## Solving and Applying Polynomial Functions

## Math Background

**Previously, you**
- Solved linear and quadratic functions
- Identified key features of polynomial functions
- Identified polynomials through degree and number of terms
- Factored and divided polynomials
- Applied the Factor and Remainder Theorems

**In this unit you will**
- Find the possible number of real solutions of polynomial functions
- Use the Fundamental Theorem of Algebra to find all zeros
- Use the Rational Root Theorem to find all possible rational roots of polynomials
- Solve problems by modeling with polynomial functions

**You can use the skills in this unit to**
- Show the three possible solutions of a quadratic polynomial (ex: two real, two complex, or a repeated root).
- Use finite differences to determine the degree of a polynomial.
- Use technology to approximate solutions of polynomials.
- Model and solve real-world problems with polynomials.
- Write an appropriate function to model real-world data.

**Vocabulary**
- **Degree of a polynomial** – The highest power of the variable in a polynomial.
- **Even Function –** A function $f(x)$ which satisfies the property that $f(-x) = f(x)$. The functions with even degree are even functions.
- **Fundamental Theorem of Algebra** – Every equation which can be put in the form with zero on one side of the equal sign and a polynomial of degree greater than or equal to one with real or complex coefficients on the other has at least one root which is a real or complex number.
- **Multiplicity** – It is how often a certain root or zero is part of the factoring. It is the number of times the root is a zero of the function.
- **Odd Function –** Any function $f(x)$ that satisfies the property that $f(-x) = -f(x)$. The functions with odd degree are odd functions.
- **Parent Polynomial Function –** The simplest function of a family of functions, $f(x) = x^n$, where $n$ is a positive integer.
- **Rational Root (Zero) Theorem –** A theorem that provides a complete list of possible rational roots of the polynomial equation $a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0 = 0$ where all coefficients are integers.
- **Zeros of a function** – An input value that produces an output of zero. It is also known as a root and is where the graph meets the x-axis, also known as an x-intercept.

**Essential Questions**

- How does the Fundamental Theorem of Algebra apply to the solutions of a polynomial function?
- Why is it useful to model real-world problems with equations and graphs?

**Overall Big Ideas**

The degree of a polynomial determines the number of roots a function possesses.

Equations and graphs can help to predict a future outcome of a real world problem or provide insight in to the problems past.

**Skill**

To apply the Fundamental Theorem of Algebra to predict the number of real and imaginary solutions.

To solve polynomial equations by factoring.

To use the Fundamental Theorem of Algebra to write polynomial equations of least degree with given roots.

To transform polynomial functions.

To use polynomial functions to model data.

To use finite differences to determine the degree of a polynomial function that will fit a set of data.

**Related Standards**

**N.CN.C.9**

Know the Fundamental Theorem of Algebra; show that it is true for quadratic polynomials.

**A.APR.B.3**

Identify zeros of polynomials when suitable factorizations are available, and use the zeros to construct a rough graph of the function defined by the polynomial.

**A.REI.D.11-2**

Explain why the x-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find solutions to $f(x) = g(x)$ approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, radical, absolute value, exponential, and logarithmic functions. *(Modeling Standard)

**F.BF.B.3-2**

Identify the effect on the graph of replacing $f(x)$ by $f(x)+k$, $kf(x)$, $f(kx)$, and $f(x+k)$ for specific values of $k$ (both positive and negative) and find the value of $k$ given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. Include recognizing even and odd functions from their graphs and algebraic expressions for them. Include simple radical, rational, an exponential functions, note the effect of multiple transformations on a single graph, and emphasize common effects of transformations across function types.

**A.CED.A.2-2**

Create equations in two or more variables to represent relationships between quantities and graph equations on coordinate axes with labels and scales. Use all types of equations. *(Modeling Standard)

**A.CED.A.3-2**

Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or non-viable options in a modeling context. Use all types of equations. For example, represent inequalities describing nutritional and cost constraints on combinations of different foods. *(Modeling Standard)

## Notes, Examples, and Exam Questions

### Units 4.8 – 4.9  Solving Polynomial Functions

**Review:  Zero-Product Property**          If $ab = 0$, then $a = 0$ or $b = 0$.

**Solving Polynomial Equations by Factoring**

**Ex 1**: Solve the equation $4x^6 - 20x^4 = -24x^2$.

Step One:  Set the equation equal to zero.          $4x^6 - 20x^4 + 24x^2 = 0$

Step Two:  Factor the polynomial.

$$4x^2\left(x^4 - 5x^2 + 6\right) = 0$$
$$4x^2\left(x^2 - 3\right)\left(x^2 - 2\right) = 0$$

Step Three: Set each factor equal to zero and solve.

$$\begin{array}{lll} 4x^2 = 0 & x^2 - 3 = 0 & x^2 - 2 = 0 \\ x^2 = 0 & x^2 = 3 & x^2 = 2 \\ x = 0 & x = -\sqrt{3}, \sqrt{3} & x = -\sqrt{2}, \sqrt{2} \end{array}$$

Solutions:  $\boxed{\left\{-\sqrt{3}, -\sqrt{2}, 0, \sqrt{2}, \sqrt{3}\right\}}$

**Ex 2**: Find the real-number solutions of the equation          $x^3 - 5x^2 + 5x - 25 = 0$.

Step One:  Set the equation equal to zero.          $x^3 - 5x^2 + 5x - 25 = 0$

Step Two:  Factor the polynomial.

$$x^2\left(x - 5\right) + 5\left(x - 5\right) = 0$$
$$\left(x - 5\right)\left(x^2 + 5\right) = 0$$

Step Three:  Set each factor equal to zero and solve.

$$\begin{array}{ll} & x^2 + 5 = 0 \\ x - 5 = 0 & x^2 = -5 \\ x = 5 & x = -5i, 5i \end{array}$$

Real-Number Solution:  $\boxed{\{5\}}$

### Application Problem

**Ex 3**:  An optical company is going to make a glass prism that has a volume of 15 cm$^3$.  The height will be $h$ cm, and the base will be a right triangle with legs of length $(h - 2)$ cm and $(h - 3)$ cm.  What will be the height?

Volume of a Prism = Area of the Base x Height          $15 = \dfrac{1}{2}(h - 2)(h - 3)h$

$$15 = \frac{1}{2}\left(h^2 - 5h + 6\right)h$$

To solve this equation for $h$, we must set it equal to zero.

$$15 = \frac{1}{2}h^3 - \frac{5}{2}h^2 + 6h$$

$$0 = \frac{1}{2}h^3 - \frac{5}{2}h^2 + 6h - 15$$

Before factoring, we can multiply both sides of the equation by 2 to eliminate (clear) the fractions.

$$2 \cdot 0 = 2 \cdot \frac{1}{2}h^3 - 2 \cdot \frac{5}{2}h^2 + 2 \cdot 6h - 2 \cdot 15$$

$$0 = h^3 - 5h^2 + 12h - 30$$

Factor by grouping.

$$0 = h^2\left(h - 5\right) + 6\left(h - 5\right)$$

$$0 = \left(h - 5\right)\left(h^2 + 6\right)$$

Solve by setting each factor equal to 0.

$$h - 5 = 0 \qquad\qquad h^2 + 6 = 0$$

$$\boxed{h = 5 \text{ cm}} \qquad h^2 = -6 \text{ No real solution}$$

The height of the prism will be 5 cm.

