Graphing Polynomial Functions

Math Background

Previously, you

- Graphed linear and quadratic functions
- Identified key features of linear and quadratic functions
- Identified polynomials through degree and number of terms
- Factored and divided polynomials
- Applied the Factor and Remainder Theorems

In this unit you will

- Identify key features of the graphs of polynomial functions
- Graph polynomial functions using end behavior and zeros
- Solve problems using the graphs of polynomial functions

You can use the skills in this unit to

- State the end behavior of given polynomials.
- Identify zeros based on graphs of polynomials.
- Identify relative maximums and minimums of the graphs.
- Model and solve real-world problems with polynomial graphs.

Vocabulary

- **Degree of a polynomial** – The highest power of the variable in a polynomial.
- **End behavior** – The behavior of the graph of a polynomial function as *x* approaches positive or negative infinity.
- **Global (or Absolute) Maximum –** A value of a given function that is greater than or equal to any value of the given function. An absolute maximum is the greatest of all values.
- **Global (or Absolute) Minimum –** A value of a given function that is less than or equal to any value of the given function. An absolute minimum is the lowest of all values.
- **Leading Coefficient –** The coefficient of the term with the highest degree.
- **Local (or Relative) Maximum –** A value of a function that is greater than those values of the function at the surrounding points, but is not the greatest of all values.
- **Local (or Relative) Minimum** – A value of a function that is less than those values of the function at the surrounding points, but is not the lowest of all values.
- **Multiplicity** – It is how often a certain root or zero is part of the factoring. It is the number of times the root is a zero of the function.
- **Zeros of a function** – An input value that produces an output of zero. It is also known as a root and is where the graph meets the x-axis, also known as an x-intercept.

Essential Questions

- What does a polynomial function look like? How do I identify the zeros and the end behavior?
- Why are graphs of polynomials important?

Overall Big Ideas

The sketch of a polynomial is a continuous graph with possible changes in shape and direction. The zeros are located where the graph crosses the x-axis. The end behavior is determined by the leading coefficient and the degree of the polynomial.

Polynomial equations provide some of the most classic problems in all of algebra. Finding zeros and extrema have many real-world applications. Real-life situations are modeled by writing equations based on data and using those equations to determine or estimate other data points (speed, volume, time, profits, patterns, etc.).

Skill

To use properties of end behavior to describe, analyze and graph polynomial functions.

Identify and use maxima and minima of polynomial functions to solve problems.

Related Standards

F.IF.C.7c

Graph polynomial functions, identifying zeros when suitable factorizations are available, and showing end behavior. *(Modeling Standard)

F.IF.B.4-2

For any function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity. *(Modeling Standard)

Notes, Examples, and Exam Questions

Graphs of Polynomials:

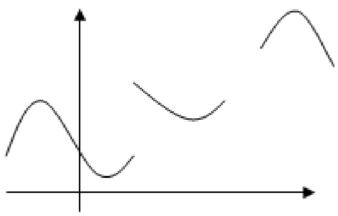
*The graphs of polynomials of degree 0 or 1 are lines.

*The graphs of polynomials of degree 2 are parabolas.

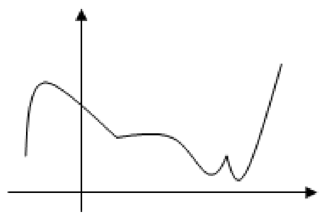
*The greater the degree of the polynomial, the more complicated its graph can be.

*The graph of a polynomial function is always a smooth curve; that is it has no breaks or corners.

Examples of graphs of non-polynomial functions:

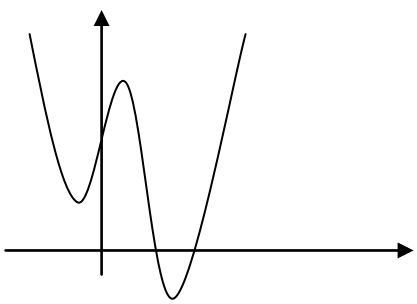


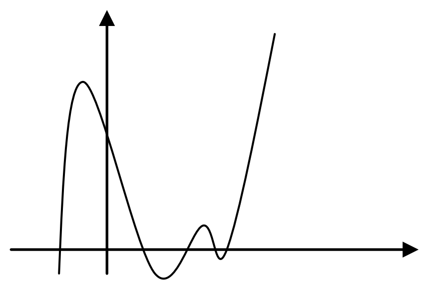
There is a break and a hole.



There is a corner and a sharp turn.

