OBJECTIVE 

• Prove theorems about triangles. Theorems include: measures of interior angles of a triangle sum to \( 180^\circ \); base angles of isosceles triangles are congruent; the segment joining midpoints of two sides of a triangle is parallel to the third side and half the length; the medians of a triangle meet at a point.

BIG IDEA (Why is this included in the curriculum?)

• All polygons can be divided into triangles – thus the proof and use of properties and relationships of triangles is essential to geometric study.

PREVIOUS KNOWLEDGE (What skills do they need to have to succeed?)

• The student must have a strong grasp on the isometric transformations and their definitions.
• The student must understand the criteria for triangle congruence.
• The student must know the angle relationships relating to parallel lines and a transversal.

VOCABULARY USED IN THIS OBJECTIVE (What terms will be essential to understand?)

PREVIOUS VOCABULARY (Terms used but defined earlier)

• Angle: A geometric figure formed by rotating a ray about its initial point.
• Angle Bisector: A line/segment/ray that divides an angle into two congruent parts.
• Complementary Angles: Two angles whose measures have the sum \( 90^\circ \).
• Linear Pair: Two adjacent angles whose non-common sides are opposite rays, thus making them supplementary.
• Parallel Lines: If two parallel lines are cut by a transversal, then…
  ▪ Alternate interior angles are congruent.
  ▪ Alternate exterior angles are congruent.
  ▪ Consecutive interior angles (same side interior angles) are supplementary.
  ▪ Corresponding angles are congruent.
• Perpendicular Bisector: A perpendicular line/segment/ray that intersects a segment at its midpoint.
• Segment Addition Postulate: If \( B \) is between \( A \) and \( C \), then \( AB + BC = AC \).
• Supplementary Angles: Two angles whose measures have the sum \( 180^\circ \).
• Triangle: A figure formed by three segments joining three non-collinear points, called vertices.
  ▪ Acute Triangle: A triangle with three acute angles.
  ▪ Right Triangle: A triangle with exactly one right angle.
  ▪ Obtuse Triangle: A triangle with exactly one obtuse angle.
  ▪ Equiangular Triangle: A triangle with three congruent angles.
  ▪ Scalene Triangle: A triangle with no congruent sides.
  ▪ Isosceles Triangle: A triangle with at least two congruent sides.
  ▪ Equilateral Triangle: A triangle with three congruent sides.

NEW VOCABULARY (New Terms and definitions introduced in this objective)

• Auxiliary line: A line added to a diagram.
• Base Angle of an Isosceles Triangle: The angle formed between the base and a leg of an isosceles triangle.

• Base of an Isosceles Triangle: The side opposite the vertex angle of an isosceles triangle.

• Exterior Angle of a Polygon: An angle formed between a side of a polygon and the extension of an adjacent side.

• Interior Angle of a Polygon: An angle inside a polygon formed by two adjacent sides.

• Legs of an Isosceles Triangle: The congruent sides of an isosceles triangle.

• Median of a Triangle: The segment from a vertex to the midpoint of the opposite side.
• Midsegment (Midline) of a Triangle: The segment/line joining the midpoints of two sides of a triangle.

![Midsegment](image)

• Point of Concurrency: A point where three or more lines meet.

![Point of Concurrency](image)

• Vertex Angle of an Isosceles Triangle: The angle formed between the two congruent legs of an isosceles triangle.

![Vertex Angle](image)

**SKILLS** (What will they be able to do after this objective?)

- The student will be able to prove and apply that the sum of the interior angles of a triangle is 180°.
- The student will be able to prove and apply that the base angles of an isosceles triangle are congruent.
- The student will be able to prove and apply the midsegment (midline) of triangle theorem.
- The student will be able to prove that the medians of a triangle meet at a point, a point of concurrency.

