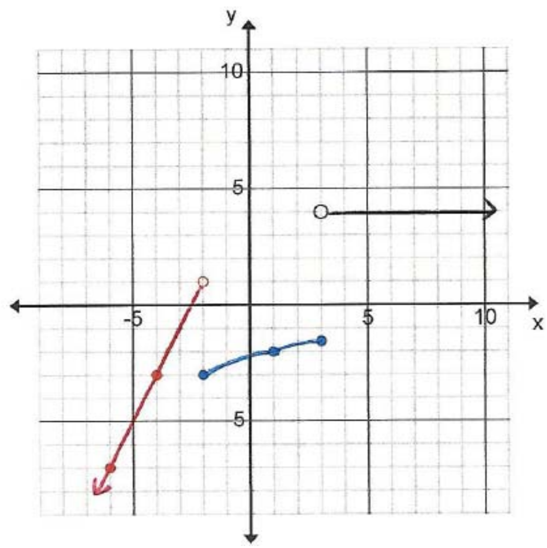
$f(x)=\sqrt{x}$	$f(x)=x^2-3$
(1,1)	(1,-2)
(4,2)	(0,-3)
(9,3)	(-2,1)
	(-3,6)



Ex 8: Graph the function: $f(x)=\begin{cases}2x+5 & x<-2\\ \sqrt{x+3}-1 & -2\le x\le 3\\ 4 & x>3\end{cases}$

There are three portions. We have two linear functions and a square root.

$f(x)=2x+5$	$f(x)=\sqrt{x+3}-4$	$f(x)=4$
(-2,1)	(-2,-3)	(3,4)
(-4,-3)	(1,-2)	(5,4)
(-6,-7)	(3,-1.55)	(7,4)



Creating and Solving Radical Functions – Unit 5.8

Powers, roots, and radicals are found in many real-life situations. We will explore several.

Note: Often, real-life applications do not yield "nice" numbers. We will use our calculator to approximate our results in these problems.

Solving Equations Using *n*th Roots

Ex 9: A basketball has a volume of about 455.6 cubic inches. The formula for the volume of a basketball is $V = \frac{4}{3}\pi r^3$. Find the radius of the basketball.

Substitute the known value(s) into the formula. $V \approx 455.6$ $\quad 455.6 = \frac{4}{3}\pi r^3$

Solve for the remaining variable. $\frac{3}{4}\cdot 455.6 = \frac{3}{4}\cdot\frac{4}{3}\pi r^3$ $\quad \sqrt[3]{\frac{341.7}{\pi}} = \sqrt[3]{r^3}$

$341.7 = \pi r^3$ $\quad 4.773 \approx r$

**Calculator Note: Be sure not to round too soon in the problem. Type in the expression $\sqrt[3]{\frac{341.7}{\pi}}$. Do not use an approximation (i.e. 3.14) for π.

Be sure to answer the question and use appropriate units. The radius is **4.773 inches**.

Operations with Functions

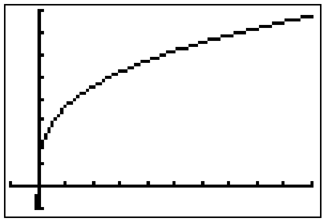
Ex 10: A professor performs an experiment on bacteria and finds that the growth rate *G* of the bacteria can be modeled by $G(t) = 82t^{0.25}$, and that the death rate *D* is $D(t) = 10.8t^{0.25}$, where *t* is the time in hours. Find an expression for the number *N* of bacterial living at a time *t*.

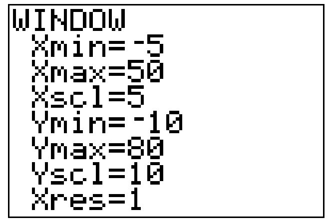
The expression for the number of bacteria living will equal the number of bacteria growing minus the number of bacteria dying. $N(t) = G(t) - D(t)$

Using the expressions given, we have $N(t) = 82t^{0.25} - 10.8t^{0.25}$ or $\boxed{N(t) = 71.2t^{0.25}}$

Ex 11: The length of a whale can be modeled by $L = 21.04\sqrt[3]{W}$, where L is the length in feet, and W is the weight in tons. Graph the model, then use the graph to find the weight of a whale that is 60 ft long.