**QOD:** Give an example of a binomial that can be factored either as the difference of two squares or as the difference of two cubes. Show the complete factorization of your binomial.

**The Fundamental Theorem of Algebra**

- If $f(x)$ is a polynomial of degree $n$ where $n > 0$, then the equation $f(x) = 0$ has at least one root in the set of complex numbers
- A polynomial function of degree $n$ has $n$ complex zeros (some zeros may repeat)
- Every polynomial function of odd degree with real coefficients has at least one real zero.
- Complex zeros come in conjugate pairs
  Review: $a + bi \Rightarrow$ conjugate $\Rightarrow a - bi$

**Finding the number of solutions or zeros**

**Review:** Find the solutions of the following examples. State how many solutions each has and classify each zero as rational, irrational, or complex (imaginary).

**Ex 4:**    $3x - 1 = 0$

One rational zero: $x = \frac{1}{3}$

**Ex 5:**    $x^2 - 9 = 0$

<p style="color:red; text-align:center">Two rational zeros:   $x = -3, 3$</p>

**Ex 6:**    $x^3 - 1 = 0$   (Hint: Use the factorization for the difference of cubes, then use the quadratic formula for the quadratic factor.)

<p style="color:red; text-align:center">Three roots, one rational zero and two complex:   $x = 1, \dfrac{-1 + i\sqrt{3}}{2}, \dfrac{-1 - i\sqrt{3}}{2}$</p>
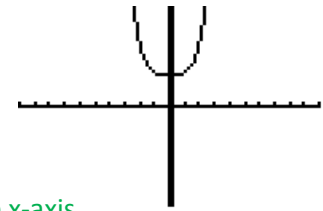
**Do you notice a pattern with the degree of the polynomial and the number of solutions each has?**

**Ex 7:**    How many different solutions are there to $x^4 - 16 = 0$?    <span style="color:red">Four, the degree is four.</span>

How do you explain this number?    <span style="color:red">There are two rational zeros, -2 and 2. There are two imaginary roots, -2i and 2i.</span>

**Ex 8:**    How many different solutions are there to $x^2 + 16 = 0$?

Solution:   $\boxed{x = 4i, \, -4i}$

<span style="color:green">Note: On the graph, the imaginary roots do not cross the x-axis.</span>

⭐ **Note:** $4i, \, -4i$ are complex conjugate pairs. $1 + i\sqrt{2}, 1 - i\sqrt{2}$ are complex conjugate pairs. The complex roots of polynomial functions with real coefficients always occur in complex conjugate pairs. <u>Is this also the case for irrational zeros?</u>

## Using the Rational Zero Theorem

**Review:** A rational zero is a rational number that produces a function value of 0. It can be visualized as $f(x) = 0$ where x is a rational number (can be written as a ratio). On the graph it is an x-intercept. The Rational Zero Theorem will help to identify possible rational zeros.

**The Rational Zero Theorem**    If $f(x) = a_n x^n + \ldots a_1 x + a_0$ has integer coefficients, then every rational zero of

$f$ has the following form:   $\dfrac{p}{q} = \dfrac{\text{factor of constant term } a_0}{\text{factor of leading coefficient } a_n}$

The first important step is to list the possible rational zeros. After they are listed we can test them using synthetic division to determine if they are rational zeros. If the value of the possible rational zeros = 0, they are called zeros.

**List the possible rational zeros:**

**Ex 9**:   Find the possible rational zeros of $f(x) = x^2 + 7x + 12$

   Step 1:  The leading coefficient is 1.  1 is the only factor of 1.

   Step 2:  The constant is 12.  All of the factors of 12 are $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$.

   Step 3:  List the possible factors - $\pm\dfrac{1}{1}, \pm\dfrac{2}{1}, \pm\dfrac{3}{1}, \pm\dfrac{4}{1}, \pm\dfrac{6}{1},$ and $\pm\dfrac{12}{1}$
   *If we factored the quadratic, we would find that $-3$ and $-4$ are zeros.


**Ex 10**:   Find the possible rational zeros of $f(x) = 3x^3 + 7x^2 + 12x - 5$

   Step 1:  The leading coefficient is 3.  The factors of 3 are 1 and 3.

   Step 2:  The constant is -5.  All of the factors of -5 are $\pm 5, \pm 1$.

   Step 3:  List the possible factors - $\pm\dfrac{1}{1}, \pm\dfrac{5}{1}, \pm\dfrac{1}{3}, \pm\dfrac{5}{3}$
   *We will not test for actual zeros for this example.


⭐ When the leading coefficient is not 1, the list of possible zeros can increase dramatically.  There are many tools that are used to find the rational zeros.  Examples that follow will demonstrate some of them.


**Ex 11**:   Find all the real zeros of $f(x) = x^3 + 72 - 5x^2 - 18x$

   Step 1:  Put the function in standard order.  $f(x) = x^3 - 5x^2 - 18x + 72$

   Step 2:  List possible rational zeros  $\pm(1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36, 72)$

   Step 3:  Try the possible zeros until you find one.

$$
\begin{array}{r|rrrr}
1 & 1 & -5 & -18 & 72 \\
  &   & 1  & -4  & -22 \\
\hline
  & 1 & -4 & -22 & 50
\end{array}
\qquad
\begin{array}{r|rrrr}
-1 & 1 & -5 & -18 & 72 \\
   &   & 1  & 4   & 14 \\
\hline
   & -1 & -4 & -14 & 86
\end{array}
\qquad
\begin{array}{r|rrrr}
3 & 1 & -5 & -18 & 72 \\
  &   & 3  & -6  & -72 \\
\hline
  & 1 & -2 & -24 & \boxed{0}
\end{array}
$$

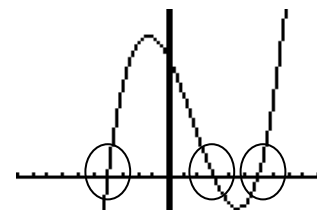   Since the polynomial degree is reduced, the function can be written as:

   Then factored:    $f(x) = (x-3)(x^2 - 2x - 24)$
                     $f(x) = (x-3)(x-6)(x+4)$

   $(x-3) = 0 \qquad (x-6) = 0 \qquad (x+4) = 0$
   Zeros:    $\boxed{x = 3 \qquad\qquad x = 6 \qquad\qquad x = -4}$

⭐ **Note:** Finding rational zeros is also referred to as finding real zeros.  Rational numbers are also real numbers. There is a distinction between *listing* possible rational zeros and *finding* rational (real) zeros.

