Create, Graph and Solve Radical Functions

Math Background

Previously, you

- Worked with exponents and used exponent properties
- Evaluated expressions with exponents
- Identified the domain, range and x-intercepts of real-life functions
- Graphed linear, quadratic and polynomial functions
- Recognized special polynomials (perfect squares, difference of squares, sum and difference of cubes)
- Solved linear, quadratic and polynomial functions
- Transformed parent functions of linear, quadratic and polynomial functions

In this unit you will

- Use the properties of exponents to convert between radical notation and terms with rational exponents
- Evaluate expressions with rational exponents
- Graph square root and cube root functions with and without technology
- Solve problems involving radical equations and inequalities

You can use the skills in this unit to

- Use the structure of an expression to identify ways to rewrite it.
- Interpret the domain and its restrictions of a real-life function.
- Describe how a square root or cube root graph is related to its parent function.
- Model and solve real-world problems with square root and cube root functions using graphs and laws of exponents.
- Identify extraneous solutions

Vocabulary

- **Base** – A number that is used as a repeated factor. In 4^3, the "4" is the base.
- **Cube Root Function** – A function whose value is the cube root of its argument. $f(x)=\sqrt[3]{x}$
- **End Behavior** – The end behavior of a function is the behavior of the graph of *f(x)* as *x* approaches positive infinity or negative infinity.
- **Extraneous Solutions** – A root of a transformed equation that is not a root of the original equation because it was excluded from the domain of the original equation.
- **Exponent –** A number used to indicate the number of times a term is used as a factor to multiply itself. The exponent is normally placed as a superscript after the term.
- **Power –** A short way of writing the same number multiplied by itself several times, written as a^n. It includes the base and the exponent.
- **Radical –** The taking of a root of a number. The symbol is $\sqrt{\ }$.
- **Radical Equation** – An equation containing radical expressions with variables in the radicands.
- **Radicand –** A number or expression inside the radical symbol.

- **Root Index** – A number written as a superscript to the left of the radical symbol giving the n^{th} root to be found. If there is no index, it is implied to find the square root. If the index is a "3", it tells us to find the cube root of the expression.
- **Square Root Function** – A function that maps the set of non-negative real numbers onto itself and when graphed is a half of a parabola with a vertical directrix.

Essential Questions

- How can numbers with rational exponents be written in other notations?
- What do the key features of the graphs of square root and cube roots tell you about the function?
- How do I solve a radical equation? How are extraneous solutions generated from a radical equation?

Overall Big Ideas

Using the properties of exponents, we can rewrite radical expressions to exponential form and vice versa.

The graph shows the solutions for the functions illustrating domain and range.

We solve radical equations by transforming the equation into a simpler form to solve. However, this can produce solutions that do not exist in the original domain.

Skill

To use properties of rational exponents to re-write radicals.

To evaluate expressions with rational exponents.

To graph radical functions and inequalities.

To transform radical functions.

To solve radical equations and inequalities.

Related Standards

N.RN.A.2
Rewrite expressions involving radicals and rational exponents using the properties of exponents.

A.SSE.A.2-2
Use the structure of an expression, including polynomial and rational expressions, to identify ways to rewrite it. For example, see $x^4 - y^4$ as $\left(x^2\right)^2 - \left(y^2\right)^2$, thus recognizing it as a difference of squares that can be factored as $\left(x^2 - y^2\right)\left(x^2 + y^2\right)$.

F.IF.B.5-2
Relate the domain of any function to its graph and, where applicable, to the quantitative relationship it describes. For example, if the function *h(n)* gives the number of person-hours it takes to assemble *n* engines in a factory, then the positive integers would be an appropriate domain for the function. *(Modeling Standard)

F.IF.C.7b-2
Graph square root and cube root functions. *(Modeling Standard)

F.BF.B.5
Understand the inverse relationship between exponents and logarithms and use this relationship to solve problems involving logarithms and exponents.

F.BF.B.3-2
Identify the effect on the graph of replacing $f(x)$ by $f(x) + \mathrm{k}$, $k\, f(x)$, $f(\mathrm{k}x)$, and $f(x + \mathrm{k})$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. Include recognizing even and odd functions from their graphs and algebraic expressions for them. Include simple radical, rational, and exponential functions, note the effect of multiple transformations on a single graph, and emphasize common effects of transformations across function types.

A.REI.A.2
Solve simple rational and radical equations in one variable, and give examples showing how extraneous solutions may arise.

Notes, Examples, and Exam Questions

<u>REVIEW: Properties of Exponents</u> (Allow students to come up with these on their own.)

Let *a* and *b* be real numbers, and let *m* and *n* be integers.

Product of Powers Property $a^m \cdot a^n = a^{m+n}$

Quotient of Powers Property $\frac{a^m}{a^n} = a^{m-n}$ or $\frac{a^m}{a^n} = \frac{1}{a^{n-m}}, a \neq 0$

Power of a Power Property $\left(a^m\right)^n = a^{mn}$

Power of a Product Property $\left(ab\right)^m = a^m b^m$

Power of a Quotient Property $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}, b \neq 0$

Negative Exponent Property $a^{-m} = \frac{1}{a^m}$ or $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n, a \neq 0$

Zero Exponent Property $a^0 = 1, a \neq 0$

<u>Evaluating Numerical Expressions with Exponents</u>

Ex: Evaluate $\left(2^{-3}\right)^2$.