**SHORT NOTES** (A short summary of notes so that a teacher can get the basics of what is expected.)

Proof has always been a difficult area in geometry to clearly define what it should look like, the level of rigor that should be applied and the format in which it should be given. Textbooks for years have developed chapters to establish the basic concepts of logic and language so that students can organize their proofs with consistent terminology and format. The formal learning of conditional statements, the establishment of properties such as transitive, additive, substitution, etc… were all discussed so that students could justify their statements consistently and logically. Other than establishing precise definitions, none of these things are mentioned specifically in any common core objectives. This has left many teachers wondering the role of proof as the core curriculum rolls out.

The absence of emphasis on these formal concepts and the introduction of transformations as the basis for congruence and similarity lead us to believe that proof needs to be established from a transformational approach. It wouldn’t make sense to transform (no pun intended) our teaching to a transformational approach and then only return to previous proofs for justification.

To ensure teachers’ understanding and success in making this transition, we will attempt to provide traditional methods as well as transformational approaches. A combination of the following proofs should be demonstrated and practiced to ensure the students’ understanding of the transition from the tradition approach to the new transformational approach.
1) Prove that the interior angles of a triangle sum to 180°.

Suggested Proofs:   Regular Geometry 1a & 1c   /   Honors Geometry 1a, 1c, & 1d

a) Triangle Dissection (Informal – Classic Approach)

An informal proof that is often used is the process of having our students create a triangle on a piece of paper, naming the three angles \(A\), \(B\), and \(C\) and then cutting out the triangle. When the triangle is cut out, the student should rip off the three angles, placing them together, vertex to vertex. They will see that the three angles form a straight line. Therefore, the sum of the three interior angles of a triangle is 180°.

b) An Auxiliary Parallel Line (Formal – Classic Proof)

A formal two column proof of this theorem is done using the angle relationships found with parallel lines and a transversal. Since these angle relationships were established in objective G.CO.9, we are now able to apply them.

**Given:** \(\Delta ABC\) 
**Prove:** \(m\angle 1 + m\angle 2 + m\angle 3 = 180°\)

<table>
<thead>
<tr>
<th>STATEMENT</th>
<th>REASON</th>
</tr>
</thead>
<tbody>
<tr>
<td>(AC \parallel BD)</td>
<td>Given (Auxiliary Line)</td>
</tr>
<tr>
<td>(m\angle 4 + m\angle 2 + m\angle 5 = 180°)</td>
<td>Angles of a Straight Angle</td>
</tr>
<tr>
<td>(m\angle 1 = m\angle 4)</td>
<td>If (\parallel), Alternate Interior (\angle)’s (\cong)</td>
</tr>
<tr>
<td>(m\angle 3 = m\angle 5)</td>
<td>If (\parallel), Alternate Interior (\angle)’s (\cong)</td>
</tr>
<tr>
<td>(m\angle 1 + m\angle 2 + m\angle 3 = 180°)</td>
<td>Substitution Property (Twice)</td>
</tr>
</tbody>
</table>

Construct an auxiliary line parallel to \(AC\) through \(B\).

c) Proof by Translation (Formal – Transformational Approach)

**Given:** \(\Delta ABC\) 
**Prove:** \(m\angle 1 + m\angle 2 + m\angle 3 = 180°\)
Translate $\triangle ABC$ by vector $\overrightarrow{AB}$ to form straight $\angle ABB'$ along the vector. The isometric properties of translation preserve angles, thus $m\angle 1 = m\angle B'BC'$. Since $\angle ABB'$ is a straight angle, we know that $m\angle 2 + m\angle CBC' + m\angle B'BC' = 180^\circ$. Translations also preserve parallelism, therefore ensuring that $\overline{AC} \parallel \overline{B'C'}$. Since $\overline{AC} \parallel \overline{B'C'}$, $m\angle 3 = m\angle CBC'$ because alternate interior angles are congruent. By making two substitutions into the straight angle relationship of $m\angle 2 + m\angle CBC' + m\angle B'BC' = 180^\circ$ we arrive at the proof that $m\angle 1 + m\angle 2 + m\angle 3 = 180^\circ$.