**Ex 12:** Find all the zeros of the function and write the polynomial as a product of linear factors.

$$f(x) = x^4 - 6x^3 + 10x^2 - 6x + 9$$

Step One:  Check for any rational zeros.

Possible rational zeros: $\dfrac{\text{factors of } 9}{\text{factors of } 1} = \pm\dfrac{1}{1}, \pm\dfrac{3}{1}, \pm\dfrac{9}{1}$ = -1, 1, -3, 3, -9, 9

Use synthetic substitution:

$$\begin{array}{r|rrrrr} 3 & 1 & -6 & 10 & -6 & 9 \\ & & 3 & -9 & 3 & -9 \\ \hline & 1 & -3 & 1 & -3 & 0 \end{array}$$

So 3 is a zero, and $x - 3$ is a factor.

Step Two:  Rewrite the last line as a polynomial and find the zeros of the polynomial.

$$f(x) = (x-3)(x^3 - 3x^2 + x - 3)$$

$x^3 - 3x^2 + x - 3 = 0 \longrightarrow$ Factor by grouping: $x^2(x-3) + 1(x-3) = 0 \Rightarrow (x^2+1)(x-3) = 0$

$$\begin{aligned} x^2 + 1 &= 0 \\ \sqrt{x^2} &= \sqrt{-1} \\ x &= \pm i \end{aligned} \qquad\qquad \begin{aligned} x - 3 &= 0 \\ x &= 3 \end{aligned}$$

Step Three: List the zeros.  Check that you have the correct number according to the Fundamental Theorem of Algebra (FTA).

$\boxed{x = 3, -i, i}$     The polynomial was degree 4, and there are 4 zeros (3 is repeated).

Note that $i$ and $-i$ are complex conjugates.

Step Four:  Write the polynomial as a product of linear factors.

$$\boxed{f(x) = (x-3)(x-3)(x+i)(x-i)}$$

**Ex 13**:  Find all of the real zeros of $f(x) = 3x^3 + 12x^2 + 3x - 18$.

Step 1: Notice that each term contains a common factor of 3.  The problem can be factored to $f(x) = 3(x^3 + 4x^2 + x - 6)$ and since $3 \neq 0$ only $(x^3 + 4x^2 + x - 6)$ can be $= 0$.

Step 2: List possible rational zeros $\pm(1, 2, 3, 6)$  (Since the leading coefficient is now 1)

Step 3: Try the possible zeros until you find one.

$$
\begin{array}{r|rrrr}
1 & 1 & 4 & 1 & -6 \\
  &   & 1 & 1 & 2 \\
\hline
  & 1 & 1 & 2 & -4
\end{array}
\qquad
\begin{array}{r|rrrr}
-1 & 1 & 4 & 1 & -6 \\
   &   & -1 & -3 & 2 \\
\hline
   & 1 & 3 & -2 & -4
\end{array}
\qquad
\begin{array}{r|rrrr}
-2 & 1 & 4 & 1 & -6 \\
   &   & -2 & -4 & 6 \\
\hline
   & 1 & 2 & -3 & \boxed{0}
\end{array}
$$

The function can be written as:

$$f(x) = (x+2)\left(x^2 + 2x - 3\right)$$

Then factored:

$$f(x) = (x+2)(x+3)(x-1)$$

$$x + 2 = 0 \qquad x + 3 = 0 \qquad x - 1 = 0$$

$$\boxed{\text{Zeros: } x = -2 \qquad\qquad x = -3 \qquad\qquad x = 1}$$
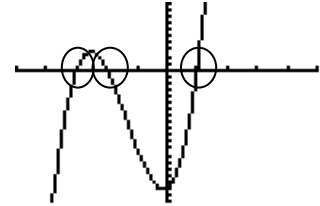
**Ex 14**: Find all the real zeros of $f(x) = 3x^5 + x^4 - 243x - 81$

Step 1: Maybe this could be graphed first.

Step 2: Look at the graph for reasonable choices

It appears they might be $-3$, 3, and $-\dfrac{1}{3}$
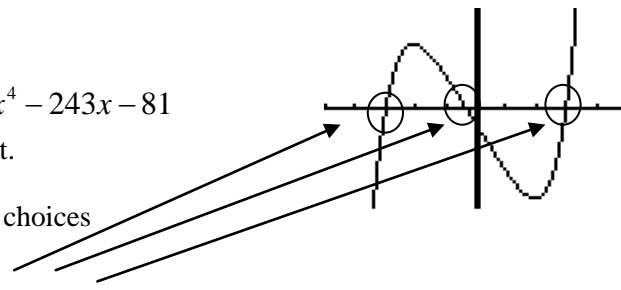
Step 3: Check the chosen values using synthetic division. Start with -3.

$$
\begin{array}{r|rrrrrr}
-3 & 3 & 1 & 0 & 0 & -243 & -81 \\
   &   & -9 & 24 & -72 & 216 & 81 \\
\hline
   & 3 & -8 & 24 & -72 & -27 & \boxed{0}
\end{array}
$$

It is a root (zero).

The factored form so far is   $f(x) = (x+3)(3x^4 - 8x^3 + 24x^2 - 72x - 27)$

Step 4: Repeat the steps above using a different reasonable choice.
Try 3.

$$
\begin{array}{r|rrrrr}
3 & 3 & -8 & 24 & -72 & -27 \\
  &   & 9 & 3 & 81 & 27 \\
\hline
  & 3 & 1 & 27 & 9 & \boxed{0}
\end{array}
$$

It is a root (zero).

Step 5: Repeat the steps above using a different reasonable choice.  Try $-\dfrac{1}{3}$

$$
\begin{array}{r|rrrr}
-\frac{1}{3} & 3 & 1 & 27 & 9 \\
   &   & -1 & 0 & -9 \\
\hline
   & 3 & 0 & 27 & \boxed{0}
\end{array}
$$

It is a root (zero).

Yes, all three work!  And each time, the function (polynomial) is reduced by one degree.

Step 6: The function $3x^2 + 27$ is left to be factored.

$3(x^2 + 9)$ has no real factors.

Solution:  There are 3 real zeros: $-3$, $3$, and $-\dfrac{1}{3}$

**QOD:** If the leading coefficient of a polynomial with integer coefficients is 1, what type of numbers must any possible rational zeros be?

### Finding the zeros of a polynomial function

This activity involves finding the rational zeros as learned in the previous section, then using other tools, such as the quadratic formula or technology, to find the irrational or complex roots.

**Ex 15:** Find all zeros of $f(x) = x^4 + 4x^3 - 6x^2 - 36x - 27$

Using the rational root theorem and synthetic division, it can be shown that $-3$ is a repeated root and 3 and -1 are roots.