Use the power of a power property: $2^{-3\cdot 2} = 2^{-6}$

Use the negative exponent property: $\frac{1}{2^6} = \boxed{\frac{1}{64}}$

Ex: Evaluate $\left(\frac{2}{3}\right)^{-5}\left(\frac{2}{3}\right)^3$.

Use the power of a product property: $\left(\frac{2}{3}\right)^{-5+3} = \left(\frac{2}{3}\right)^{-2}$

Use the negative exponent property: $\left(\frac{2}{3}\right)^{-2} = \left(\frac{3}{2}\right)^2$

Use the power of a quotient property: $\frac{3^2}{2^2} = \boxed{\frac{9}{4}}$

Ex: Evaluate $\frac{5^3 \cdot 5^0}{5^6}$.

Use the zero exponent property: $\frac{5^3 \cdot 1}{5^6} = \frac{5^3}{5^6}$

Use the quotient of a power property: $5^{3-6} = 5^{-3}$

Use the negative exponent property: $\frac{1}{5^3} = \boxed{\frac{1}{125}}$

Note: We can use the quotient of a power property to keep the exponent positive.

Use the quotient of a power property: $\frac{1}{5^{6-3}} = \frac{1}{5^3} = \boxed{\frac{1}{125}}$

REVIEW Radicals:

n^{th} Root: If $b^n = a$, then *b* is the n^{th} root of *a*. This is written $\sqrt[n]{a} = b$. *n* is called the **index** of the radical. *a* is called the **radicand**.

Note: The square root has an implied index of 2.

Ex: $\sqrt{9} = \boxed{3}$, because $3^2 = 9$

Ex: $\sqrt[3]{8} = \boxed{2}$, because $2^3 = 8$

Multiple Roots

Ex: Find the real *n*th root(s) of *a* if *a* = 16 and *n* = 4.

$\pm\sqrt[4]{16} = \boxed{\pm 2}$, because $(-2)^4 = 16$ and $2^4 = 16$.

Evaluate Radical Expressions Units 5.1, 5.2

Roots as Rational Exponents: The n^{th} root, $\sqrt[n]{a}$, can be written as an exponent $a^{\frac{1}{n}}$.

$a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m$ - Notice the placement of the *m* and *n*. The root index is the denominator and the exponent is the numerator.

Ex 1: Evaluate $81^{\frac{1}{4}}$.

$81^{\frac{1}{4}} = \boxed{3}$, because $3^4 = 81$. Note: This could also be written as $\sqrt[4]{81} = \boxed{3}$.

Ex 2: Evaluate $\sqrt[3]{-125}$.

$\sqrt[3]{-125} = (-125)^{\frac{1}{3}} = \boxed{-5}$, because $(-5)^3 = -125$.

Ex 3: Evaluate $16^{\frac{3}{2}}$.

It often helps to find the root first (the denominator) and then do the multiplication (the numerator).

$$16^{\frac{3}{2}} = \left(\sqrt{16}\right)^3 = (4)^3 = \boxed{64}$$

Ex 4: Evaluate $(32)^{-\frac{2}{5}}$.

$$(32)^{-\frac{2}{5}} = \frac{1}{(32)^{\frac{2}{5}}} = \frac{1}{\left(\sqrt[5]{32}\right)^2} = \frac{1}{(2)^2} = \boxed{\frac{1}{4}}$$

Finding Roots on the Calculator

Ex 5: Use the graphing calculator to approximate $\left(\sqrt[3]{12}\right)^4$.

We will rewrite $\left(\sqrt[3]{12}\right)^4$ with a rational exponent: $(12)^{\frac{4}{3}}$. Type this into your calculator using the carrot key ^ for the exponent. Be sure to put the exponent in parentheses.

1 2 ^ (4 ÷ 3) ENTER

```
12^(4/3)
        27.47314182
```

Question: What expression is your calculator evaluating if you do not use parentheses?

Ex 6: Use the properties of exponents to simplify the expression $\left(5^4 \cdot 2^4\right)^{-\frac{1}{4}}$.

Power of a Product: $\left(5^4 \cdot 2^4\right)^{-\frac{1}{4}} = \left[(5 \cdot 2)^4\right]^{-\frac{1}{4}}$

Power of a Power: $\left[(5 \cdot 2)^4\right]^{-\frac{1}{4}} = (5 \cdot 2)^{-1} = 10^{-1} = \boxed{\frac{1}{10}}$

****Note: The product and quotient properties for exponents can be extended to radicals, as we now know that a radical is simply a rational exponent.**

Product Property: $\sqrt[n]{a \cdot b} = \sqrt[n]{a}\sqrt[n]{b}$

Quotient Property: $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$

Ex 7: Use the properties of exponents to simplify the expression $\left(\frac{\sqrt[3]{12}}{\sqrt[3]{4}}\right)^2$.

Power of a Quotient: $\left(\frac{\sqrt[3]{12}}{\sqrt[3]{4}}\right)^2 = \left(\sqrt[3]{\frac{12}{4}}\right)^2 = \left(\sqrt[3]{3}\right)^2$

Rewrite with Rational Exponents: $\left(\sqrt[3]{3}\right)^2 = \left(3^{\frac{1}{3}}\right)^2$

Power of a Power: $\left(3^{\frac{1}{3}}\right)^2 = \boxed{3^{\frac{2}{3}}}$

Simplest Form of a Radical: all perfect *n*th powers are removed and all denominators are rationalized

Ex 8: Simplify the expression $(250)^{\frac{1}{3}}$.