**d) Proof by Rotation (Formal – Transformational Approach)**

**Given:** $\triangle ABC$

**Prove:** $m\angle 1 + m\angle 2 + m\angle 3 = 180^\circ$

Rotate $\triangle ABC$ $180^\circ$ about the midpoint of $\overline{BC}$ forming image $\triangle DCB$ with congruent corresponding angles (CPCTC). Rotate $\triangle DCB$ about the midpoint of $\overline{BD}$ forming image $\triangle BED$ with congruent corresponding angles (CPCTC). Because of the congruent alternate interior angles formed $(\angle ABC \cong \angle DCB$ and $\angle EBD \cong \angle CDB)$, $\overline{AB} \parallel \overline{DC}$ and $\overline{DC} \parallel \overline{BE}$, respectively. In addition, because there is only one line parallel to $\overline{DC}$ through point $B$, $\angle ABE$ is a straight $\angle$ formed by $\angle 1$, $\angle 2$ and $\angle 3$. Thus, $m\angle 1 + m\angle 2 + m\angle 3 = 180^\circ$. 

(2) Prove that Base Angles of an Isosceles are Equal (Isosceles Triangle Theorem)

Suggested Proofs: Regular Geometry 1c, 1d & 1e / Honors Geometry 1c, 1d & 1e

a) Proof by Angle Bisector (Formal -- Classic Approach)

Given: ΔABC is an isosceles triangle, with base AC.
Prove: ∠A ≅ ∠C

Construct an auxiliary line that is the angle bisector of ∠B.

<table>
<thead>
<tr>
<th>STATEMENT</th>
<th>REASON</th>
</tr>
</thead>
<tbody>
<tr>
<td>∆ABC is an isosceles triangle.</td>
<td>Given</td>
</tr>
<tr>
<td>BD is the angle bisector of ∠B</td>
<td>Given (Auxiliary Line)</td>
</tr>
<tr>
<td>BA ≅ BC</td>
<td>Definition of an Isosceles Δ</td>
</tr>
<tr>
<td>∠ABD ≅ ∠CBD</td>
<td>Definition of an Angle Bisector</td>
</tr>
<tr>
<td>BD ≅ BD</td>
<td>Reflexive Prop. (Common Side)</td>
</tr>
<tr>
<td>∆ABD ≅ ∆CBD</td>
<td>SAS</td>
</tr>
<tr>
<td>∠A ≅ ∠C</td>
<td>CPCTC (Corresponding Parts of Congruent Triangles are Congruent)</td>
</tr>
</tbody>
</table>

b) Proof by Perpendicular Bisector (Formal – Classic Approach)

Given: ΔABC is an isosceles triangle, with base AC.
Prove: ∠A ≅ ∠C

Construct an auxiliary line that is a perpendicular bisector of AC. (B is on the perpendicular bisector of AC because AB ≅ BC)

<table>
<thead>
<tr>
<th>STATEMENT</th>
<th>REASON</th>
</tr>
</thead>
<tbody>
<tr>
<td>∆ABC is an Isosceles.</td>
<td>Given</td>
</tr>
<tr>
<td>BD is the ⊥ bisector of ∠B</td>
<td>Given (Auxiliary Line)</td>
</tr>
<tr>
<td>AD ≅ CD</td>
<td>Definition of a ⊥ Bisector</td>
</tr>
<tr>
<td>(D is the midpoint of AC)</td>
<td></td>
</tr>
<tr>
<td>∠ADB &amp; ∠CDB are right</td>
<td>Definition of a ⊥ Bisector</td>
</tr>
<tr>
<td>∠'s</td>
<td>(BD is ⊥ to AC)</td>
</tr>
<tr>
<td>∠ADB ≅ ∠CDB</td>
<td>All Right Angles are ≅</td>
</tr>
<tr>
<td>BD ≅ BD</td>
<td>Reflexive Property (Common Side)</td>
</tr>
<tr>
<td>∆ABD ≅ ∆CBD</td>
<td>SAS</td>
</tr>
<tr>
<td>∠A ≅ ∠C</td>
<td>CPCTC (Corresponding Parts of Congruent Triangles are Congruent)</td>
</tr>
</tbody>
</table>

Alternative methods to prove congruent triangles: HL (AB ≅ CB, Definition of Isosceles Triangle) or SSS (AB ≅ CB, Definition of Isosceles Triangle and BD ≅ BD, Reflexive Property)
c) Proof by Paper Folding (Informal Proof – Transformational Approach)

Prove that the base angles of an isosceles triangle are congruent.