The factored form looks like this: $(x + 3)^2(x - 3)(x + 1)$. The graph is shown.
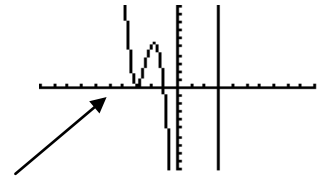
**When a factor $x - k$ is raised to an odd power, the graph crosses through the $x$-axis. When a factor $x - k$ is raised to an even power, the graph is tangent to the $x$-axis.**

Solution:  There are four real zeros  $-3$ is a repeated root and 3 and -1  are roots.

**Ex 16:** Find all zeros of $f(x) = x^4 - x^3 - x^2 - 5x - 30$

Using the rational root theorem and synthetic division, it can be shown that 3 and -2 are roots.

Rewrite the polynomial in factored form:  $f(x) = x^4 - x^3 - x^2 - 5x - 30$

factors to $(x + 2)(x - 3)(x^2 + 5)$

Factor or quadratic formula:  $x^2 + 5 = 0$

yields zeros of $x = \pm i\sqrt{5}$

Solution:

There are four zeros, 3 and -2 and $\pm i\sqrt{5}$. Two are real and two are complex conjugates

## Unit 4.10  Using Zeros to Write Polynomial Functions

**Ex 17:** Write a polynomial function $f$ of least degree that has real coefficients, a leading coefficient of 1, and 3, 2 and -5 as zeros.
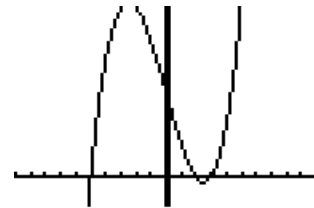
Step 1:          Write $f(x)$ in factored form: $f(x) = (x-3)(x-2)(x+5)$

Step 2:          Review - Multiply the polynomials two at a time.  Because they are binomials, we can use FOIL to multiply the first two.  $f(x) = (x^2 - 5x + 6)(x + 5)$

$$f(x) = x^3 - 19x + 30$$

**Solution:**          $\boxed{f(x) = x^3 - 19x + 30}$

**Ex 18:** Write a polynomial function $f$ of least degree that has real coefficients, a leading coefficient of 1, and 1, -2, and $1-i$ as zeros.

Step 1: Since $1-i$ is a zero, so is $1+i$

Step 2: Write $f(x)$ in factored form: $f(x) = (x-1)(x+2)(x-(1-i))\ (x-(1+i))$

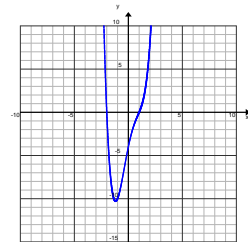Step 3: Regroup:  $f(x) = (x-1)(x+2)\left[(x-1)-i\right]\left[(x-1)+i\right]$

Step 4: Expand, multiply polynomials, and combine like terms.

$$f(x) = (x^2 + x - 2)\left[(x-1)^2 - i^2\right]$$
$$f(x) = (x^2 + x - 2)\left[(x^2 - 2x + 1) + 1\right]$$
$$f(x) = (x^2 + x - 2)(x^2 - 2x + 2)$$
$$\boxed{f(x) = x^4 - x^3 - 2x^2 + 6x - 4}$$

Note: This graph only has two x-intercepts. Why?

**Ex 19:** Write a cubic function that passes through the following points:  (1, 0) (-2, 0) (4, 0) and (2, 4).

Step 1:  Using the three zeros, write the cubic function in factored form, leaving $a$ as the unknown leading coefficient:

$$f(x) = a(x-1)(x+2)(x-4)$$

Step 2:  Use the other ordered pair to solve for $a$.

$$4 = a(2-1)(2+2)(2-4)$$
$$4 = -8a$$
$$a = -\frac{1}{2}$$

Step 3:  Multiply the polynomial:

$$f(x) = -\frac{1}{2}(x-1)(x+2)(x-4)$$

$$= -\frac{1}{2}(x-1)\left(x^2 - 2x - 8\right) = -\frac{1}{2}\left(x^3 - 3x^2 - 6x + 8\right)$$

$$f(x) = -\frac{1}{2}x^3 + \frac{3}{2}x^2 + 3x - 4$$

**Ex 20:**  Find a polynomial function in standard form with a leading coefficient of 3 that has the zeros $-2$ and $1+2i$.

Write the polynomial as a product of linear factors using the zeros.

$$f(x) = 3(x+2)\left(x-(1+2i)\right)\left(x-(1-2i)\right)$$

Note:  If $1+2i$ is a zero, then its complex conjugate, $1-2i$, must also be a zero.

$$f(x) = 3(x+2)(x-1-2i)(x-1+2i)$$
$$= 3(x+2)\left(x^2 - 2x + 5\right)$$

Write the polynomial in standard form.

$$= 3(x^3 + x + 10)$$

$$\boxed{f(x) = 3x^3 + 3x + 30}$$

Notice the pattern in the last couple of examples.  The product of $x-(a+bi)$ and $x-(a-bi)$ is:

$$\left[x-(a+bi)\right]\left[x-(a-bi)\right] = x^2 - (a-bi)x - (a+bi)x + (a+bi)(a-bi)$$
$$= x^2 - 2ax + \left(a^2 + b^2\right)$$

Such a quadratic expression **with real coefficients but no real zeros is irreducible over the reals.**

<u>**Irreducible**</u> <u>Quadratic Factor</u>:  A quadratic factor of a polynomial function is *irreducible over the reals* if it has real coefficients but no real zeros.

**Ex 21:**  Write the polynomial as the product of factors that are irreducible over the reals.  Then, write the polynomial in completely factored form.  $f(x) = 3x^5 + x^4 - 3x^3 - x^2 - 60x - 20$

Step One:  Check for any rational zeros.