Step One: Factor out the perfect cube (3rd root). $\sqrt[3]{125 \cdot 2}$

Step Two: Rewrite using the product property. $\sqrt[3]{125} \cdot \sqrt[3]{2}$

Step Three: Simplify by taking the cube root of the perfect cube. $\boxed{5\sqrt[3]{2} = 5(2)^{\frac{1}{3}}}$

Ex 9: Simplify the expression $\left(\frac{8}{3}\right)^{\frac{1}{4}}$.

Step One: Rewrite in radical form. $\frac{\sqrt[4]{8}}{\sqrt[4]{3}}$

Step Two: Multiply the numerator and denominator by a root that will make the denominator's radicand a perfect 4th root. (We can multiply 3 by 27 for a product of 81, which is a perfect 4th root.

$$\frac{\sqrt[4]{2}}{\sqrt[4]{3}} \cdot \frac{\sqrt[4]{27}}{\sqrt[4]{27}} = \frac{\sqrt[4]{2 \cdot 27}}{\sqrt[4]{3 \cdot 27}} = \frac{\sqrt[4]{54}}{\sqrt[4]{81}}$$

Step Three: Simplify by taking the 4th root of the denominator.

$$\boxed{\frac{\sqrt[4]{54}}{3} = \frac{1}{3}(54)^{\frac{1}{4}}}$$

Ex 10: Simplify the expression $\frac{\sqrt{x^2-7x+12}}{\sqrt{x^2+2x-15}}$.

Step One: Factor the numerator and denominator. $\frac{\sqrt{(x-3)(x-4)}}{\sqrt{(x-3)(x+5)}}$

Step Two: Rewrite using the product property. $\frac{\sqrt{(x-3)}\sqrt{(x-4)}}{\sqrt{(x-3)}\sqrt{(x+5)}}$

Step Three: Cancel the common factor. $\frac{\sqrt{(x-4)}}{\sqrt{(x+5)}}$

Step Four: Rationalize the denominator. $\frac{\sqrt{(x-4)}}{\sqrt{(x+5)}}\cdot\frac{\sqrt{(x+5)}}{\sqrt{(x+5)}}=\boxed{\frac{\sqrt{x^2+x-20}}{x+5}}$

Simplifying Variable Expressions: Note: For the following exercises, we must assume all variables are positive.

Ex 11: Simplify the expression $\sqrt{xy}\cdot\sqrt{x^3y^5}$.

Step One: Rewrite using rational exponents. $(xy)^{\frac{1}{2}}(x)^{\frac{3}{2}}(y)^{\frac{5}{2}}$

Step Two: Use the product of powers property. $\left(x^{\frac{1}{2}+\frac{3}{2}}\right)\left(y^{\frac{1}{2}+\frac{5}{2}}\right)$

Step Three: Simplify. $\left(x^{\frac{4}{2}}\right)\left(y^{\frac{6}{2}}\right)=\boxed{x^2y^3}$

Ex 12: Simplify the expression $\sqrt[3]{\frac{8x^6}{y^9}}$.

Step One: Rewrite using rational exponents. $\frac{8^{\frac{1}{3}}\left(x^{\frac{6}{3}}\right)}{y^{\frac{9}{3}}}$

Step Two: Simplify. $\boxed{\frac{2x^2}{y^3}}$

Ex 13: Simplify the expression $\sqrt[3]{x}\cdot\sqrt[4]{x}$.

Step One: Rewrite using rational exponents. $(x)^{\frac{1}{3}}\cdot(x)^{\frac{1}{4}}$

Step Two: Use the product of powers property. $\left(x^{\frac{1}{3}+\frac{1}{4}}\right)=\left(x^{\frac{4}{12}+\frac{3}{12}}\right)$

Step Three: Simplify. $\left(x^{\frac{7}{12}}\right)=\boxed{\sqrt[12]{x^7}}$

SAMPLE EXAM QUESTIONS

1. **Write the radical expression $\left(\sqrt[3]{7}\right)^4$ in exponential notation.**

A. 7^1 **B.** 7^{12}

C. $7^{\frac{3}{4}}$ **D.** $7^{\frac{4}{3}}$

Ans: D

2. **If $f(x)=\sqrt{x^2-1}$ and $g(x)=\sqrt{x-1}$, which expression represents $\dfrac{f(x)}{g(x)}$ for $x>1$?**

A. $\sqrt{x}$

B. $\sqrt{x-1}$

C. $\sqrt{x+1}$

D. $\dfrac{1}{\sqrt{x+1}}$

Ans: C

3. **If $x>1$ and $\dfrac{\sqrt{x}}{x^3}=x^m$, what is the value of *m*?**

(A) $-\dfrac{7}{2}$

(B) -3

(C) $-\dfrac{5}{2}$

(D) -2

(E) $-\dfrac{3}{2}$

Ans: E

Graphing Radical Functions – Units 5.3, 5.4

Parent Function - Standard Form for a Radical Function: $y = a\sqrt[n]{x-h}+k$

In this section, we will concentrate on graphing the square root and cube root functions. The square root function is the inverse of a quadratic function, and the cube root function is the inverse of a cubic function. (We will look at their inverses in section 5.6.)

We will use the "parent functions" $y=\sqrt{x}$ and $y=\sqrt[3]{x}$.

