



Big Idea(s): Operations on sets of numbers are performed according to properties or rules. An operation works to change numbers. There are six operations in arithmetic that "work on" numbers: addition, subtraction, multiplication, division, raising to powers, and taking roots.

A binary operation requires two numbers. Addition is a binary operation, because " $5 +$ " doesn't mean anything by itself. Multiplication is another binary operation. Are all operations binary?

A.APR.A.1 Perform operations on polynomials. Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials.

N.RN.B.3 Use properties of rational and irrational numbers. Explain why the sum or product of two rational numbers is rational; that the sum of a rational number and an irrational number is irrational; and that the product of a nonzero rational number and an irrational number is irrational.

Skills:

- **Identify and apply real number properties using variables, including distributive, commutative, associative, identity, inverse, and absolute value to expressions or equations.**
- **Identify, understand, and explain the sets of numbers that are closed under given operations**

Math Background

PREREQUISITE KNOWLEDGE/SKILLS:

- properties of real numbers.
- properties of closure for sets of real numbers.

In this unit you will

- extend closure properties to polynomials.
- review or extend closure properties to irrational numbers.

You can use the skills in this unit to

- understand and predict the nature of possible solutions.

Essential Questions

- **Why are all sets not closed under all operations?**
- **How can closure help understand the type of solution you might expect with operations?**
- **Is there a pattern for closure in sets?**

Note: A review of the skills is included at the end of these notes.

Sample problems are provided to give teachers examples that guide instruction.

Sample Questions* (A.APR.A.1 and N.RN.B.3)

1. Determine if the set of polynomials is closed under division. Explain why or why not.

a. The set of polynomials is closed under division.

Just as multiplication is repeated addition, division is repeated subtraction. Since polynomials are closed under subtraction, they are also closed under division.

b. The set of polynomials is not closed under division.

Let $f(x)$ and $g(x)$ be polynomial expressions where $g(x)$ is not equal to zero.

Then $\frac{f(x)}{g(x)}$ is undefined if $g(x) = 0$. In this case, $\frac{f(x)}{g(x)}$ is not a rational expression, so

the set of polynomials is not closed under division.

c. The set of polynomials is closed under division.

Since the set of polynomials is closed under multiplication, and division is the inverse operation for multiplication, the set of polynomials is also closed under division.

d. The set of polynomials is not closed under division.

Let $f(x)$ and $g(x)$ be polynomial expressions where $g(x)$ is not equal to zero.

By the definition of polynomial expressions, $\frac{f(x)}{g(x)}$ is not a polynomial expression, so

the set of polynomials is not closed under division. (The quotient of two polynomial expressions is a rational expression.)

Solution: D DOK 3

2. Which number in the set $\left\{\frac{1}{2}, 6.1, \sqrt{15}, 92.65\right\}$ is irrational?

a. 94.05

c. $\frac{1}{2}$

b. $6.\bar{1}$

d. $\sqrt{17}$

Solution: D DOK 1

3. Which number in the set $\sqrt{16}, \sqrt{3}, \pi, 5\sqrt{3}$ is rational?

a. $\sqrt{16}$

c. $\sqrt{3}$

b. $5\sqrt{3}$

d. π

Solution: A DOK 1



4. Which word best describes the product of $\sqrt{7}$ and $\sqrt{11}$?
- Rational
 - Natural
 - Imaginary
 - irrational

Solution: D DOK 1

5. Which word best describes the sum of $\sqrt{19}$ and $\sqrt{3}$?
- Irrational
 - Rational
 - Natural
 - imaginary

Solution: A DOK 1

6. What is the sum of $3\sqrt{5}$ and $17\sqrt{5}$? Is the sum rational or irrational?
- | | |
|------------------------------|-------------------|
| a. $20\sqrt{5}$; irrational | c. 25; rational |
| b. $20\sqrt{5}$; rational | d. 25; irrational |

Solution: A DOK 2

7. What is the product of $\frac{4}{7}$ and $\frac{11}{16}$? Is the product rational or irrational?
- | | |
|----------------------------------|--------------------------------|
| a. $\frac{15}{112}$; irrational | c. $\frac{15}{112}$; rational |
| b. $\frac{11}{28}$; irrational | d. $\frac{11}{28}$; rational |