Create an isosceles triangle by using your compass to construct a circle. Then draw two radii (all radii of the same circle are congruent) and connect the endpoints with a segment. Label the radii, $\overline{AB}$ and $\overline{CB}$. Then fold the paper until point $A$ maps to point $C$. Crease the paper. Notice that when you do this $\angle A \cong \angle C$. Therefore, the base angles of an isosceles triangle are congruent.

d) Proof by Symmetry (Informal – Transformational Approach)

Given: $\triangle ABC$ is an isosceles triangle, with base $\overline{AC}$.
Prove: $\angle A \cong \angle C$

Construct an auxiliary line, $\overline{BD}$, such that $\overline{BD}$ is the perpendicular bisector of $\overline{AC}$. Established in G.CO.3, an isosceles triangle has reflectional symmetry about the perpendicular bisector of its base. Thus $\angle A \cong \angle C$ because $\angle A$ reflects onto $\angle C$.

e) Proof by Transform (Formal – Transformational Approach)

Given: $\triangle ABC$ is an isosceles triangle, with base $\overline{AC}$.
Prove: $\angle A \cong \angle C$

Construct an auxiliary line that is a perpendicular bisector of $\overline{AC}$. By the definition of isosceles triangle, $\overline{AB} \cong \overline{CB}$. Because point $B$ is equidistant to points $A$ and $C$, $B$ is on the perpendicular bisector of $\overline{AC}$. A reflection over the perpendicular bisector would map $A$ onto $C$, $B$ onto $B$, and $D$ onto $D$. Thus the isometric properties of a reflection then give us $\triangle ABD \cong \triangle CBD$. Therefore, $\angle A \cong \angle C$ by corresponding parts of congruent triangles are congruent.

3. Prove the Midsegment Theorem (that the segment joining midpoints of two sides of a triangle is parallel to the third side and half the length)

It seems that the three most popular ways to prove this theorem use concepts presented in objectives; G.CO.11 (parallelogram properties), G.SRT.3 (Similar Triangles), and G.GPE.4 (Coordinate Proof). Therefore, this theorem may be revisited numerous times throughout the course.
Suggested Proofs:   Regular Geometry 1e, 1f, 1b & 1c   /   Honors Geometry 1a, 1b & 1c

a) Proof by Parallelogram Properties (Formal – Classic Approach)

(This proof could be used after G.CO.10 has been taught. Determining that a quadrilateral is a parallelogram and then applying its properties is used to complete this proof.)

**Given:** \( \triangle ABC \), where \( D \) is the midpoint of \( AB \) and \( E \) is the midpoint of \( BC \).

**Prove:** \( DE \parallel AC \) and \( DE = \frac{1}{2} AC \)

Construct an auxiliary line that is parallel to \( AB \) through \( C \) and extend \( DE \) until it intersects the new parallel line at point \( F \).