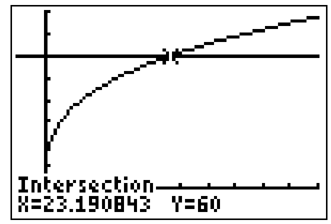
Graph the model in Y_1.





Graph the function $L = 60$ in Y_2 and find the point of intersection to solve the problem.

The weight of the whale is approximately **23.19 tons**.



<u>You Try</u>: The strings of guitars and pianos are under tension. The speed v of a wave on the string depends on the force (tension) F on the string and the mass M per unit length L according to the formula $v = \sqrt{\frac{F \cdot L}{M}}$. A wave travels through a string with a mass of 0.2 kilograms at a speed of 9 meters per second. It is stretched by a force of 19.6 Newtons. Find the length of the string.

Ex 12: A biologist is studying two species of animals in a habitat. The population, p_1, of one of the species is growing according to $p_1 = 500t^{\frac{3}{2}}$ and the population, p_2, of the other species is growing according to $p \ = 100t^2$ where time, t, is measured in years. After how many years will the two species be equal?

<u>Solve Algebraically:</u>

Set the equations equal to each other. $500t^{\frac{3}{2}} = 100t^2$

$$\frac{500t^{\frac{3}{2}}}{100t^2} = \frac{100t^2}{100t^2}$$

Divide and use exponent rules. $5t^{\frac{3}{2}-\frac{4}{2}} = 1$

$$5t^{-\frac{1}{2}} = 1$$

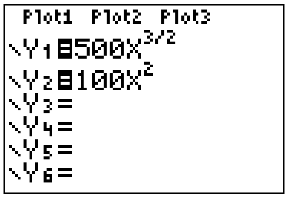
$$\frac{5t^{-\frac{1}{2}}}{5} = \frac{1}{5}$$

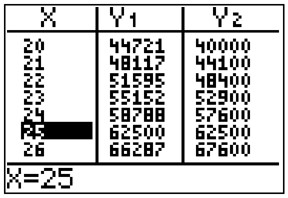
Solve for t. $\left(t^{-\frac{1}{2}}\right)^{-2} = \left(\frac{1}{5}\right)^{-2}$

$\boxed{t = 25 \text{ years}}$

Solve Graphically:

Input the function p_1 into Y_1, input function p_2 into Y_2 and find the point of intersection. Use the Table to find the shared ordered pair.





Note that when $t = 25$, both functions equal 62,500. So, after 25 years, the two populations will be the same at 62,500.

Ex 13: The distance a person can see on a clear day is estimated by the formula $V = \dfrac{1225\sqrt{a}}{1000}$, where v = visibility (in miles) and a = altitude. If Jerome can see for 150 miles, about how far above the ground is he?

Step 1: Substitute 150 miles in for visibility. $150 = \dfrac{1225\sqrt{a}}{1000}$

Step 2: Multiply by the reciprocal. $\left(\dfrac{1000}{1225}\right)150 = \dfrac{1225\sqrt{a}}{1000}\left(\dfrac{1000}{1225}\right)$

Step 3: Square both sides. $\left(\dfrac{6000}{49}\right)^2 = \left(\sqrt{a}\right)^2$

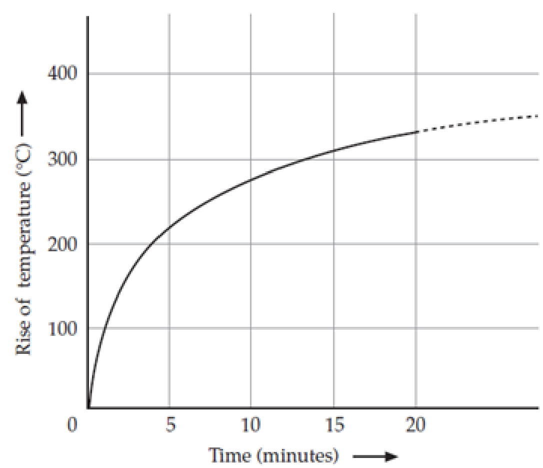
$\boxed{14993.7526 = a}$

Jerome is about **14,993.75 miles** above the ground.

Ex 14: The graph below shows the change in temperature of a burning house over time.

a) Describe the graph.

The graph looks like the square root function. It passes through (0,0) and increases as the domain increases.



b) This graph was found in an old math book and next to it was written:

Rise of temperature = $t^{0.25}$
Show that this function does not describe the graph correctly.

Substitute values into the function $t^{\frac{1}{4}}$.