Solution: C DOK 2

8. What is the product of 1.6 and $2\sqrt{7}$? Is the product rational or irrational?
- | | |
|-------------------------------|---------------------|
| a. $3.2\sqrt{7}$; rational | c. 22.4; rational |
| b. $3.2\sqrt{7}$; irrational | d. 22.4; irrational |

Solution: B DOK 2



9. What is the product of $\sqrt{5}$ and $\sqrt{2}$? Is the product rational or irrational?
- a. 10; rational c. 7; rational
 b. $\sqrt{10}$; irrational d. $2\sqrt{5}$; irrational

Solution: B DOK 2

SHORT ANSWER

10. Prove that the quotient of two nonzero rational numbers is rational.

Solution: DOK 4

Let x and y be two nonzero rational numbers.

By the definition of rational numbers, x can be written as $\frac{a}{b}$ and y can be written as $\frac{c}{d}$, where a , b , c , and d are nonzero integers.

$$\frac{x}{y} = \frac{\frac{a}{b}}{\frac{c}{d}} \quad \text{Divide } x \text{ by } y.$$

$$\frac{x}{y} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc} \quad \text{To divide, multiply by the reciprocal.}$$

Because the product of two nonzero integers is also a nonzero integer, the expressions ad and bc are both nonzero integers.

Therefore, the quotient $\frac{x}{y}$ is the ratio of two integers. So, by definition, the quotient is a rational number.

11. Use an indirect proof to prove that the difference of a rational number and an irrational number is irrational.

Solution: DOK 4

Let a be a rational number and b be an irrational number. Let c be the difference of b and a .

Assume that c is rational. Then:

$$\begin{aligned} b - a &= c \\ b &= c + a \quad \text{Add } a \text{ to both sides.} \end{aligned}$$



Because the sum of two rational numbers is rational, $c + a$ is a rational number. But it is equal to b , which is an irrational number.

Therefore, our assumption is wrong, and the difference of a rational number and an irrational number is irrational.

12. How can you determine whether the product of $\frac{2}{7}$ and $\frac{9}{11}$ is rational or irrational without multiplying?

Solution: DOK 2

Both $\frac{2}{7}$ and $\frac{9}{11}$ are rational numbers. The product of two rational numbers is rational, so the product of $\frac{2}{7}$ and $\frac{9}{11}$ is rational.

ESSAY

13. Let $q = \sqrt{9} \cdot p$, where p is a rational number. What type of number is q ? Explain your reasoning.

Solution: The number represented by q is a rational number. DOK 3

Sample explanation: $\sqrt{9}$ can be simplified to 3. Since we know p is a rational number, we know it can be written as $\frac{a}{b}$, where a and b are integers and b is not zero. So:

$$\begin{aligned} q &= \sqrt{9} \cdot p \\ &= 3p \\ &= \frac{3a}{b} \end{aligned}$$

Since a is an integer, $3a$ is also an integer, so q is a rational number.

14. What kind of number is $-\frac{5}{9}m$ if m is a rational number? Explain your reasoning.

Solution: A rational number DOK 3



Sample explanation: $-\frac{5}{9}$ is a rational number. m is also rational number, so it can be written as $\frac{a}{b}$, where a and b are integers and b is not zero. So:

$$\begin{aligned} -\frac{5}{9}m & \\ &= -\frac{5}{9} \cdot \frac{a}{b} \\ &= -\frac{5a}{9b} \end{aligned}$$

Since a and b are integers, $-5a$ and $9b$ are also integers and $9b$ is not equal to zero, so $-\frac{5}{9}m$ is a rational number.

15. Consider the equation $m = k\sqrt{17}$. k is a non-zero rational number. What type of number is m ? Explain your reasoning.

Solution: The number m must be irrational DOK 3

Sample explanation: Assume that m is rational. The number k is a non-zero rational number and $\sqrt{17}$ is an irrational number. Therefore:

$$\begin{aligned} m &= k\sqrt{17} \\ \frac{m}{k} &= \sqrt{17} \end{aligned}$$

If m and k are both rational, then their quotient is also rational. However, $\sqrt{17}$ is not a rational number. The assumption that m is a rational number is incorrect, and therefore it must be irrational.