<table>
<thead>
<tr>
<th>STATEMENT</th>
<th>REASON</th>
</tr>
</thead>
<tbody>
<tr>
<td>( D ) is the midpoint of ( AB ) ( \quad )</td>
<td>Given</td>
</tr>
<tr>
<td>( E ) is the midpoint of ( BC ) ( \quad )</td>
<td>Given</td>
</tr>
<tr>
<td>( DB \parallel FC ) ( \quad )</td>
<td>Given (Auxiliary Line)</td>
</tr>
<tr>
<td>( \angle B \cong \angle FCE ) ( \quad )</td>
<td>If ( \parallel ), Alternate Interior ( \angle )’s ( \cong )</td>
</tr>
<tr>
<td>( \angle BDE \cong \angle CEF ) ( \quad )</td>
<td>Vertical ( \angle )’s ( \cong )</td>
</tr>
<tr>
<td>( CE \cong EB ) ( \quad )</td>
<td>Definition of Midpoint</td>
</tr>
<tr>
<td>( \triangle BDE \equiv \triangle CEF ) ( \quad )</td>
<td>ASA</td>
</tr>
<tr>
<td>( DE \cong FE ) ( \quad )</td>
<td>CPCTC</td>
</tr>
<tr>
<td>( DB \cong FC ) ( \quad )</td>
<td>CPCTC</td>
</tr>
<tr>
<td>( AD \cong DB ) ( \quad )</td>
<td>Definition of Midpoint</td>
</tr>
<tr>
<td>( AD \cong FC ) ( \quad )</td>
<td>Transitive Property</td>
</tr>
<tr>
<td>( \overrightarrow{ADFC} ) is a Parallelogram ( \quad )</td>
<td>( \overrightarrow{AD} ) &amp; ( \overrightarrow{FC} ) are both ( \cong ) and ( \parallel )</td>
</tr>
<tr>
<td>( DE \parallel AC ) ( \quad )</td>
<td>Parallelogram Definition</td>
</tr>
<tr>
<td>( AC = DF ) ( \quad )</td>
<td>Opposite Sides of ( \parallel )gram =</td>
</tr>
<tr>
<td>( DE = FE ) ( \quad )</td>
<td>Definition of Congruence</td>
</tr>
<tr>
<td>( DE + FE = DF ) ( \quad )</td>
<td>Segment Addition Postulate</td>
</tr>
<tr>
<td>( DE + DE = DF, 2DE = DF ) ( \quad )</td>
<td>Substitution &amp; Simplify</td>
</tr>
<tr>
<td>( 2DE = AC ) ( \quad )</td>
<td>Substitution</td>
</tr>
<tr>
<td>( DE = \frac{1}{2} AC ) ( \quad )</td>
<td>Multiplication/Division Property of Equality</td>
</tr>
</tbody>
</table>

b) Proof by Similarity (Formal – Classic Approach)

(This proof could be used after G.SRT.3 has been taught. The similarity criteria of SAS is used to complete the proof.)
Given: \( \triangle ABC \), where \( D \) is the midpoint of \( \overline{AB} \) and \( E \) is the midpoint of \( \overline{BC} \).

Prove: \( \overline{DE} \parallel \overline{AC} \) and \( DE = \frac{1}{2} AC \)

<table>
<thead>
<tr>
<th>STATEMENT</th>
<th>REASON</th>
</tr>
</thead>
<tbody>
<tr>
<td>( D ) is the midpoint of ( \overline{AB} )</td>
<td>Given</td>
</tr>
<tr>
<td>( E ) is the midpoint of ( \overline{BC} )</td>
<td></td>
</tr>
<tr>
<td>( \overline{CE} \cong \overline{EB}, \overline{AD} \cong \overline{DB} )</td>
<td>Definition of Midpoint</td>
</tr>
<tr>
<td>( CE = EB, AD = DB )</td>
<td>Definition of Congruence</td>
</tr>
<tr>
<td>( AD + DB = AB, CE + EB = \overline{BC} )</td>
<td>Segment Addition Postulate</td>
</tr>
<tr>
<td>( DB + DB = AB, EB + EB = \overline{BC} )</td>
<td>Substitution Property</td>
</tr>
<tr>
<td>( 2DB = AB, 2EB = BC )</td>
<td>Substitution &amp; Simplify</td>
</tr>
<tr>
<td>( DB = \frac{1}{2} AB, EB = \frac{1}{2} BC )</td>
<td>Multiplication/Division Property of Equality</td>
</tr>
<tr>
<td>( \angle B \cong \angle B )</td>
<td>Reflexive Angle (Common ( \angle ))</td>
</tr>
<tr>
<td>( \triangle DBE \sim \triangle ABC )</td>
<td>SAS Similarity</td>
</tr>
<tr>
<td>( \angle BDE \cong \angle BAC )</td>
<td>Corresponding angles are ( \cong ) in Similar ( \triangle )</td>
</tr>
<tr>
<td>( \overline{DE} \parallel \overline{AC} )</td>
<td>If Corresponding ( \angle )'s are ( \cong ), then ( \parallel )</td>
</tr>
<tr>
<td>( DE = \frac{1}{2} AC )</td>
<td>Corresponding sides are proportional in Similar ( \triangle )</td>
</tr>
</tbody>
</table>

c) Proof by Coordinates (Formal – Classic Approach)

(This proof could be used after G.GPE.4 has been taught. Coordinate proof is used to complete this proof.)