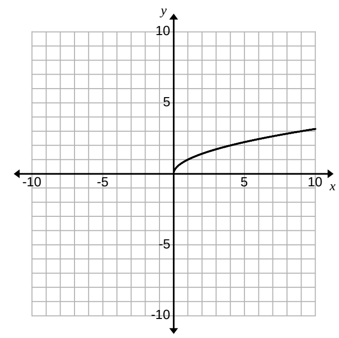
Ex: Graph of $y=\sqrt{x}$ or $y=x^{\frac{1}{2}}$:

Domain: $x \ge 0$; Range: $y \ge 0$

x-intercept, y-intercept: (0, 0)

End behavior: $f(x) \to +\infty$ as $x \to +\infty$

x	$f(x)=\sqrt{x}$
0	0
1	1
4	2
9	3



Ex: Graph of $y=\sqrt[3]{x}$ or $y=x^{\frac{1}{3}}$:

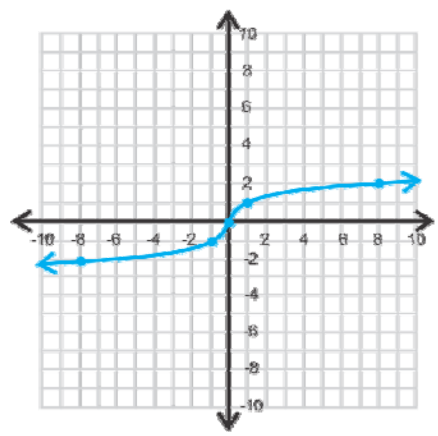
Domain: all real numbers

Range: all real numbers

x-intercept, y-intercept: (0, 0)

End behavior: $f(x) \to +\infty$ as $x \to +\infty$
$f(x) \to -\infty$ as $x \to -\infty$

X	Y1
-5	-1.71
-4	-1.587
-3	-1.442
-2	-1.26
-1	-1
0	0
1	1
2	1.2599
3	1.4422
4	1.5874
5	1.71



Explore on a graphing calculator: Graph the following on the graphing calculator and make note of the changes to the appropriate parent function.

$y=-\sqrt{x}$ $\quad y=2\sqrt{x}$ $\quad y=-3\sqrt{x}$ $\quad y=\frac{1}{3}\sqrt{x}$

$y=\sqrt{x+3}$ $\quad y=\sqrt{x-4}$ $\quad y=\sqrt{x}+5$ $\quad y=\sqrt{x}-2$

Do the same activity with cube roots in place of the square roots.

<u>Summary of the Transformations on Square Root and Cube Root Functions:</u>

$$y = a\sqrt{x-h}+k \qquad y = a\sqrt[3]{x-h}+k$$

a: If $a<0$, the graph is reflected over the *x*-axis. If $0<|a|<1$, the graph is compressed vertically (or stretched horizontally. If $|a|>1$, the graph is stretched vertically.

h: The graph is shifted *h* units horizontally.

k: The graph is shifted *k* units vertically.

For the Square root function:

The domain is: $\{x \mid x \ge h\}$

The range is: If $a>0$, then the range is $\{f(x) \mid f(x) \ge k\}$

If $a<0$, then the range is $\{f(x) \mid f(x) \le k\}$

For the Cube root function:

The domain is: All Real Numbers

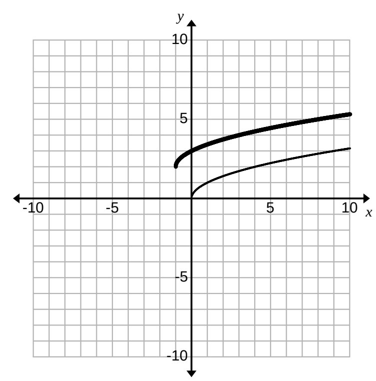
The range is: All Real Numbers

Ex 14: Describe how the graph of $y=\sqrt{x+1}+2$ compares to $y=\sqrt{x}$. Then, sketch the graph and state its domain and range.

$h=-1$ and $k=2$, so the graph will be shifted left 1 and up 2.

$a=1$, so the graph will not be stretched.

Note: It helps to sketch the graph of the parent function first. The graph of $y=\sqrt{x+1}+2$ is the bold graph.



Domain: $x \ge -1$; Range: $y \ge 2$ (Note: These can be found by looking at the graph.) They can also be found algebraically. The radicand must always be positive, so set $\sqrt{x+1} \ge 0$ and solve the inequality, so we get $x+1 \ge 0 \rightarrow x \ge -1$. To find the range algebraically, let the radicand equal zero and solve for y: $y=\sqrt{x+1}+2 \rightarrow y=0+2 \rightarrow y=2$ and since *a* is positive, we have $y \ge 0$.

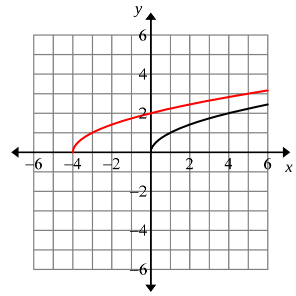
Ex 15: Graph $y_1 = \sqrt{x}$ and $y_2 = \sqrt{x+4}$ on the same coordinate plane. Describe the transformation.

Horizontal Shift LEFT 4 units. (graph is in red)

Domain: $x \geq -4$; Range: $y \geq 0$

x-intercept: $0 = \sqrt{x+4} \rightarrow 0 = x+4 \rightarrow \boxed{-4 = x}$

y-intercept: $y = \sqrt{0+4} \rightarrow y = \sqrt{4} = \boxed{2}$



Ex 16: Sketch the graph of $f(x) = -\sqrt{x} - 2$.

Transformations: Reflect over the x-axis, shift down 2 units. Note: It helps to do the reflection first before doing other translations.