16. Let $t = s + 4.7$, where s is an irrational number. What type of number must t be? Explain your reasoning.

Solution: t is an irrational number. DOK 3

Sample explanation: Assume that t is rational. That means it can be written as $\frac{a}{b}$, where a and b are integers and b does not equal zero. So:



$$\frac{a}{b} = s + 4.7$$

$$\frac{a}{b} - 4.7 = s$$

$$\frac{a}{b} - \frac{47}{10} = s$$

$$\frac{10a - 47b}{10b} = s$$

However, since a and b are integers, then $10a - 47b$ and $10b$ are also integers. This means that s can be written as the quotient of two integers, which contradicts the fact that s is irrational. Therefore t must also be irrational.

17. Let $g = h + \frac{4}{5}$, where h is an irrational number. What type of number must g be? Explain your reasoning.

Solution: g is an irrational number. : DOK 3

Sample explanation: Assume that g is rational. That means it can be written as $\frac{a}{b}$, where a and b are integers and b does not equal zero. So:

$$\frac{a}{b} = h + \frac{4}{5}$$

$$\frac{a}{b} - \frac{4}{5} = h$$

$$\frac{5a - 4b}{5b} = h$$

However, since a and b are integers, then $5a - 4b$ and $5b$ are also integers. This means that h can be written as the quotient of two integers, which contradicts the fact that h is irrational. Therefore g must also be irrational.

18. Let p be a rational number that is not zero, and q be an irrational number. If $pq = r$, could r be a rational number? Explain your reasoning.

Solution: No, r must be irrational. DOK 4

Sample explanation: Assume that r is rational. p is a rational number, so it can be written as $\frac{a}{b}$, where a and b are integers and b does not equal zero. If r is also a rational number,

so it can be written as $\frac{c}{d}$, where c and d are integers and d does not equal zero. So:



$$pq = r$$

$$\frac{a}{b}q = \frac{c}{d}$$

$$q = \frac{bc}{ad}$$

So q must also be rational, but this contradicts what we are given. Therefore, r must be irrational.

19. Let x and y be rational numbers. Prove that the sum of x and y is also a rational number.

Solution: DOK 4

Because x and y are rational, then they can be written as $x = \frac{a}{b}$ and $y = \frac{c}{d}$, where a , b , c , and d are integers and b and d are not zero. Their sum is

$\frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{bc}{bd} = \frac{ad+bc}{bd}$. Because a , b , c , and d are integers, $ad + bc$ and bd are also integers, and bd is not zero because b and d are not zero. Therefore, the sum of x and y is also a rational number.

20. Let m and n be rational numbers. Prove that their product, mn , is also a rational number.

Solution: DOK 4

Because m and n are rational numbers, $m = \frac{a}{b}$ and $n = \frac{c}{d}$, where a , b , c , and d are integers and b and d are not zero. The product of m and n is $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$. Because a , b , c , and d are integers, ac and bd are also integers, and bd is not zero because b and d are not zero. Therefore, the product of m and n is a rational number.



For Algebra I, this section may be a review as needed.

A **set** is a collection of objects. Each object in a set is called an **element** of the set. A set may have not elements, a finite number of elements, or an infinite number of elements. For example $N = \{1,2,3,\dots\}$ describes the set of natural numbers.

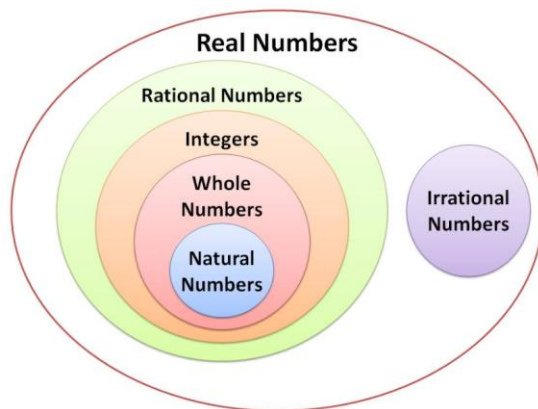
A **subset** is a set contained entirely within another set. For example $A = \{2,6,1,50\}$ is a subset of N above.