Examples of graphs of polynomial functions:





Graphing Polynomial Functions: To graph a polynomial function, make a table of values using synthetic substitution, plot the points, and determine the *end behavior* to draw the rest of the graph.

End Behavior: the behavior of the graph as x gets very large (approaches positive infinity $(+\infty)$) OR as x gets very small (or approaches negative infinity $(-\infty)$).

Notation: $x \to +\infty$ (x approaches positive infinity) (The very far right end of a graph).

$x \to -\infty$ (x approaches negative infinity) (The very far left end of a graph).

Exploration Activity: Graph each function on the calculator. Determine the end behavior of $f(x)$ as x approaches negative and positive infinity. Fill in the table and write your conclusion regarding the degree of the function and the end behavior. (Teacher Note: Answers are in red.)

$f(x)$	Degree	Sign of Leading Coefficient	$x \to$ ____	$f(x) \to$ ____
$f(x)=x^2$	2	$+$	$+\infty$ $-\infty$	$+\infty$ $+\infty$
$f(x)=-x^2$	2	$-$	$+\infty$ $-\infty$	$-\infty$ $-\infty$
$f(x)=x^3$	3	$+$	$+\infty$ $-\infty$	$+\infty$ $-\infty$
$f(x)=-x^3$	3	$-$	$+\infty$ $-\infty$	$-\infty$ $+\infty$
$f(x)=x^4$	4	$+$	$+\infty$ $-\infty$	$+\infty$ $+\infty$
$f(x)=-x^4$	4	$-$	$+\infty$ $-\infty$	$-\infty$ $-\infty$
$f(x)=x^5$	5	$+$	$+\infty$ $-\infty$	$+\infty$ $-\infty$
$f(x)=-x^5$	5	$-$	$+\infty$ $-\infty$	$-\infty$ $+\infty$
$f(x)=x^6$	6	$+$	$+\infty$ $-\infty$	$+\infty$ $+\infty$
$f(x)=-x^6$	6	$-$	$+\infty$ $-\infty$	$-\infty$ $-\infty$

Conclusion: The graph of a polynomial function

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$$

has the following end behavior. These patterns are very predictable.

Degree	Leading Coefficient	End Behavior
Even	Positive	as $x \to -\infty,\ f(x) \to +\infty$ as $x \to +\infty,\ f(x) \to +\infty$
Even	Negative	as $x \to -\infty,\ f(x) \to -\infty$ as $x \to +\infty,\ f(x) \to -\infty$
Odd	Positive	as $x \to -\infty,\ f(x) \to -\infty$ as $x \to +\infty,\ f(x) \to +\infty$
Odd	Negative	as $x \to -\infty,\ f(x) \to +\infty$ as $x \to +\infty,\ f(x) \to -\infty$

Think of end behavior as what happens on either end of the graph. There can be a lot of curves, etc. in the middle, but polynomial functions either increase or decrease at the far ends (as $x \to \pm\infty$, $f(x) \to \pm\infty$).

Ex 1: Graph the polynomial function $f(x) = x^4 - 5x^3 + x - 1$ by hand. Check your graph on the graphing calculator.

Step One: Make a table of values using synthetic substitution.

x	-2	-1	0	1	2
$f(x)$	53	4	-1	-4	-23

$$\begin{array}{r|rrrrr} -2 & 1 & -5 & 0 & 1 & -1 \\ & & -2 & 14 & -28 & 54 \\ \hline & 1 & -7 & 14 & -27 & \boxed{53} \end{array} \qquad \begin{array}{r|rrrrr} -1 & 1 & -5 & 0 & 1 & -1 \\ & & -1 & 6 & -6 & 5 \\ \hline & 1 & -6 & 6 & -5 & \boxed{4} \end{array}$$

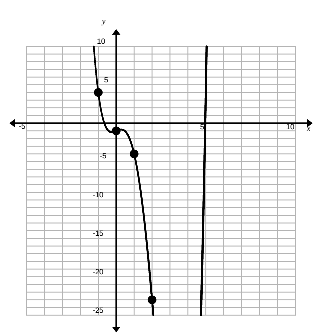
$$\begin{array}{r|rrrrr} 1 & 1 & -5 & 0 & 1 & -1 \\ & & 1 & -4 & -4 & -3 \\ \hline & 1 & -4 & -4 & -3 & \boxed{-4} \end{array} \qquad \begin{array}{r|rrrrr} 2 & 1 & -5 & 0 & 1 & -1 \\ & & 2 & -6 & -12 & -22 \\ \hline & 1 & -3 & -6 & -11 & \boxed{-23} \end{array}$$

Why didn't we use synthetic substitution to find $f(0)$?