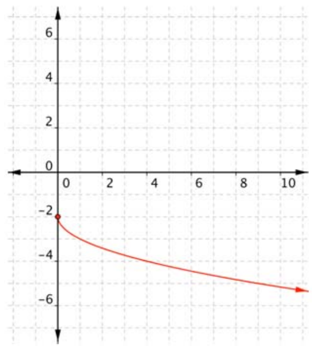
Domain: $x \geq 0$; Range: $y \leq -2$

Work: $\sqrt{x} \geq 0$, so $x \geq 0$. $y = -\sqrt{x} - 2 \rightarrow$ $y \leq 0 - 2 \rightarrow y \leq -2$.

Note that it is $\leq$ and not $\geq$ because the graph is reflected over the x-axis.

x-intercept: None

y-intercept: -2



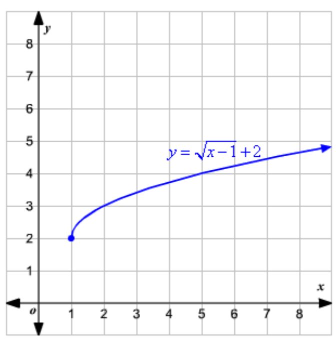
Ex 17: Sketch the graph of $f(x) = \sqrt{x-1} + 2$.

Transformations: Shift right 1 and shift up 2 units

Domain: $x \geq 1$; Range: $y \geq 2$

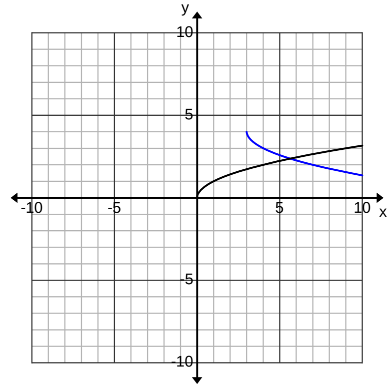
x-intercept: None

y-intercept: 2



Ex 18: Sketch the graph the function $y = 4 - (x-3)^{\frac{1}{2}}$ and state its domain and range.

(Hint: Write it in the form $y = a\sqrt{x-h}+k$ first.)



Standard Form: $f(x) = -\sqrt{x-3}+4$

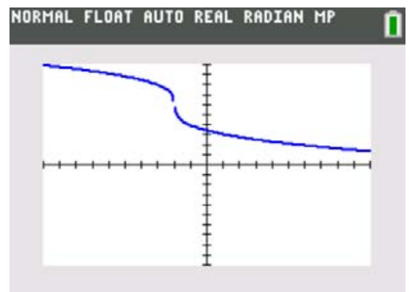
Transformations: Reflect over the x-axis, shift right 3 and down 4 units

Domain: $x \ge 3$

Range: $y \le 4$

Parent function in black, transformed in blue.

Ex 19: Using your findings from the exploration and knowledge of transformations, predict how the graph of $y = -2\sqrt[3]{x+2}+6$ would compare to the graph of $y = \sqrt[3]{x}$. Use the graphing calculator to verify your conjecture.



It would be reflected over the *x*-axis, stretched vertically by 2, shifted to the left 2 and shifted up 6.

Ex 20: Sketch the graph the function $y = \sqrt{4-x}$ and state its domain and range.

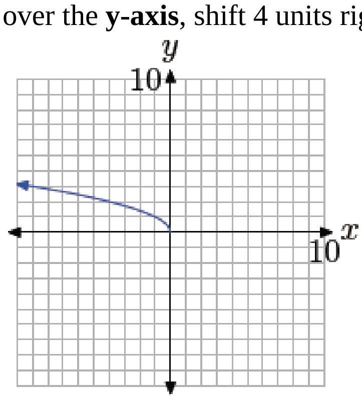
(Hint: Write it in the form $y = a\sqrt{x-h}+k$ first.)

Standard Form: $f(x) = \sqrt{-(x-4)}$

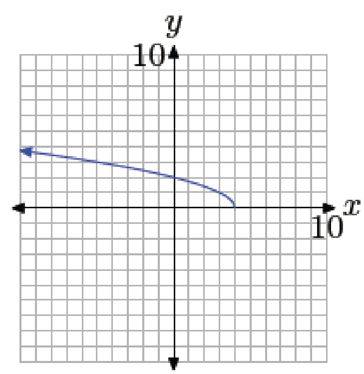
Transformations: Reflect over the **y-axis**, shift 4 units right

Domain: $x \le 4$

Range: $y \le 0$



$f(x) = \sqrt{-x}$



$f(x) = \sqrt{-(x-4)}$

QOD: If $f(x) = \sqrt{x}$, what would the graph of $f(-x)$ look like? What is the domain and range of $f(-x)$?

SAMPLE EXAM QUESTIONS

4. Identify the x and y intercepts of the function $f(x) = \sqrt[3]{x-8}$.

A. (8,0) and (0,-2)

B. (2,0) and (0,2)

C. (8,0) and (0,8)

D. (-2,0) and (0,8)

Ans: A

5. Which is the domain of the function $f(x) = 5\sqrt{x-4}+3$?

A. $\{x \mid x \geq 4\}$

B. $\{x \mid x \geq 3\}$

C. $\{x \mid x \geq 0\}$

D. $\{x \mid x \in \mathbb{R}\}$

Ans: A

6. Compare the graph of $y = 6 - \sqrt[3]{x}$ with the graph of its parent function $f(x) = \sqrt[3]{x}$.

A. Shifts 6 units down

B. Reflects across the x-axis and shifts 6 units down

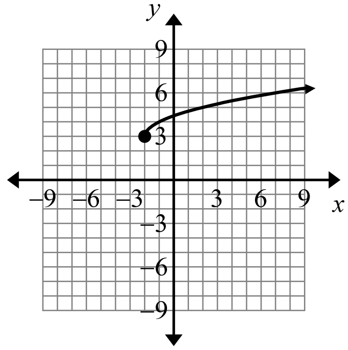
C. Reflects across the x-axis and shifts 6 units up