When you first start dealing with numbers, you learn about the four main sets, or groups, of numbers, which nest inside one another:

- **Counting numbers (also called natural numbers):** The set of numbers beginning 1, 2, 3, 4, . . . and going on infinitely
- **Integers:** The set of counting numbers, zero, and negative counting numbers
- **Rational numbers:** The set of integers and fractions
- **Real numbers:** The set of rational and irrational numbers (which can't be written as simple fractions)

The sets of counting numbers, integers, rational, and real numbers are nested, one inside another, similar to the way that a city is inside a state, which is inside a country, which is inside a continent. The set of counting numbers is inside the set of integers, which is inside the set of rational numbers, which is inside the set of real numbers.

The diagram below shows other subsets of the set of real numbers.





Counting, or natural, numbers

The set of counting numbers is the set of numbers you first count with, starting with 1. (so 0 is not a counting number.) Because they seem to arise naturally from observing the world, they're also called the *natural numbers*: 1, 2, 3, 4, 5, 6, 7, 8, 9, . . .

When you add two counting numbers, the answer is always another counting number. Similarly, when you multiply two counting numbers, the answer is always a counting number. Another way of saying this is that the set of counting numbers is closed under both addition and multiplication.

Whole numbers

The set of counting numbers and the number 0 make up the whole numbers. So you see the natural numbers are a subset of the whole numbers. Are whole numbers closed under addition? Are whole numbers closed under subtraction? No, because $1 - 3 = -2$, which is not a whole number.

Integers

The set of integers arises when you try to subtract a larger counting number from a smaller one. For example, $4 - 6 = -2$. The set of integers includes the following:

- The counting numbers
- Zero
- The negative counting numbers

Here's a partial list of the integers:

. . . -4, -3, -2, -1, 0, 1, 2, 3, 4, . . .

Like the counting numbers, the integers are closed under addition and multiplication. Similarly, when you subtract one integer from another, the answer is always an integer. That is, the integers are also closed under subtraction.

Rational numbers

The set of rational numbers includes all integers and all fractions. Like the integers, the rational numbers are closed under addition, subtraction, and multiplication. Furthermore, when you divide one rational number by another, the answer is always a rational number. Another way to say this is that the rational numbers are closed under division.

Real numbers

Even if you filled in all the rational numbers on the number line, you'd still have points left unlabeled. These points are the irrational numbers.

An **irrational number** is neither a whole number nor a fraction. In fact, an irrational number can only be approximated as a non-repeating decimal. In other words, no matter how many decimal places you write down, you can always write down more; furthermore, the digits in this decimal never become repetitive or fall into any pattern.

The most famous irrational number is π ,

3.14159265358979323846264338327950288419716939937510 . . .

Together, the rational and irrational numbers make up the **real numbers**, which comprise every point on the number line.

**CLOSURE**

A set of numbers is closed, or has closure, under a given operation if the result of the operation on any two numbers in the set is also in the set. For example, the set of even numbers is closed under addition, since the sum of two even numbers is also an even number.

Note: A method to show that a set is NOT closed is to find a counterexample.

There are many examples that can be used to demonstrate closure. Fill in the table with other examples. It is important to allow students to express the properties in their terms (as long as they are correct).

Closure project

Identify if the set is closed under the operations. Give examples or counterexamples.

	Add	Subtract	Multiply	Divide
Real numbers				
Rational numbers				
Irrational numbers				
Integers				
Whole numbers				
Polynomials				

Special casesWhat type of number is the . . .

Sum of a rational number and an irrational number?	
Product of rational number and an irrational number?	
Difference of a rational number and an irrational number?	
Quotient of rational number and an irrational number?	