Step Two: Determine end behavior using the degree and sign of the leading coefficient.

The degree is even, and the leading coefficient is positive. So, as $x \to -\infty,\ f(x) \to +\infty$; as $x \to +\infty,\ f(x) \to +\infty$.

Step Three: Graph the polynomial function.



Ex 2: Graph the polynomial function $f\left(x\right)=-x^3+2x^2-4$ by hand. Check on the graphing calculator.

Step One: Make a table of values using synthetic substitution.

x	-2	-1	0	1	2
$f\left(x\right)$	12	-1	-4	-3	-4

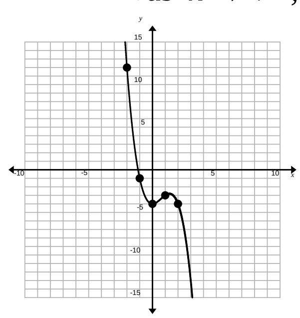
$$\begin{array}{r|rrrr} -2 & -1 & 2 & 0 & -4 \\ & & 2 & -8 & 16 \\ \hline & -1 & 4 & -8 & \boxed{12} \end{array} \qquad \begin{array}{r|rrrr} -1 & -1 & 2 & 0 & -4 \\ & & 1 & -3 & 3 \\ \hline & -1 & 3 & -3 & \boxed{-1} \end{array}$$

$$\begin{array}{r|rrrr} 1 & -1 & 2 & 0 & -4 \\ & & -1 & 1 & 1 \\ \hline & -1 & 1 & 1 & \boxed{-3} \end{array} \qquad \begin{array}{r|rrrr} 2 & -1 & 2 & 0 & -4 \\ & & -2 & 0 & 0 \\ \hline & -1 & 0 & 0 & \boxed{-4} \end{array}$$

Step Two: Determine end behavior using the degree and sign of the leading coefficient.

The degree is odd and the leading coefficient is negative. So, $\begin{cases} \text{as } x\to-\infty, & f\left(x\right)\to+\infty \\ \text{as } x\to+\infty, & f\left(x\right)\to-\infty \end{cases}$

Step Three: Graph the polynomial function.



Analyzing polynomial graphs

Concept Summary:

Let $f\left(x\right)=a_nx^n+a_{n-1}x^{n-1}+...+a_2x^2+a_1x+a_0$ be a polynomial function.

The following statements are equivalent:

Zero: ***k* is a zero of the function *f*.**

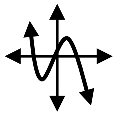
Factor: ***x* − *k* is a factor of polynomial *f(x)*.**

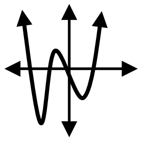
Solution: ***k* is a solution of the polynomial function *f(x)=0*.**

***x* - Intercept:** ***k* is an *x*-intercept of the graph of the polynomial function *f*.**

<u>**QOD**</u>: Which term of the polynomial function is most important when determining the end behavior of the function?

Ex 3: Indicate if the degree of the polynomial function shown in the graph is odd or even and indicate the sign of the leading coefficient.

a)  Odd degree; Negative leading coefficient

b)  Even degree; Positive leading coefficient

Using Zeros to Graph a Polynomial: We use the values of x that make the polynomial equal to zero.

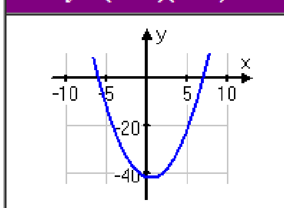
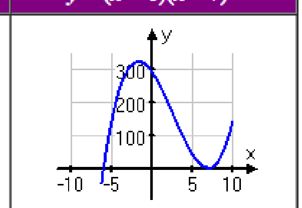
If $P(x) = (x-2)(x+3)$ then the zeros of the polynomials are 2 and -3.

This means that the graph of this polynomial crosses the x-axis at 2 and -3 (2 and -3 are the x intercepts)

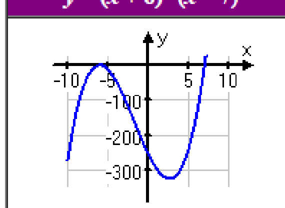
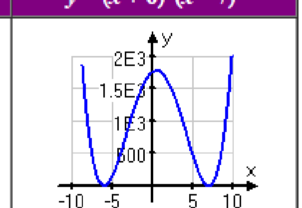
Zeros of Polynomial Functions

Multiplicity: This refers to the number of times the root is a zero of the function. We can have "repeated" zeros. If a polynomial function *f* has a factor of $(x-c)^m$, and not $(x-c)^{m+1}$, then c is a zero of **multiplicity** *m* of *f*.