D. Reflects across the y-axis and shifts 6 units up

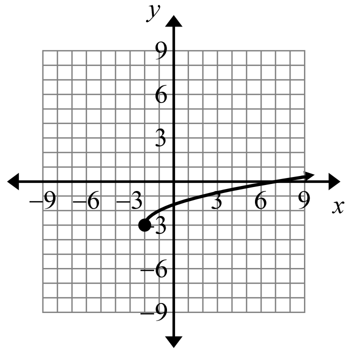
Ans: C

7. **Which is the graph of** $f(x)=\sqrt{x-2}+3$ **?**

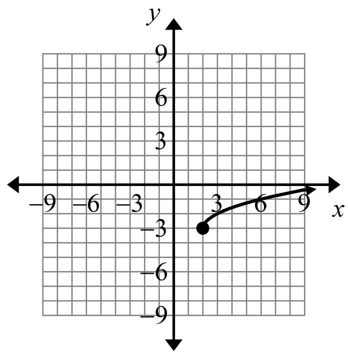
A.



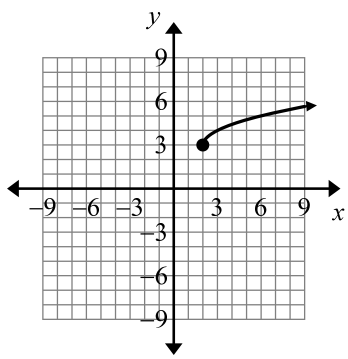
B.



C.



D.



Ans: D

8. Compare the graph of $y = 6 - \sqrt[3]{x}$ **with the graph of its parent function** $f(x) = \sqrt[3]{x}$ **.**

A. Shifts 6 units down

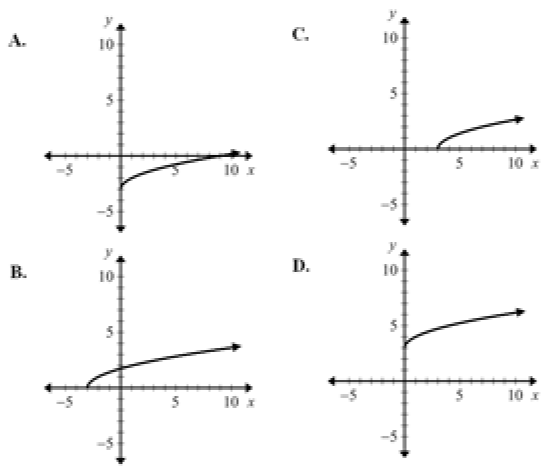
B. Reflects across the x-axis and shifts 6 units down

C. Reflects across the x-axis and shifts 6 units up

D. Reflects across the y-axis and shifts 6 units up

Ans: C

9. What is the graph of $y = \sqrt{x} - 3$ **?**



Ans: A

<u>Solving Radical Equations and Inequalities – Unit 5.5</u>

Solving radical equations are very similar to solving other types of equations. The objective is to get the variable by itself. However, now there are radicals within the equations. Recall that the opposite of the square root of something is to square it and the opposite of the cube root of something is to cube it. Make sure to ALWAYS check your answers when solving radical equations. Sometimes you will solve an equation, get a solution, and then plug it back in and it will not work. These types of solutions are called **extraneous solutions** and are not actually considered solutions to the equation.

Steps for Solving a Radical Equation:

- Isolate the radical.
- Raise both sides to the *n*th power, where *n* is the index of the radical.
- Isolate the variable.
- Check your solution in the original equation. This is crucial, as you may obtain extraneous solutions – solutions that do not work in the original equation.

Ex 21: Solve the equation $5-\sqrt[3]{2x}=8$.

Step One:
$$-\sqrt[3]{2x}=3$$
$$\sqrt[3]{2x}=-3$$

Step Two:
$$\left(\sqrt[3]{2x}\right)^3=(-3)^3$$
$$2x=-27$$

Step Three:
$$x=-\frac{27}{2}$$

Step Four:
$$5-\sqrt[3]{2\left(-\frac{27}{2}\right)}=8$$
$$5-\sqrt[3]{-27}=8$$
$$5-(-3)=8$$

The solution works in the original equation, so $\boxed{x=-\frac{27}{2}}$

Ex 22: Solve the equation $2\sqrt{6x-5}+20=6$.

Step One:
$$2\sqrt{6x-5}=-14$$
$$\sqrt{6x-5}=-7$$

Step Two:
$$\left(\sqrt{6x-5}\right)^2=(-7)^2$$
$$6x-5=49$$

Step Three:
$$6x=54$$
$$x=9$$

Step Four:
$$2\sqrt{6(9)-5}+20=6$$
$$2\sqrt{49}+20=6$$
$$14+20\neq 6$$

The solution does not work in the original equation. Therefore, it is an extraneous solution, and this equation has NO SOLUTION.

Question: Could you have determined earlier in the process of solving that this equation had no solution? Explain.

Ex 23: Solve the equation $\sqrt{x-2} = x-2$.