Given: \( \triangle ACB \), where \( D \) is the midpoint of \( \overline{AC} \) and \( E \) is the midpoint of \( \overline{BC} \).

Prove: \( \overline{DE} \parallel \overline{AB} \) and \( DE = \frac{1}{2} AB \)

Midpoints

\[
\text{Midpoint } \left( \overline{AC} \right) = \left( \frac{2g + 0}{2}, \frac{2h + 0}{2} \right) = (g, h)
\]

\[
\text{Midpoint } \left( \overline{BC} \right) = \left( \frac{2g + 2x}{2}, \frac{2h + 0}{2} \right) = (g + x, h)
\]
Slope
\[ m(\overline{DE}) = \frac{h-h}{(g+x)-g} = \frac{0}{x} = 0 \quad \text{and} \quad m(\overline{AB}) = \frac{0-0}{2x-0} = \frac{0}{2x} = 0 \]

Thus, \( \overline{DE} \parallel \overline{AB} \) because the slopes are equal.

Distance
\[
\begin{align*}
\overline{DE} &= \sqrt{((g+x)-g)^2 + (h-h)^2} \\
&= \sqrt{(x)^2} \\
&= x \\
\overline{AB} &= \sqrt{(2x-0)^2 + (0-0)^2} \\
&= \sqrt{(2x)^2} \\
&= 2x
\end{align*}
\]

Thus, \( DE = \frac{1}{2} AB \).

d) Proof by Translation (Formal – Transformational Approach)

**Given:** \( \triangle ABC \), where \( D \) is the midpoint of \( \overline{AB} \) and \( E \) is the midpoint of \( \overline{BC} \).

**Prove:** \( \overline{DE} \parallel \overline{AC} \) and \( DE = \frac{1}{2} AC \)

Translate \( \triangle DBE \) by vector \( \overrightarrow{DA} \). If the translation of \( E \) by vector \( \overrightarrow{DA} \) lies on \( \overline{AC} \), then \( \overline{DE} \parallel \overline{AC} \) by the properties of a translation. Using an indirect proof, we assume that \( L \) (the image of \( E \)) does not lie on \( \overline{AC} \). A contradiction occurs (provided below), so \( L \) is the image of \( E \) and lies on \( \overline{AC} \). In addition, by the properties of translations, \( \overline{DE} \parallel \overline{AC} \).

Translate \( \triangle DBE \) by vector \( \overrightarrow{EC} \) (vector \( \overrightarrow{EC} \) is the same as vector \( \overrightarrow{DL} \) because they are parallel and the same distance). We know that \( DE = AL \) by the previous translation, and we know the \( DE = LC \) by the current translation, therefore, \( AL = LC \) (transitive). Thus, \( AL + LC = AC \) (Segment Addition Postulate), \( DE + DE = AC \) (Substitution), and \( 2DE = AC \) (Simplify). Lastly, by the Multiplication/Division Property, \( DE = \frac{1}{2} AC \).
**Indirect Proof:** That the translation of $E$ by vector $\overrightarrow{DA}$ will lie on $\overline{AC}$.

The way an indirect proof works is to assume the opposite and prove that it creates a contradiction so that the opposite must be true.

So let us assume that point $L$, the image of $E$ translated by vector $\overrightarrow{DA}$ does not lie on $\overline{AC}$.

The contradiction comes when we prove that $\triangle DBE \cong \triangle LEG$ but then $EG$ must equal $EC$ but it doesn’t… thus the contradiction verifies that $L$ must be on $\overline{AC}$.