- Odd Multiplicity: *f* crosses the *x*-axis at *c*; $f(x)$ changes signs
- Even Multiplicity: *f* "kisses" or is tangent to the *x*-axis at *c*; $f(x)$ doesn't change signs

$y = (x+6)(x-7)$	$y = (x+6)(x-7)^2$
	
$x = -6$ once $x = 7$ once	$x = -6$ once $x = 7$ twice

All four graphs have the same zeros, at *x* = 6 and *x* = 7, but the multiplicity of the zero determines whether the graph crosses the *x*-axis at that zero or if it instead turns back the way it came.

$y = (x+6)^2(x-7)$	$y = (x+6)^2(x-7)^2$
	
$x = -6$ twice $x = 7$ once	$x = -6$ twice $x = 7$ twice

<u>Guidelines for Sketching Graphs Polynomial Functions:</u>

✓ <u>Zeros</u>: Factor the polynomial to find all its real zeros; these are the x-intercepts of the graphs.

✓ <u>Test points:</u> Make a table of values for the polynomial. Include test points to determine whether the graph of the polynomial lies above or below the x-axis on the intervals determined by the zeros. Include the y-intercept in the table.

✓ <u>End Behavior:</u> Determine the end behavior of the polynomial.

✓ <u>Graph:</u> Plot the intercepts and the other points you found in the table. Sketch a smooth curve that passes through these points and exhibits the required end behavior.

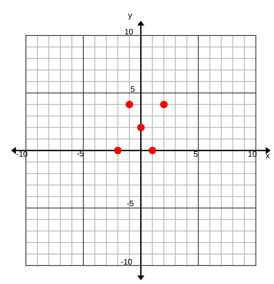
Ex 4: Graph the function $f(x) = (x+2)(x-1)^2$

Step 1: Plot the *x*-intercepts. Since *x* + 2 and *x* − 1 are factors, −2 and 1 are zeros (*x*-intercepts)

Note: *x* + 2 is raised to an odd power so the graph crosses the *x*-axis at *x* = −2. *x* −1 is raised to an even power so the graph is tangent to the *x*-axis at *x* = 1.

☆ **When a factor $x-k$ is raised to an odd power, the graph crosses through the x-axis.**
When a factor $x-k$ is raised to an even power, the graph is tangent to the x-axis.

Step 2: Plot a few points between the x-intercepts.
$f(-3) = -16$; $f(-1) = 4$; $f(0) = 2$; $f(2) = 4$

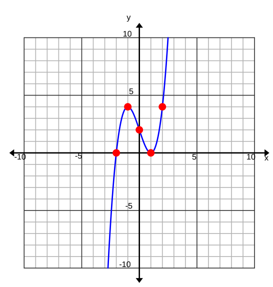


Step 3: Determine the end behavior of the graph.
Cubic function (odd degree) with positive leading coefficient

$$\text{as } x \to -\infty,\ f(x) \to -\infty$$

$$\text{as } x \to +\infty,\ f(x) \to +\infty$$

Step 4: Sketch the graph



Ex 5: Sketch the graph of $f\left(x\right)=\frac{1}{4}\left(x+1\right)^2\left(x-4\right)$. Describe the multiplicity of the zeros.

Plot the intercepts. Because -1 and 4 are zeros of *f*, plot (-1, 0) and (4, 0).

$x=-1$: multiplicity 2 $\qquad$ $x=4$: multiplicity 1

This means that it will "kiss" or is tangent to the x-axis at -1 (even multiplicity) and it will cross the x-axis at 4 (odd multiplicity)

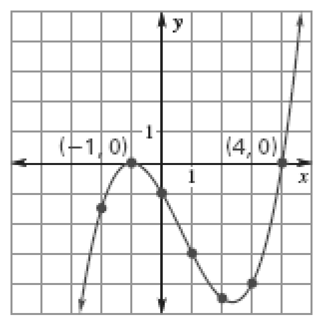
Plot points between and beyond the *x*-intercepts.

X	-2	0	1	3	5
Y	-3/2	-1	-3	-4	9

Determine the end behavior. Because *f* has three factors of the form *x* – *k* it is a cubic function. It has a positive leading coefficient.