Step One: Done (radical is isolated)

Step Two: $\left(\sqrt{x-2}\right)^2 = (x-2)^2$

$x-2 = x^2-4x+4$

Step Three: Because this is a quadratic equation, you may use one of the methods for solving quadratic equations (quadratic formula, factoring, or completing the square).

$0 = x^2-5x+6$

$0 = (x-2)(x-3)$ This can be factored, so we will solve using this method.

$x = 2,3$

Step Four:

$\sqrt{(2)-2} = (2)-2$

$\sqrt{0} = 0$

$\sqrt{(3)-2} = (3)-2$

$\sqrt{1} = 1$

Solution Set: $\{2,3\}$

Check:

$\sqrt{x-2} = x-2$

$x=2: \sqrt{2-2} = 2-2$ ☺

$\sqrt{x-2} = x-2$

$x=3: \sqrt{3-2} = 3-2$ ☺

<u>Equations with Two Radicals</u>: To solve, we will move the radicals to opposite sides, then raise both sides to the *n*th power, where *n* is the index of the radical.

Ex 24: Solve the equation $\sqrt[4]{2x+10} - 2\sqrt[4]{x} = 0$.

Step One: Move one of the radicals to the other side. $\sqrt[4]{2x+10} = 2\sqrt[4]{x}$

Step Two: Raise both sides to the 4^{th} power. $\left(\sqrt[4]{2x+10}\right)^4 = \left(2\sqrt[4]{x}\right)^4 \rightarrow 2x+10 = 16x$

Step Three: Solve for *x*. $10 = 14x \rightarrow \frac{5}{7} = x$

Step Four:

$$\sqrt[4]{2\left(\frac{5}{7}\right)+10} - 2\sqrt[4]{\left(\frac{5}{7}\right)} = 0$$

$$\sqrt[4]{\frac{80}{7}} - 2\sqrt[4]{\frac{5}{7}} = 0$$

These are not perfect 4^{th} roots, so we will check on the calculator.

```
(80/7)^(1/4)-2(5
/7)^(1/4)
                0
```

The solution is $x = \frac{5}{7}$.

Solving Equations with Rational Exponents

- Isolate the expression with the rational exponent.
- Raise both sides to the reciprocal power (see below).
- Isolate the variable.
- Check your solution in the original equation. This is crucial, as you may obtain extraneous solutions.

To solve an equation in the form $x^{\frac{m}{n}} = a$, raise both sides to the reciprocal power: $\left(x^{\frac{m}{n}}\right)^{\frac{n}{m}} = (a)^{\frac{n}{m}} \Rightarrow x = a^{\frac{n}{m}}$

Ex 25: Solve the equation $-3x^{\frac{2}{3}} = -12$.

Step One: Isolate the variable. $\frac{-3x^{\frac{2}{3}}}{-3} = \frac{-12}{-3} \rightarrow x^{\frac{2}{3}} = 4$

Step Two: Raise both sides to the reciprocal power. $\left(x^{\frac{2}{3}}\right)^{\frac{3}{2}} = (4)^{\frac{3}{2}}$

Step Three: Simplify. $(4)^{\frac{3}{2}} = \sqrt{4^3} = \left(\sqrt{4}\right)^3 = 2^3 = 8$

Step Four: Check your answer. $-3(8)^{\frac{2}{3}} = -3\sqrt[3]{8^2} = -3\sqrt[3]{64} = -3(4) = -12$

The solution works in the original equation so are solution is $\boxed{8}$.

Ex 26: Solve the equation $-125 = 5x^{\frac{2}{5}}$.

Step One: Isolate the variable.

$$\frac{-125}{5} = \frac{5x^{\frac{2}{5}}}{5}$$

$$-25 = x^{\frac{2}{5}}$$

Step Two: Raise both sides to the reciprocal power.

$$(-25)^{\frac{5}{2}} = \left(x^{\frac{2}{5}}\right)^{\frac{5}{2}}$$

$$\left(\sqrt{-25}\right)^5 = x$$

Because there is no real square root of -25, this equation has $\boxed{\text{no real solution}}$.

Ex 27: Solve the equation $3(x+1)^{\frac{4}{3}} = 48$.

Step One: $(x+1)^{\frac{4}{3}} = 16$

Step Two: $\left[(x+1)^{\frac{4}{3}}\right]^{\frac{3}{4}} = (16)^{\frac{3}{4}} \rightarrow x+1=8$

Step Three: $x = 7$

Step Four:
$3((7)+1)^{\frac{4}{3}} = 48$
$3(8)^{\frac{4}{3}} = 48$
$3(16) = 48$

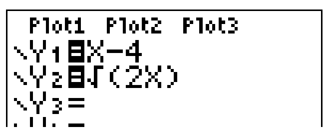
The solution is $\boxed{x=7}$.

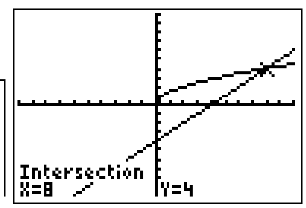
Solving a Radical Equation on the Graphing Calculator: We will solve equations by graphing. You may either graph both sides of the equation as two functions and find the x-coordinate of the point of intersection, or set the equation equal to zero and find the x-intercept of the resulting function.

Ex 28: Solve the equation $x - 4 = \sqrt{2x}$ by graphing.

Method 1: Graph both sides of the equation.

The solution is $\boxed{x=8}$.