Possible rational zeros:  $\pm\dfrac{1}{1}, \pm\dfrac{2}{1}, \pm\dfrac{4}{1}, \pm\dfrac{5}{1}, \pm\dfrac{10}{1}, \pm\dfrac{20}{1}, \pm\dfrac{1}{3}, \pm\dfrac{2}{3}, \pm\dfrac{4}{3}, \pm\dfrac{5}{3}, \pm\dfrac{10}{3}, \pm\dfrac{20}{3}$

$$\begin{array}{r|rrrrrr} -\dfrac{1}{3} & 3 & 1 & -3 & -1 & -60 & -20 \\ & & -1 & 0 & 1 & 0 & 20 \\ \hline & 3 & 0 & -3 & 0 & -60 & 0 \end{array}$$

Use synthetic substitution:

**Step Two:** Rewrite the last line as a polynomial and find the zeros of the polynomial.

$$f(x) = \left(x + \frac{1}{3}\right)\left(3x^4 - 3x^2 - 60\right) = 3\left(x + \frac{1}{3}\right)\left(x^4 - x^2 - 20\right) = (3x+1)\left(x^4 - x^2 - 20\right)$$

$$x^4 - x^2 - 20 = 0 \implies \left(x^2 - 5\right)\left(x^2 + 4\right) = 0 \qquad \begin{array}{ll} x^2 - 5 = 0 & x^2 + 4 = 0 \\ x = \pm\sqrt{5} & x = \pm 2i \end{array}$$

Product of factors irreducible over the reals:     $\boxed{f(x) = (3x+1)\left(x+\sqrt{5}\right)\left(x-\sqrt{5}\right)\left(x^2+4\right)}$

Completely factored form:     $\boxed{f(x) = (3x+1)\left(x+\sqrt{5}\right)\left(x-\sqrt{5}\right)(x-2i)(x+2i)}$

**Application examples:**

**Ex 22:** The monthly average high temperatures in degrees Fahrenheit at Daytona Beach can be modeled by $f(x) = 0.0151x^4 - 0.438x^3 + 3.60x^2 - 6.49x + 72.5$ where $x = 1$ corresponds to January and $x = 12$ represents December.

a) Find the average high temperature during March and July.

    For March, $x = 3$ and for July, $x = 7$. Substitute the values into the function to find the avg. high temp.

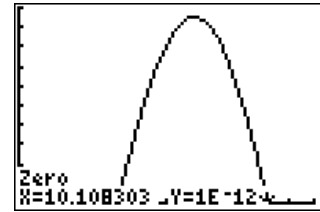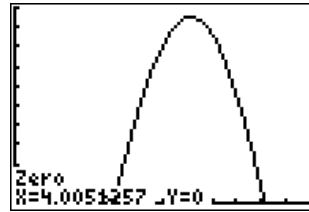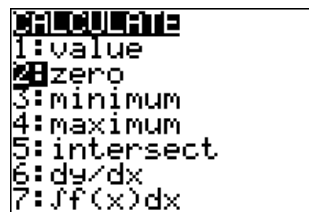$$f(x) = 0.0151(3)^4 - 0.438(3)^3 + 3.60(3)^2 - 6.49(3) + 72.5 = 74.827$$
$$f(x) = 0.0151(7)^4 - 0.438(7)^3 + 3.60(7)^2 - 6.49(7) + 72.5 = 89.491$$

b) Estimate when the average high temperature is 80 degrees Fahrenheit.

    F(x) is given as 80, so $80 = 0.0151x^4 - 0.438x^3 + 3.60x^2 - 6.49x + 72.5$

    Get zero on one side, $0 = 0.0151x^4 - 0.438x^3 + 3.60x^2 - 6.49x - 7.5$

    Graph the function and find its zeros. The zeros are: $x \approx 4, 10$

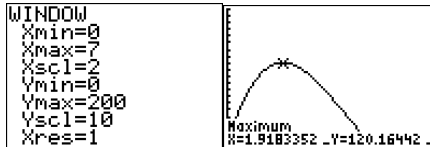**The average high temperature is 80 degrees in April and October.**

**Ex 23:** An open box is formed by cutting squares from the corners of a 10 ft by 14 ft rectangular piece of cardboard and folding up the flaps.  What is the maximum volume of the box?

Let $x$ be the length of the sides of the squares cut from the corners.

Write the volume $V$ of the box as a function of $x$.          $V = lwh = (14 - 2x)(10 - 2x)x$

Graph $V$ on the calculator and find the maximum value.

```
WINDOW
 Xmin=0
 Xmax=7
 Xscl=2
 Ymin=0
 Ymax=200
 Yscl=10
 Xres=1
```

```
Maximum
X=1.9183352  Y=120.16442
```

The maximum volume is approximately $\boxed{120 \text{ ft}^3}$ .

### Unit 4.13  Finite Differences:

To decide whether y-values for equally-spaced x-values can be modeled by a polynomial function, you can use **finite differences.**

**Ex 24:** The first three triangular numbers are shown at the right.  A formula for the $n$th triangular number is $f(n) = \dfrac{1}{2}(n^2 + n)$.  Show that this function has constant second-order differences.

**Triangular Numbers**

$f(1) = 1$

$f(2) = 3$

$f(3) = 6$
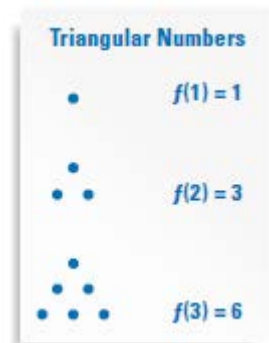
Let's write the first several triangular numbers.  Find the first-order differences by subtracting consecutive triangular numbers.  Then, find the second-order differences by subtracting consecutive first-order differences.

$$f(1) \quad f(2) \quad f(3) \quad f(4) \quad f(5) \quad f(6) \quad f(7)$$
$$1 \quad\ \ 3 \quad\ \ 6 \quad\ \ 10 \quad 15 \quad 21 \quad 28$$
$$2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7$$
$$1 \quad 1 \quad 1 \quad 1 \quad 1$$

**Function values for equally-spaced *n*-values**

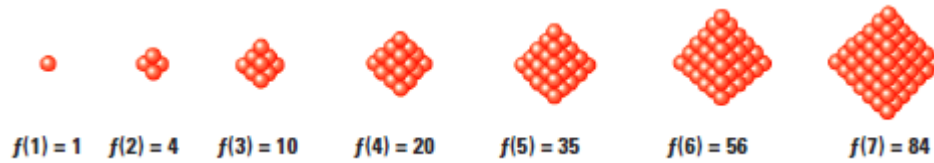**First-order differences**

**Second-order differences**
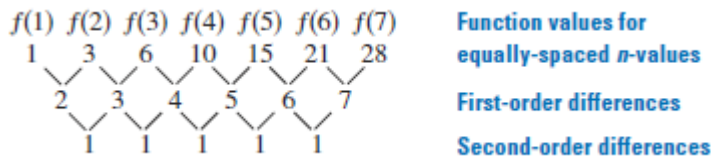
Notice that the second-order differences are constant.  Also note that the function has a degree of two.  This illustrates the property of finite differences.

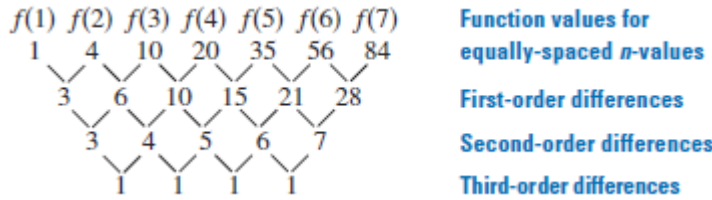**PROPERTIES OF FINITE DIFFERENCES**

1. If a polynomial function $f(x)$ has degree $n$, then the $n$th-order differences of function values for equally spaced $x$-values are nonzero and constant.
2. Conversely, if the $n$th-order differences of equally-spaced data are nonzero and constant, then the data can be represented by a polynomial function of degree $n$.