$DA = BD$ because $D$ is the midpoint and $DA = EL$ because translation vector $\overrightarrow{DA}$ thus $BD = EL$ by transitive.

$m \angle BDE = m \angle LED$ and $m \angle LED = m \angle ELG$ because corresponding $\angle$’s = (CPCTC) thus $m \angle BDE = m \angle ELG$, by transitive.

$m \angle B = m \angle LEG$ because corresponding angles (CPCTC).

$\triangle DBE \cong \triangle LEG$ but then $EG$ must equal $EC$ but it doesn’t… thus the contradiction verifies that $L$ must be on $\overline{AC}$.

e) **Proof by Double Translations (Informal – Transformational Approach)**

This isn’t necessarily proving the relationship but we are going to work backwards to show that it works.

Translate $\triangle ABC$ by vector $\overrightarrow{AB}$ resulting in $\triangle BB'D$. Translate $\triangle ABC$ by vector $\overrightarrow{AC}$ resulting in $\triangle CDE$.

These two transformations create $\triangle AB'E$, where $B$ is the midpoint of $\overline{AB'}$ and $D$ is the midpoint of $\overline{B'E}$, thus $\overline{BD}$ is the midsegment.

$\overline{BD} \parallel \overline{AE}$ because of the translation by vector $\overrightarrow{AB}$ and $BD = \frac{1}{2} AE$ because of the translation of $AC$ to $BD$ and to $CE$. 

f) Proof by Construction (Informal – Investigative Approach)

This isn’t really a proof but it simply a way to check the relationship using a compass and straightedge.

Have students construct the midpoints of $\overline{AB}$ and $\overline{CB}$, $D$ and $E$ respectively. Create the $\overline{DE}$ midsegment.

Using compass and straightedge copy $\angle BDE$ onto $\angle DAC$ and see if they are the same. If corresponding angles are $\cong$, then $\overline{DE} \parallel \overline{AC}$. Using a compass and straightedge copy $2\overline{DE}$ onto $\overline{AC}$ and see if $2DE = AC$. If they are, then $DE = \frac{1}{2}AC$ by inspection.

4. Prove that the medians of a triangle meet at a point.

Suggested Proofs: Regular Geometry 1a / Honors Geometry 1a

a) Proof by Midpoint (Formal – Classic Approach)

This approach uses the proportional parts relationship (G.SRT.4) and properties of a parallelogram (G.CO.11). Therefore, this approach must wait to be presented until these concepts have been covered.

**Given:** $\triangle ABC$, where $\overline{BF}$ and $\overline{CD}$ are medians.

**Prove:** That the 3rd median goes through point $G$.

It is not unique that two lines intersect at a point, but it is very special when three do. That point is known as a point of concurrency.

Create $\overrightarrow{AG}$ and mark where the ray meets $\overline{BC}$ as point $E$. Draw an auxiliary line parallel to median $\overline{BF}$ through $C$ and let it intersect $\overrightarrow{AG}$ at $H$.

$F$ is the midpoint of $\overline{AC}$ and $\overline{FG} \parallel \overline{CH}$, therefore $G$ is the midpoint of $\overline{AH}$ by proportional parts.

$D$ is the midpoint of $\overline{AB}$ and $G$ is the midpoint of $\overline{AH}$, thus $\overline{DG}$ is the midsegment of $\triangle ABH$ making $\overline{BH} \parallel \overline{DG}$ (or $\overline{GC}$ its extension).

Therefore, $\overline{BG} \parallel \overline{HC}$ and $\overline{BH} \parallel \overline{GC}$ making quadrilateral $BHCG$ a parallelogram. In any parallelogram, diagonals bisect, thus $BE = EC$ making $E$ the midpoint of $\overline{BC}$ and $\overrightarrow{AG}$ a median concurrent with the other two medians.
b) Proof by Midsegment (Formal – Classic Approach)

This approach uses properties of a parallelogram (G.CO.11). Therefore, this approach must wait to be presented until these concepts have been covered.