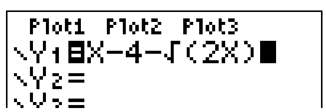


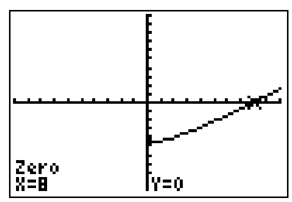
Question: What does the y-value represent in the point of intersection?

Method 2: Set the equation equal to zero.

$x - 4 - \sqrt{2x} = 0$

The solution is $\boxed{x=8}$.





QOD: Explain using rational exponents why raising a radical to the nth power, where n is the index of the radical, will eliminate the radical.

QOD #2: Can you evaluate a radical if the radicand is negative and the index is odd? Explain. Can you evaluate a radical if the radicand is negative and the index is even? Explain.

Solving Radical Inequalities:

The steps for solving radical inequalities is similar to that of equations, however, we must add a step. The first thing we must do is identify the values of the variable for which the radicand is nonnegative. After finding the domain, follow the steps used above to solve the inequality algebraically and remembering to verify any solutions.

Ex 29: Solve $3+\sqrt{5x-10}\le 8$.

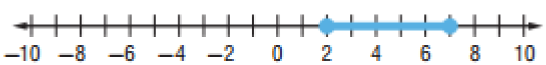
Since the radicand of a square root must be greater than or equal to zero, first solve $5x-10\ge 0$ to identify the values of x for which the left side of the inequality is defined. In this case, $x\ge 2$. Now isolate the variable, square both sides, simplify and check.

$$3+\sqrt{5x-10}\le 8$$
$$\sqrt{5x-10}\le 5$$
$$5x-10\le 25 \rightarrow 5x\le 35$$
$$x\le 7$$

Using the domain, we have $x\ge 2$ and $x\le 7$. We can test some x-values to confirm the solution. Try one less than 2, one between 2 and 7 and one greater than 7.

$x=0$	$x=4$	$x=9$
$3+\sqrt{5(0)-10}\overset{?}{\le} 8$ $3+\sqrt{-5}\le 8$ ✗ Since $\sqrt{-5}$ is not a real number, the inequality is not satisfied.	$3+\sqrt{5(4)-10}\overset{?}{\le} 8$ $6.16\le 8$ ✓ Since $6.16\le 8$, the inequality is satisfied.	$3+\sqrt{5(9)-10}\overset{?}{\le} 8$ $8.92\le 8$ ✗ Since $8.92\not\le 8$, the inequality is not satisfied.

The solution checks. Only values in the interval $2\le x\le 7$ satisfy the inequality. You can summarize the solution with a number line.



Ex 30: Solve $\sqrt{x+7}>\sqrt{2x-1}$.

We cannot have the square root of a negative number which means $x\ge -7$ and $x\ge \frac{1}{2}$, so $x\ge \frac{1}{2}$.

Step One – isolate the radicals: done

Step Two – square both sides: $x+7>2x-1$

Step Three – Solve: $8>x \rightarrow x<8$

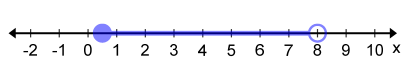
Step Four – Check: Substitute 0, 2 and 9

$\sqrt{0+7}>\sqrt{2(0)-1} \rightarrow \sqrt{7}>\sqrt{-1}$ invalid domain

$\sqrt{2+7}>\sqrt{2(2)-1} \rightarrow \sqrt{9}>\sqrt{3}$ true

$\sqrt{9+7}>\sqrt{2(9)-1} \rightarrow \sqrt{16}>\sqrt{17}$ not true

Solution: $\boxed{\frac{1}{2}\le x<8}$



SAMPLE EXAM QUESTIONS

10. Which value of x makes this equation true? $9(x-7)^{\frac{4}{3}}=9$

A. 1 **B.** 7 **C.** 8 **D.** 34

Ans: C

11. Solve for x: $\sqrt[3]{4x+1}=5$

A. $x=-31$ **B.** $x=6$ **C.** $x=31$ **D.** No real solution

Ans: C

12. Solve for x. $x-7=\sqrt{x-1}$

A. $x=5$ and $x=10$ **B.** $x=5$

C. $x=10$ **D.** No real solutions

Ans: A

13. Solve for x. $\sqrt{x-3}-\sqrt{x}=3$

A. $x=4$ **B.** $x=6$

C. $x=9$ **D.** No real solutions

Ans: D

13. What is the value of x in the equation $\sqrt{2x+1}+3=6$**?**

A. 1 **C.** 4

B. 13 **D.** 16

Ans: C

14. If $\sqrt[3]{12x+28}=4$, what is the value of x^3?

A. -8 **B.** 3 **C.** 12 **D.** 27

Ans: D

15. The relationship between the weight of a whale in tons, W, and the length in feet, L, is given by $W=0.000137L^{3.18}$. Which expression below would be used to find the length of a whale that weighs 50 tons?

A. $\sqrt[3.18]{\dfrac{50}{0.000137}}$

B. $0.000137(50)^{3.18}$

C. $\dfrac{\sqrt[3.18]{50}}{0.000137}$

D. $\sqrt[3.18]{50}(0.000137)$

Ans: A

<u>**Sample SAT Question:**</u> **Taken from College Board online practice problems**.

16. If x and y are real numbers and the square of y is equal to the square root of x, which of the following must be true?

I. $x=y^4$
II. $x\geq 0$
III. $y\geq 0$

(A) I only
(B) I and II only
(C) I and III only
(D) II and III only
(E) I, II, and III

Ans: E