**Ex 25:** The first six triangular pyramidal numbers are shown below. Find the degree of the polynomial function that gives the *n*th triangular pyramidal number.

$f(1) = 1$   $f(2) = 4$   $f(3) = 10$   $f(4) = 20$   $f(5) = 35$   $f(6) = 56$   $f(7) = 84$

Find the finite differences.

$$f(1) \; f(2) \; f(3) \; f(4) \; f(5) \; f(6) \; f(7)$$
$$1 \quad 4 \quad 10 \quad 20 \quad 35 \quad 56 \quad 84$$

**Function values for equally-spaced *n*-values**

$$3 \quad 6 \quad 10 \quad 15 \quad 21 \quad 28$$

**First-order differences**

$$3 \quad 4 \quad 5 \quad 6 \quad 7$$

**Second-order differences**

$$1 \quad 1 \quad 1 \quad 1$$

**Third-order differences**

Because the third-order differences are constant, the degree of the polynomial would be 3 or cubic.

**Ex 26:** Determine what degree the polynomials represented by the tables below will be.

| X | Y |
|---|---|
| 10 | 1018 |
| 11 | 1349 |
| 12 | 1746 |
| 13 | 2215 |
| 14 | 2762 |
| 15 | 3393 |

| X | Y |
|---|---|
| -5 | -1250 |
| 0 | 0 |
| 5 | -1250 |
| 10 | -20,000 |
| 15 | -101,250 |
| 20 | -320,000 |

Because third-order differences are constant,

the degree of the polynomial would be 3.

Because fourth-order differences are constant,

the degree of the polynomial would be 4.

## Unit 4.11 and 4.12  Modeling Data and Transforming Polynomial Functions

**Ex 27:**    The town of Frostburg experienced a bit of a heat wave during January of this year.  The graph below shows the curve of best fit that represents the low temperature of every day in January.
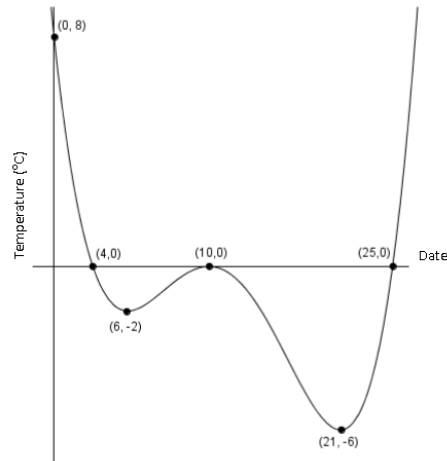


A newspaper journalist is writing a story on the weather and needs to report some information.  He needs a bit of guidance with interpreting the graph.

a)  Write a few sentences describing the key characteristics of the graphs as it relates to the context of the problem.  Be sure to include domain, range, intervals where the function increases and decreases, x and y intercepts, and any other important information

There are not many temperature changes in the month of January. First day of January brings us a temperature of 8 degrees but it will stay above zero for four days only. Starting with January 4$^{th}$ the temperature will drop below 0 and will continue in the negative zone for another three days until January 24$^{th}$.The lowest of the month will be on the 21$^{st}$ when it will be -6 degrees. For the first 6 days of the month the temperature is decreasing, reaching -2 degrees on January 6th, then it will rise for the next 4 days, reaching 0 degrees on January 10$^{th}$, then will drop again until the 21$^{st}$ when is going to reach -6 degrees the lowest of the month. After this date the temperature will only increase and after the 25$^{th}$ we will experience only positive values.

The graph below shows the curve of best fit that represents the low temperature of every day in February.
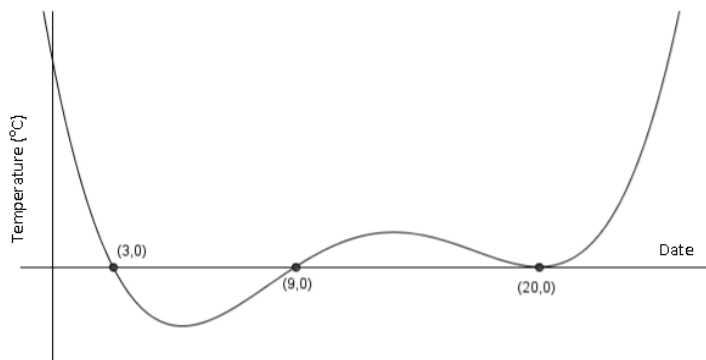
b)  Three different models have been proposed that could be used to determine the temperature for a particular date in February.  The models are given below:

Model 1:        $y = ax^2 + bx + c$

Model 2:        $y = a(x-3)(x-9)(x-20)^2$

Model 3:        $y = a(x+3)(x+9)(x+20)^2$

Which model would best describe the low temperatures for February?  Explain why you chose that model.

By looking at the end behavior of the graph, it is an even function.  This means that it cannot be Model 1 as it is a degree of 2 and the graph has 3 x-intercepts.  It is Model 2 and not Model 3 because at $x = 20$ the graph is tangent to the $x$-axis so the multiplicity of the zero must be an even number.  At $x = 3$ and $x = 9$ the graph crosses the $x$-axis so their multiplicity is an odd number.

The weather in July showed a related pattern to the weather in February.  The curve of best fit for July is shown below:
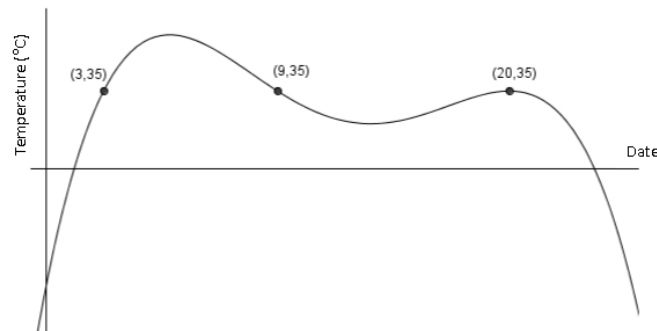


c)  Explain the relationship between the graph for February and the graph for July.  Use that relationship to create an equation for the temperatures in July.

Let's name the function that describes the temperature in February $f(x)$ and the one in July $g(x)$. Looking at the two graphs, $g(x)$ is a reflection of $f(x)$ with respect to the $x$-axis, followed by a vertical shift 35 up, so the equation that connects the two functions is:  $g(x) = -f(x) + 35$

Since  $f(x) = a(x-3)(x-9)(x-20)^2$ then  $g(x) = -a(x-3)(x-9)(x-20)^2 + 35$

**Ex 28:**  An object is dropped from a height of 2 meters.  Its distance from the ground is recorded every five hundredths of a second.  Find a polynomial function to model this data.
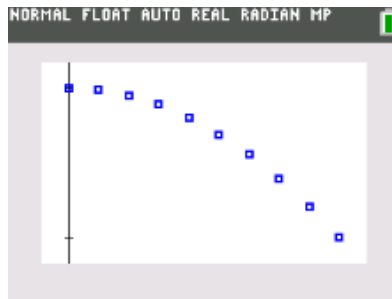
Step One:  Use finite differences to decide the degree of the polynomial that would best fit the data.  Notice that the $2^{nd}$ differences are constant, so this would best be fit with a quadratic model.