$1 \rightarrow 1^{\frac{1}{4}} = 1,\ 16 \rightarrow 16^{\frac{1}{4}} = 2,\ 1 \rightarrow 81^{\frac{1}{4}} = 3$. These coordinates are not on the graph, so the function does not describe the graph correctly.

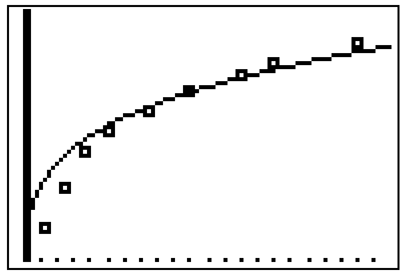
c) Assume that the power function $r = At^{0.25}$ is a good description of the graph. Find a reasonable value for A. Graph the new function.

Use an ordered pair from the graph and substitute in. Estimate the rise of temperature when the time is 10 minutes – about 275. Substitute the variables into the equation and solve for A.

$275 = A(10)^{.25}$

$275 = 1.778A$ $\quad r = 155t^{0.25}$ is a reasonable approximation of the graph.

$A \approx 154.64$

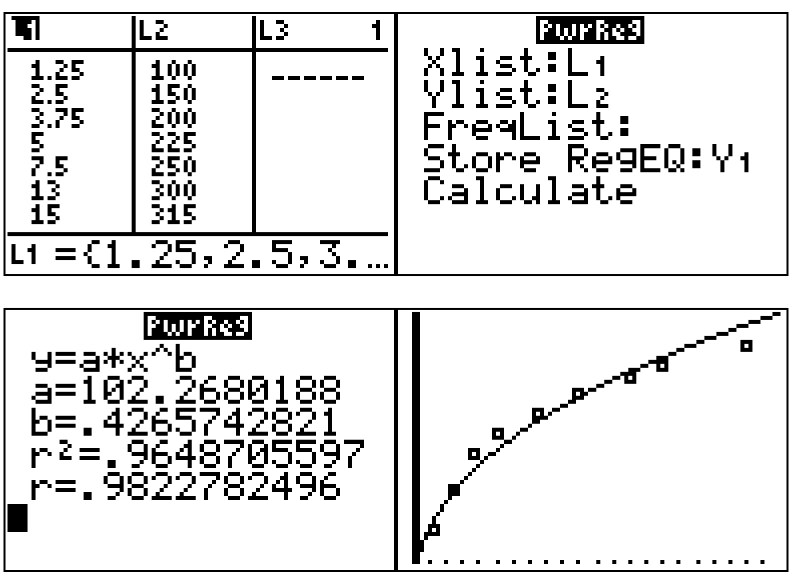


d) Compare the graph in part (c) to the original one. Do you think that a different power of t might result in a better model? Would a larger or smaller power produce a better fit? Explain.

The graph of $r = 155t^{0.25}$ is too steep in the beginning and too flat for the bigger values of t. Therefore, a power that is a little bit bigger might produce a better fit.

e) Use the original graph to find data. Carry out a power regression on the data to find a function that would produce a better fit.

Depending on the estimated ordered pairs, here is one possible answer: $r = 102.27t^{0.43}$



SAMPLE EXAM QUESTIONS

1. **Which is the inverse of $f(x)=(2x+1)^3-4$?**

   (A) $a(x)=\sqrt[3]{2x+1}+4$

   (B) $b(x)=\dfrac{\sqrt[3]{x+4}}{2}-1$

   (C) $a(x)=\dfrac{\sqrt[3]{x+4}-1}{2}$

   (D) $a(x)=\sqrt[3]{x+4}-\dfrac{1}{2}$

Ans: C

2. **Which is the inverse of the function $y=\dfrac{\sqrt{2x-3}}{3}$?**

   (A) $y=\dfrac{x^2+3}{2}$, where $x\ge 0$

   (B) $y=\dfrac{3x^2+3}{2}$, where $x\ge 0$

   (C) $y=\dfrac{9x^2+3}{2}$, where $x\ge 0$

   (D) $y=\dfrac{x^2}{2}$, where $x\ge 0$

Ans: C

3. **An animal population can be modeled over time by $P(t)=2t^{\frac{2}{3}}+10$, where t is measured in weeks. After how many weeks will the population be 18 animals?**

   (A) 8 (B) 12 (C) 23 (D) 27

Ans: A