Real Number Properties

- Commutative Property of Addition: $a + b = b + a$

Real Number Example: $-3 + 5 = 5 + -3 = 2$

- Commutative Property of Multiplication: $a \cdot b = b \cdot a$

Real Number Example: $-\frac{2}{9}\left(-\frac{3}{4}\right) = -\frac{3}{4}\left(-\frac{2}{9}\right) = \frac{1}{6}$

- Associative Property of Addition: $(a + b) + c = a + (b + c)$

Real Number Example: $(-4.2 + 5) + 3 = -4.2 + (5 + 3) = 3.8$

- Associative Property of Multiplication: $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

Real Number Example: $\left(-3 \cdot \frac{1}{2}\right) \cdot 6 = -3 \cdot \left(\frac{1}{2} \cdot 6\right) = -9$

- Identity Property of Addition: $a + 0 = a$ Note: The *additive identity* is 0.

Real Number Example: $-3 + 0 = -3$

- Identity Property of Multiplication: $a \cdot 1 = a$ Note: The *multiplicative identity* is 1.

Real Number Example: $6.2 \cdot 1 = 6.2$

- Inverse Property of Addition: $a + (-a) = 0$

Note: The *opposite* of a number is the *additive inverse*.

Real Number Example: $-4 + 4 = 0$

- Inverse Property of Multiplication: $a \cdot \frac{1}{a} = 1$

Note: The *reciprocal* of a number is the *multiplicative inverse*.

Real Number Example: $-2 \cdot \left(-\frac{1}{2}\right) = 1$

- Zero Product Property: If $a \cdot b = 0$, then either $a = 0$ or $b = 0$.



- Distributive Property: $a \cdot (b + c) = a \cdot b + b \cdot c$

Real Number Example: Use the distributive property to multiply $12 \cdot 32$.

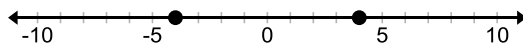
We can rewrite 32 as $30 + 2$ in order to use the distributive property so that we can perform the multiplication in our heads. $12(30 + 2) = 12 \cdot 30 + 12 \cdot 2 = 360 + 24 = \boxed{384}$

Absolute Value

Absolute Value: the distance a number is away from the origin on the number line

Ex: What two numbers have an absolute value of 4?

Solution: There are two numbers that are 4 units away from the origin, as shown in the number line.



They are $\boxed{-4 \text{ and } 4}$.

Notation: $|x| = x$ if $x \geq 0$ and $|x| = -x$ if $x < 0$

Ex: Evaluate the expression. $2|-4 + 1|$

Δ Note: Absolute value bars are like grouping symbols in the order of operations.

Evaluate the expression inside the absolute value bars. $= 2|-3|$

Evaluate the absolute value and simplify. $= 2(3) = \boxed{6}$

Ex: Evaluate the variable expression $-x^2 - 3|4 - x| + \frac{2x}{y}$ when $x = -1$ and $y = \frac{2}{3}$.

Substitute $x = -1$ and $y = \frac{2}{3}$ into the expression. $-(-1)^2 - 3|4 - (-1)| + \frac{2(-1)}{\left(\frac{2}{3}\right)}$



$$= -(-1)^2 - 3|5| + \frac{2(-1)}{\left(\frac{2}{3}\right)}$$

$$= -1 - 15 + \frac{-2}{\left(\frac{2}{3}\right)}$$

Use the order of operations to simplify.

$$= -1 - 15 - 2 \cdot \left(\frac{3}{2}\right)$$

$$= -1 - 15 - 3$$

$$= -16 - 3 = \boxed{-19}$$

You Try:

1. Identify the property shown. $(c + d) + e = (d + c) + e$
2. Show that the distributive property does not work over absolute value bars by showing that $3|5 - 8| \neq 3|5| - 3|8|$.

QOD: Determine if the statement is true or false. Explain your answer. "The reciprocal of any number is greater than zero and less than 1."

Sample Nevada High School Proficiency Exam Questions (taken from 2009 released version H):

1. Which equation illustrates the commutative property of addition?

A $(4 + x) + 3x = 3x + (4 + x)$

B $3x(4 + x) + 0 = 3x(4 + x)$

C $(4 + x) + 3x = 4 + (x + 3x)$

D $3x(4 + x) = 3x(4) + 3x(x)$

2. The equation below illustrates a property of real numbers.

$$3(7x + 9) = 21x + 27$$

Which property is illustrated by the equation?