Step Two:  Input the data in a graphing calculator.  Use quadratic regression to find the best fitting model for the data.

| Time (s) $x$ | Height (m) $y$ | $D_1$ | $D_2$ |
|---|---|---|---|
| 0.00 | 2.000 | | |
| | | −0.012 | |
| 0.05 | 1.988 | | −0.025 |
| | | −0.037 | |
| 0.10 | 1.951 | | −0.024 |
| | | −0.061 | |
| 0.15 | 1.890 | | −0.025 |
| | | −0.086 | |
| 0.20 | 1.804 | | −0.024 |
| | | −0.110 | |
| 0.25 | 1.694 | | −0.025 |
| | | −0.135 | |
| 0.30 | 1.559 | | −0.024 |
| | | −0.159 | |
| 0.35 | 1.400 | | −0.025 |
| | | −0.184 | |
| 0.40 | 1.216 | | −0.024 |
| | | −0.208 | |
| 0.45 | 1.008 | | |

The solution is $a = -4.9$, $b \approx 0$, and $c \approx 2$, so an equation that fits the data is

$$y = -4.9x^2 + 2$$

**Ex 29:** Start with a 21.5-by-29 cm sheet of paper. Orient the paper so that the long side is horizontal. Fold the upper left corner so that it touches some point on the bottom edge. Find the area, in cm², of the triangle formed in the lower left corner. What distance, $x$, along the bottom edge of the paper produces the triangle with greatest area?
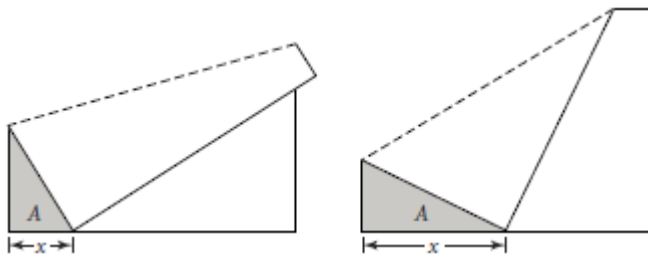


First find the areas for different values of $x$. Then, find a formula for the area of the triangle in terms of $x$. Let $h$ be the height of the triangle. Then, the hypotenuse has length $21.5 - h$.
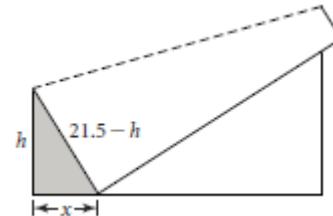


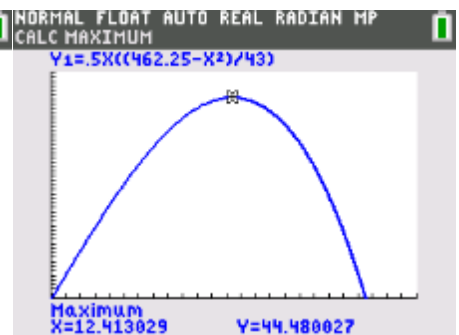Use the Pythagorean Theorem to write $h$ in terms of $x$.

$x^2 + h^2 = (21.5 - h)^2$, so $h = \dfrac{462.25 - x^2}{43}$   Now, we can write the formula for the area.   $f(x) = \dfrac{1}{2}x\left(\dfrac{462.25 - x^2}{43}\right)$

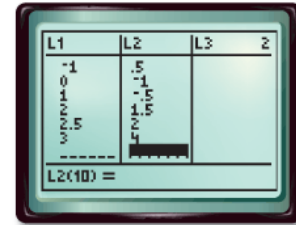Graph the function on a graphing calculator. Find the maximum value of the function.
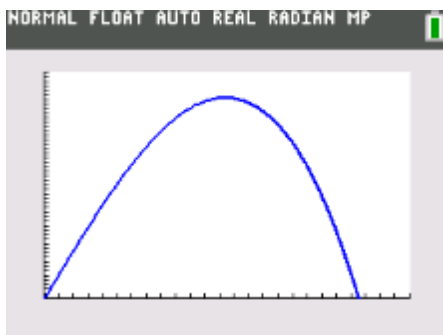






The value of $x$ that gives the greatest area is about 12.4 cm. The maximum area is about **44.5 cm²**.

**Ex 30:  Use a graphing calculator to write a polynomial function to model the set of data.**

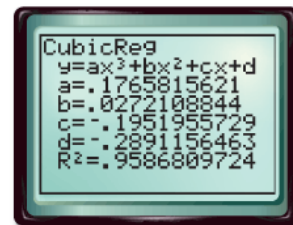| **x** | -3 | -2.5 | -2 | -1 | 0 | 1 | 2 | 2.5 | 3 |
|-------|----|------|----|----|----|----|----|-----|----|
| **f(x)** | -4 | -3 | -1 | 0.5 | -1 | -0.5 | 1.5 | 2 | 4 |

Clear the statistical memory and input the data.

Using the standard viewing window (Zoom 9), graph the
data. The scatter plot seems to change direction
two times, so a cubic function would best fit the
scatter plot.

$[-10, 10]$ scl: 1 by $[-10, 10]$ scl: 1

Press **STAT** and highlight **CALC**. Since the scatter
plot seems to be a cubic function, choose
**6:CubicReg.**

Rounding the coefficients to the nearest hundredth,
$f(x) = 0.18x^3 + 0.03x^2 - 0.20x - 0.29.$

To check the polynomial function, enter the function
in the **Y=** list. Then use the **TABLE** feature for *x* values
from -3 with an interval of 0.5. Compare the values of
*y* with those given in the data. The values are similar,
and the polynomial function seems to be a good fit.
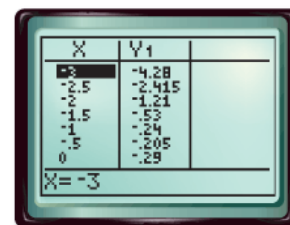
**SAMPLE EXAM QUESTIONS**

1. **Which of the following represents the solution set of the polynomial equation** $x^4 - 7x^2 + 12 = 0$ **?**