That means that *f*(*x*) → −∞ as *x* → − ∞ → as *x* and *f*(*x*) → + ∞ as *x* → + ∞.

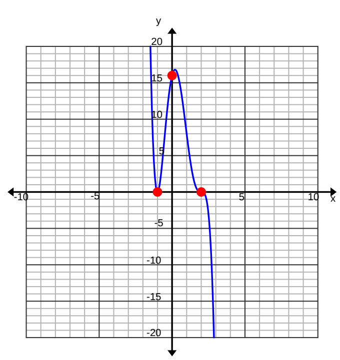
Draw the graph so that it passes through the plotted points and has the appropriate end behavior.



Ex 6: Sketch the graph of $g\left(x\right)=-2\left(x-2\right)^3\left(x+1\right)^2$. Describe the multiplicity of the zeros.

$x=2$: multiplicity 3 $\qquad$ $x=-1$: multiplicity 2 $\qquad$ *y*-intercept: $y=2\left(-8\right)\left(1\right)=16$

End behavior (odd degree of 5, negative leading coefficient): $\lim_{x\to-\infty}=\infty$; $\lim_{x\to\infty}=-\infty$

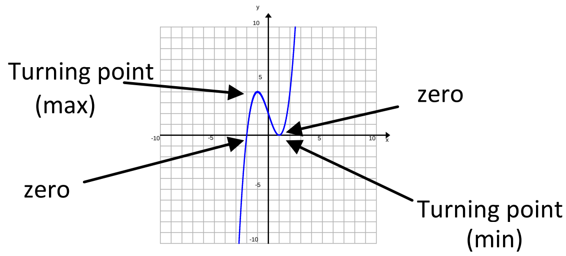


Finding Turning Points

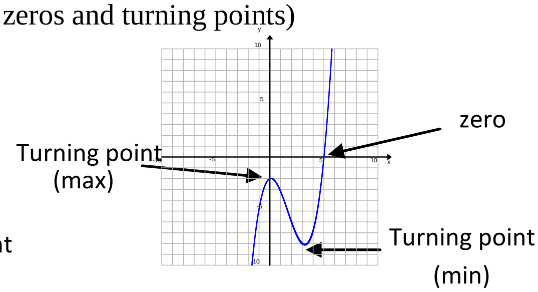
Turning points of polynomial functions: Another important characteristic of graphs of polynomial functions is that they have turning points corresponding to local maximum and minimum values. The y – coordinate of a turning point is a ***local or relative maximum*** if the point is higher than all nearby points. The y – coordinate of a turning point is a ***local or relative minimum*** if the point is lower than all nearby points. Global or absolute minimums and maximums are the greatest or least values of the entire function.

> The graph of every polynomial function of degree n has **at most** $n - 1$ turning points. Moreover, if a polynomial has n distinct real zeros, then its graph has exactly $n - 1$ turning points.

Ex 7: Identify the zeros and turning points (estimate the zeros and turning points)

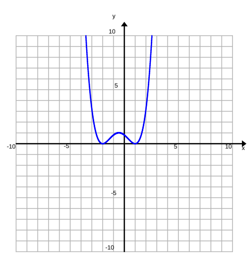


Leading coefficient positive
3 real zeros (including the double zero)
{-2, 1, 1}
2 turning points
(-1,4); (1,0)
1 local max; 1 local min



Leading coefficient positive
1 real zero, 2 imaginary zeros
{5}
2 turning points
(0,-2); (3,-8)
1 local max; 1 local min

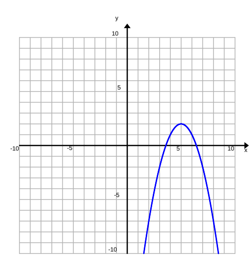
You Try:



Leading coefficient __________
____real zeros

____turning points

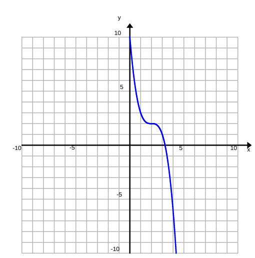
____local max; ____local min



Leading coefficient __________
____real zeros

____turning points

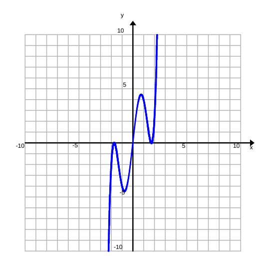
____local max; ____local min



Leading coefficient __________
____real zeros

____turning points

____local max; ____local min



Leading coefficient __________
____real zeros

____turning points

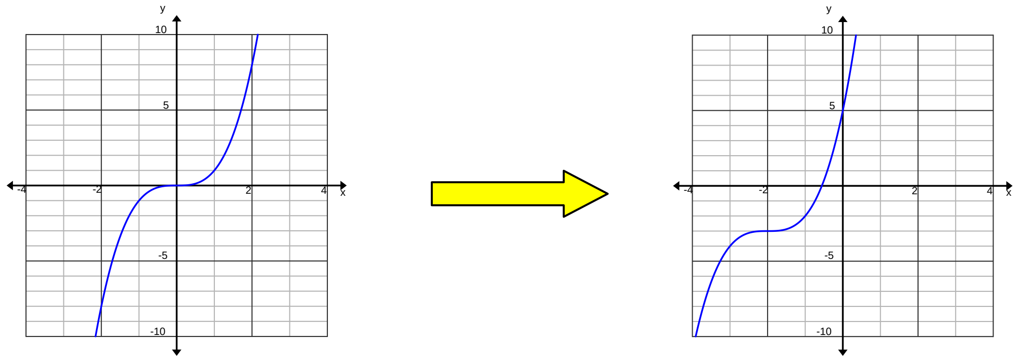
____local max; ____local min

More on Graphing Polynomial Functions:

Continue to reinforce transformations while graphing polynomial functions. Relate this to the functions previously translated in Units One and Three – quadratic, absolute value, linear and exponential.

Ex 8: Consider the cubic function: $f(x)=x^3$. How can we graph $g(x)$ if $g(x)=(x+2)^3-3$?

The graph of $f(x)$ would start in the third quadrant, increase, cross at the origin and continue increasing in the first quadrant. $g(x)$ is transformed with a horizontal translation of 2 to the left and a vertical translation of 3 down, giving us the following graph:



f(x) g(x)

Ex 9: Consider $p(x)=2x^4-x^3-11x^2+5x+5$. $h(x)$ is $p(x)$ translated 4 units right and 2 units up and a vertical stretch of 3. What is the equation of $h(x)$?

Recall from transformations the following general format: $f(x)=a(x-h)^p+k$. The variable *a* relates to vertical stretches or shrinks, *k* relates to the vertical translation and the variable *h* relates to the horizontal translation. Inputting the values 3, 2 and 4, we get:

$$h(x)=3p(x-4)+2=3\left[2(x-4)^4-(x-4)^3-11(x-4)^2+5(x-4)+5\right]+2$$

Ex 10: Consider the function $f(x)=3x^3-9x^2-3x+9$.

a) Use the leading coefficient and degree of $f(x)$ to describe the end behavior.

The leading term is $3x^3$ so as x increases without bound, $f(x)$ increases without bound and as x decreases without bound $f(x)$ decreases without bound. $\lim_{x\to\infty} f(x)\to\infty$ and $\lim_{x\to-\infty} f(x)\to-\infty$

b) Write the rule for the function $g(x)=f(-x)$, and describe the transformation.

$$g(x)=f(-x)=3(-x)^3-9(-x)^2-3(-x)+9=-3x^3-9x^2+3x+9$$

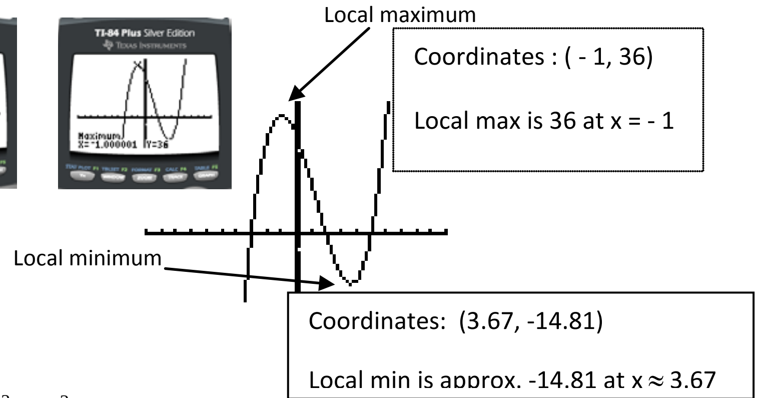
The transformation here is a reflection with respect to the *y*-axis since the input values were replaced with their opposites.

c) Describe the end behavior of $g(x)$. How does the end behavior of $g(x)$ relate to the transformation of $f(x)$?