A associative property

B commutative property



- C distributive property
- D identity property

3. Which equation shows the use of the inverse property of multiplication?

A $7x \cdot \frac{1}{7x} = 1$

B $7x \cdot 1 = 7x$

C $7x \cdot 0 = 0$

D $7x \cdot \frac{1}{7} = \frac{1}{7} \cdot 7x$



Skills: simplify algebraic expressions by adding and subtracting like terms.

Terms: algebraic expressions separated by a + or a – sign

Constant Term: a term that is a real number (no variable part)

Coefficient: the numerical factor of a term

Like Terms: terms that have the exact same variable part (must be the same letter raised to the same exponent) Note: Only *like terms* can be added or subtracted (combined).

△To add or subtract like terms, add or subtract the coefficients and keep the variable part.

Ex: Simplify $3♥ + 4♥$.

This means we are adding 3 hearts plus 4 hearts. Therefore, we have a total of 7 hearts, which we can write as $7♥$.

Ex: Use the expression $1 - 3x^2 + 4xy + x^2 + 5y^2$ to answer the following questions.

1. How many terms are in the expression?

Solution: There are 5 terms.

2. Name the constant term(s).

Solution: There is one constant term, 1.

3. How many like terms are in the expression?

Solution: There are 2 like terms. $-3x^2$ and x^2

4. What is the coefficient of the xy term?

Solution: The xy term has a coefficient of 4.

What is the coefficient of the fourth term?

The third term has a

5. Simplify the algebraic expression.

Solution: Combine like terms. $1 - 3x^2 + x^2 + 4xy + 5y^2 = \boxed{1 - 2x^2 + 4xy + 5y^2}$

6. How many terms are in the simplified form of the algebraic expression?

Solution: There are 4 terms.

Simplifying an Algebraic Expression

Ex: Simplify the expression $4(x+8) - 5x + 1$.

Step One: Eliminate the parentheses using the distributive property.

$$4x + 32 - 5x + 1$$

$$= 4x - 5x + 32 + 1$$

Step Two: Combine like terms.

$$= -1x + 33$$

$$= \boxed{-x + 33}$$

Ex: Simplify the expression $5 - 2n(3n + 8) - 4n^2$

Step One: Eliminate the parentheses using the distributive property.

$$5 - 6n^2 - 16n - 4n^2$$

Note: The expression in parentheses is being multiplied by $-2n$, which is what was “distributed”.

Step Two: Combine like terms.

$$= 5 - 6n^2 - 4n^2 - 16n$$

$$= \boxed{5 - 10n^2 - 16n}$$

Note: The answer may be written as $-10n^2 - 16n + 5$ by the commutative property.

Ex: Write a simplified expression for the perimeter of a rectangle with length $(x + 7)$ and width $(x - 2)$. Note: The formula for the perimeter of a rectangle is $P = 2l + 2w$.

Substitute the length and width into the formula.

$$2(x + 7) + 2(x - 2)$$

Simplify using the distributive property and combining like terms.

$$= 2x + 14 + 2x - 4$$

$$= \boxed{4x + 10}$$

You Try:

1. Simplify the expression $x - (4 - 3x) + 8$.
2. Write a simplified expression for the area of a triangle with a base of $4x$ and a height of $(2x + 1)$.

QOD: Explain in your own words why only like terms can be added or subtracted.



Sample Practice Question(s):

1. Simplify the expression:

$$5 + 3(x - 4) - x$$

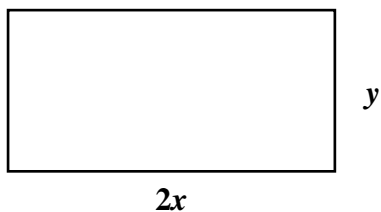
- A. $7x - 4$
- B. $7x - 32$
- C. $2x + 1$
- D. $2x - 7$

2. Simplify the expression:

$$5x^2 - 2x + 6 + 4x + 6x^2 - 1$$

- A. $-x^2 + 6x + 5$
- B. $3x^2 + 10x + 5$
- C. $11x^2 + 2x + 5$
- D. $11x^2 + 6x + 5$

3. Write an expression for the perimeter of the rectangle:



- A. $2xy$
- B. $6xy$
- C. $4x + 2y$
- D. $4x^2 + 2y^2$