   **A.** $\left\{2, -2, i\sqrt{3}, -i\sqrt{3}\right\}$

   **B.** $\left\{2, -2, \sqrt{3}, -\sqrt{3}\right\}$

   **C.** $\left\{2i, -2i, i\sqrt{3}, -i\sqrt{3}\right\}$

   **D.** $\left\{2i, -2i, \sqrt{3}, -\sqrt{3}\right\}$

   <span style="color:red">Ans: A</span>

2. **Which lists the set of all <u>real</u> zeros of the following polynomial function?**

   $$f(x) = x^3 + 3x^2 + 4x + 12$$

   **A.** $\{-3\}$

   **B.** $\{-3, 2\}$

   **C.** $\{-3, -2, 2\}$

   **D.** $\{-3, -2, 1, 2\}$

   <span style="color:red">Ans: A</span>

3. **According to the Fundamental Theorem of Algebra, how many complex zeros does the polynomial** $f(x) = -5x^4 + 2x^3 + x - 1$ **have?**

   **A.** 2

   **B.** 3

   **C.** 4

   **D.** 5

   <span style="color:red">Ans: C</span>

4. **According to the Fundamental Theorem of Algebra, how many roots does the following equation have?**        $6x^2 + 4 = 11x$

   (A)    2                    (B)    4

   (C)    6                    (D)    11

   <span style="color:red">Ans: A</span>

5.  **A 4<sup>th</sup> degree polynomial with real coefficients is found to have exactly two distinct real roots. What must be true about the other two roots?**

    A.   One root is real and the other is imaginary.

    B.   Both roots must be real.

    C.   Both roots are imaginary roots that are complex conjugates.

    D.   All the roots have been found.

    Ans:  C

6.  **If $f(x) = 2x^4 + 7x^3 + 3x^2 - 8x - 4$, find the possible rational roots of $f(x)$.**

    (A)     $x = 1,\ 4$

    (B)     $x = 1,\ \pm 4$

    (C)     $x = \pm\dfrac{1}{2},\ \pm 1,\ \pm 2$

    (D)     $x = \pm\dfrac{1}{2},\ \pm 1,\ \pm 2,\ \pm 4$

    Ans:  D

7.  **Given polynomial $q(x)$, $q(4) = 6$. Which statement is correct?**

    (A)     $x = 4$ is not a root

    (B)     $x = 4$ is a root

    (C)     $(x - 4)$ is a factor

    (D)     $(x + 4)$ is not a factor

    Ans:  A

8.  **Write a cubic function that passes through the following points:  (-2, 0) (3, 0) (-1, 0) and (1, 2).**

    (A)     $y = x^3 - 7x - 6$

    (B)     $y = -x^3 + 7x + 6$

    (C)     $y = \dfrac{1}{6}x^3 - \dfrac{7}{6}x + 1$

    (D)     $y = -\dfrac{1}{6}x^3 + \dfrac{7}{6}x + 1$

    Ans:  D

**9. How many possible rational zeros exist for the polynomial function** $y = 3x^6 + 9x^2 + 4x - 12$ **?**

(A)    9

(B)    12

(C)    18

(D)    24

<span style="color:red">Ans:  C</span>

**10.** $p(x) = 3x^5 + 13x^4 + 19x^3 + 17x^2 + 16x + 4$

a)   Show that $p(-2)$ is a root.

You could substitute $x = -2$, if $p(-2) = 0$, then it is a root.

Based on the Remainder Theorem you could use synthetic division to show that you get a remainder of zero.

$$
\begin{array}{r|rrrrrr}
-2 & 3 & 13 & 19 & 17 & 16 & 4 \\
   &   & -6 & -14 & -10 & -14 & -4 \\
\hline
   & 3 & 7 & 5 & 7 & 2 & 0
\end{array}
$$

b)   Factor $p(x)$ completely.

With the result from part a, that $-2$ is a root, we know that $(x+2)$ is a factor.  Using the Possible Rational Zero Theorem, list all the possible rational zeros for $3x^4 + 7x^3 + 5x^2 + 7x + 2$ which is the remainder after synthetically dividing $p(x)$ by $-2$.  Possible rational zeros are:

$$\frac{\pm 1 \quad \pm 2}{\pm 1 \quad \pm 3} = \pm 1, \pm \frac{1}{3} \pm 2, \pm \frac{2}{3}$$

Synthetic division show that $-2$ is a zero, so the multiplicity for $-2$ becomes 2.

$$
\begin{array}{r|rrrrr}
-2 & 3 & 7 & 5 & 7 & 2 \\
   &   & -6 & -2 & -6 & -2 \\
\hline
   & 3 & 1 & 3 & 1 & 0
\end{array}
$$

So $p(x) = (x+2)^2 (3x^3 + x^2 + 3x + 1)$

We can now factor $3x^3 + x^2 + 3x + 1$ by grouping:

$3x^3 + x^2 + 3x + 1$

$x^2 (3x+1) + (3x+1)$

$(3x+1)(x^2 + 1)$

Finally, the complete factorization is $\boxed{p(x) = (x+2)^2 (3x+1)(x^2+1)}$

c) If $f(x) = p(x-3)$, what are the real roots of $f(x)$?

$$f(x) = p(x-3) = (x-3+2)^2 \left[3(x-3)+1\right]\left[(x-3)^2+1\right] = (x-1)^2(3x-8)\left[(x-3)^2+1\right]$$

so the real roots of $f(x)$ are : $1$ and $\dfrac{8}{3}$.

**11. Given the polynomial $p(x) = x^4 + 3x^3 + 12x - 16$:**

a) Show that $p(2i)$ is a root.

$$p(2i) = (2i)^4 + 3(2i)^3 + 12(2i) - 16 = 16 - 24i + 24i - 16 = 0 \text{, so } 2i \text{ is a zero of the polynomial.}$$

b) What other root must also be a root of $p(x)$? Explain.

If a polynomial with real coefficients has a complex zero, then its complex conjugate will be a zero. Therefore, since $2i$ was a zero, its conjugate, $-2i$, will also be a zero.

c) Factor $p(x)$ completely.

Since -2i and 2i are zeros we can divide by $\left(x-(-2i)\right)(x-2i) = \left(x^2+4\right)$ knowing that it will divide evenly.

$$
\begin{array}{r}
x^2 + 3x - 4 \\
x^2 + 4 \overline{\smash{\big)}\, x^4 + 3x^3 \qquad\ +12x - 16} \\
\underline{-x^4 \qquad\ -4x^2 \qquad\qquad} \\
+3x^3 - 4x^2 + 12x - 16 \\
\underline{-3x^3 \qquad\quad -12x \qquad} \\
-4x^2 \qquad -16 \\
\underline{+4x^2 \qquad +16} \\
0
\end{array}
$$

So, $p(x) = (x-2i)(x+2i)\left(x^2+3x-4\right)$. $x^2+3x-4$ will factor into $(x+4)(x-1)$ so the complete factorization of $p(x)$ is: $\boxed{p(x) = (x-2i)(x+2i)(x+4)(x-1)}$.