Since $g(x)$ is a reflection of $f(x)$, as x increases without bound, $g(x)$ decreases without bound and as x decreases without bound, $g(x)$ increases without bound. $\lim_{x\to\infty} g(x)\to-\infty$ and $\lim_{x\to-\infty} g(x)\to\infty$

Using Technology to Approximate Zeros

Ex 11: Use a graphing calculator to graph and calculate the approximate local maximum(s) and local minimum(s) of $f(x) = (x-2)(x+3)(x-5)$. Use the Calculate: minimum or Calculate: maximum



Ex 12: Approximate the real zeros of $f(x) = x^4 - 2x^3 - x^2 - 2x - 2$

Use a graphing calculator to graph and calculate the zeros. Use the Calculate: zero

QOD: What is the difference between local and absolute maxima and minima?

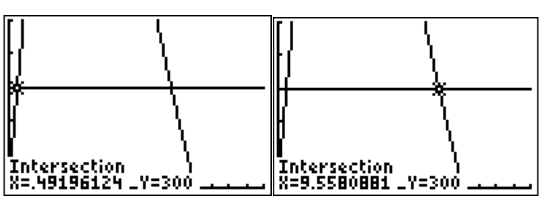
Application of Polynomial Functions

Ex 13: You cut equal squares from the corners of a 22 by 30 inch sheet of cardboard to make a box with no top. What size squares would need to be cut for the volume to be 300 cubic inches?

Define x as the length of the sides of the squares. Write a formula for the volume of the box.

$$V(x) = x(22-2x)(30-2x)$$

Solve the equation $300 = x(22-2x)(30-2x)$ by graphing.

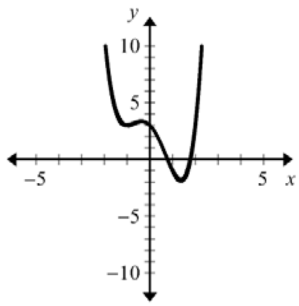


Squares with lengths of approximately 0.492 or 9.558 inches should be cut.

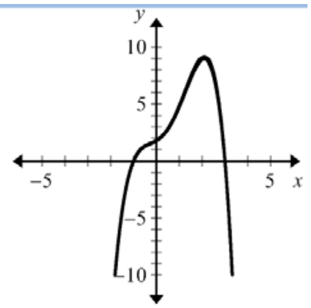
SAMPLE EXAM QUESTIONS

1. Which best represents the graph of the polynomial function $y = -x^4 + 2x^2 + 2x + 3$?

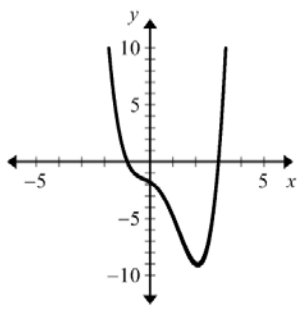
A.



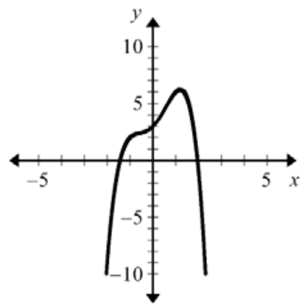
C.



B.



D.



Ans: D

2. Which describes the end behavior of the graph of $f(x) = x^4 - 5x + 2$ as $x \to -\infty$?

A. $f(x) \to -\infty$

B. $f(x) \to +\infty$

C. $f(x) \to 0$

D. $f(x) \to 2$

Ans: B

3. Use the graph of the polynomial function.

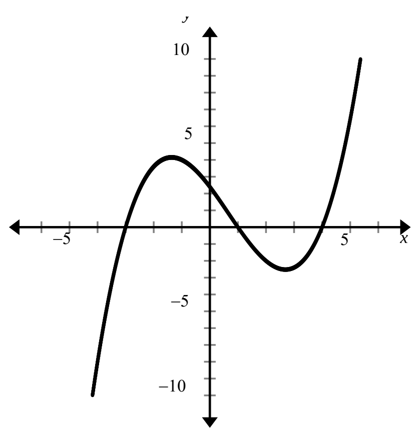
What are the zeros of the polynomial?

A. $\{2\}$

B. $\{-2\}$

C. $\{-3,\ 1,\ 4\}$

D. $\{3,\ -1,\ -4\}$



Ans: C

4. The table lists all the real roots of a 5^{th} degree polynomial $p(x)$ and the multiplicity of each root.

x	Multiplicity
–3	1
–1	1
1	2
2	1

Which general factorization correctly represents $p(x)$?

A. $a(x-3)(x-1)^2(x-2)$

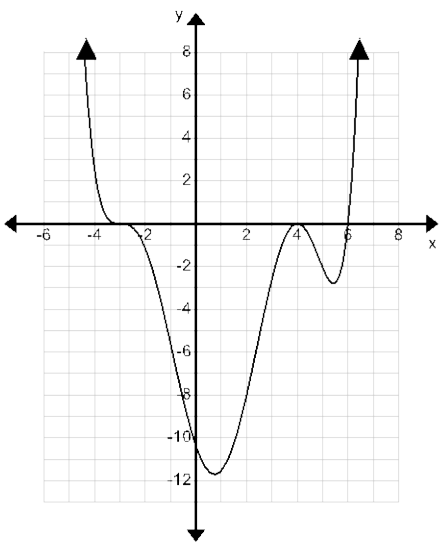
B. $a(x+3)(x-1)^3(x-2)$

C. $a(x+3)(x+1)(x-1)^2(x-2)$

D. $a(x+3)(x+1)^3(x-2)$

Ans: C

5. Use the graph of $p(x)$ to answer questions



a) **True or False: The leading term of $p(x)$, when written in standard form, is positive.**

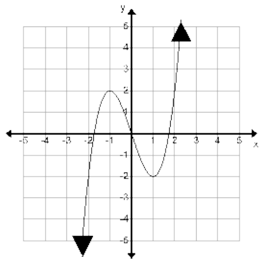
True

b) **True or False: From the graph, $p(-3)=0$. The multiplicity of the factor $(x+3)$ is even. Explain your answer.**

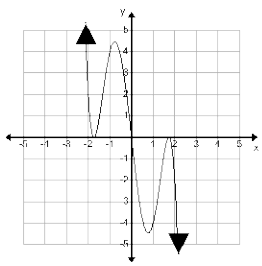
False. The multiplicity of the factor (x + 3) is odd because the graph crosses the x-axis at -3. It the multiplicity was even, it would not cross.

6. Which graph represents $f(x) = x^5 - 6x^3 + 9x$?

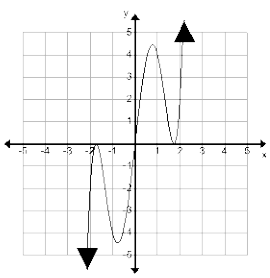
(A)



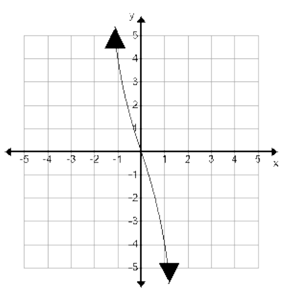
(B)



(C)

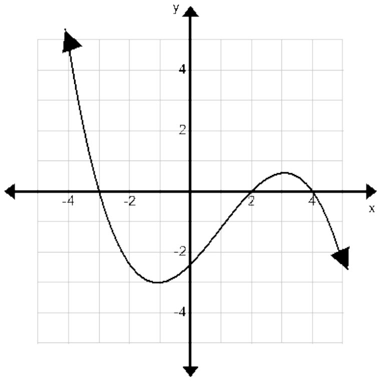


(D)



Ans: C

7. Consider the graph of $p(x)$ below. Which general factorization correctly represents $p(x)$.



Which general factorization correctly represents $p(x)$?

A. $a(x+3)(x-2)(x-4)$

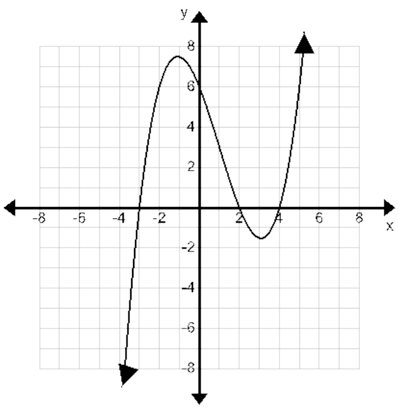
B. $a(x+3)(x+2)(x+4)$

C. $a(x-3)(x+2)(x-4)$

D. $a(x-3)(x-2)(x-4)$

Ans: A

8. The graph of $p(x)$ is shown below.



Which general factorization correctly represents $p(x)$?

A. $-4(x-3)(x-2)(x-4)$

B. $6(x-3)(x+2)(x+4)$

C. $\frac{1}{4}(x+3)(x-2)(x-4)$

D. $4(x+3)(x+2)(x+4)$

Ans: C