**Given:** $\triangle ABC$, where $\overline{AE}$ and $\overline{CD}$ are medians.
**Prove:** That the medians are concurrent.

In $\triangle ABC$, $\overline{DE}$ is parallel and half the size of $\overline{AC}$ due to the midsegment theorem. Create the midpoint of $\overline{AG}$ and $\overline{CG}$, $J$ and $K$ respectively. In $\triangle AGC$, $\overline{JK}$ is parallel and half the size of $\overline{AC}$ due to the midsegment theorem. Therefore, $\overline{DE} \parallel \overline{JK}$, by transitive, and $\overline{DE} \cong \overline{JK}$, by substitution. Therefore, quadrilateral $DEKJ$ a parallelogram, because one pair of opposite sides are both $\cong$ and $\parallel$.

Since diagonals of a parallelogram bisect each other, $DG = GK$ and $JG = GE$. We already know that $AJ = JG$ and $CK = KG$ because $J$ and $K$ are midpoints. Thus $G$ is twice as far to one endpoint of the median as it is to the other. This condition uniquely defines a point on the median. Since for two medians the points thus defined coincide, the same, by symmetry, would be true for all three medians.

The next proof has basic steps but the algebraic manipulation is at a high level of difficulty. While it has been provided here in full, I would suggest not solving this in the general case, it would be much easier to do with three coordinates.
c) Proof by Coordinates (Formal – Classic Approach)

This approach uses properties of a parallelogram (G.GPE.4). Therefore, this approach must wait to be presented until these concepts have been covered.

**Given:** \( \triangle ABC \), where \( \overline{BD} \) and \( \overline{CE} \) are medians.

**Prove:** That the medians are concurrent.

The first goal of this proof is to determine the coordinates of point \( F \). To do this we will determine the equations of \( \overline{BD} \) and \( \overline{CE} \) and then solve the system of equations between them.

<table>
<thead>
<tr>
<th>Find the slope of ( \overline{BD} )</th>
<th>Find the slope of ( \overline{CE} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{2b - 0}{2a - c} = \frac{2b}{2a - c} = m )</td>
<td>( \frac{b - 0}{a - 2c} = \frac{b}{a - 2c} = m )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Find the equation of ( \overline{BD} ) using point slope form</th>
<th>Find the equation of ( \overline{CE} ) using point slope form</th>
</tr>
</thead>
<tbody>
<tr>
<td>( (y - 0) = \frac{2b}{2a - c} (x - c) )</td>
<td>( (y - b) = \frac{b}{a - 2c} (x - a) )</td>
</tr>
<tr>
<td>( y = \frac{2b}{2a - c} x - \frac{2bc}{2a - c} )</td>
<td>( y - b = \frac{b}{a - 2c} x - \frac{ab}{a - 2c} )</td>
</tr>
<tr>
<td></td>
<td>( y = \frac{b}{a - 2c} x - \frac{ab}{a - 2c} + b )</td>
</tr>
<tr>
<td></td>
<td>( y = \frac{b}{a - 2c} x - \frac{ab}{a - 2c} + \frac{b(a - 2c)}{a - 2c} )</td>
</tr>
<tr>
<td></td>
<td>( y = \frac{b}{a - 2c} x - \frac{ab}{a - 2c} + \frac{ab - 2bc}{a - 2c} )</td>
</tr>
<tr>
<td></td>
<td>( y = \frac{b}{a - 2c} x - \frac{2bc}{a - 2c} )</td>
</tr>
</tbody>
</table>
Solve the system of equations between these two equations.

\[
\begin{align*}
\frac{2b}{2a-c} x - \frac{2bc}{2a-c} &= \frac{b}{a-2c} x - \frac{2bc}{a-2c} \\
\left( \frac{2b}{2a-c} - \frac{b}{a-2c} \right) x &= \frac{2bc}{2a-c} - \frac{2bc}{a-2c} \\
(2b(a-c) - b(2a-c))x &= 2bc(a-2c) - 2bc(2a-c) \\
(2ab-4bc-2ab+bc)x &= 2abc-4bc^2-4abc+2bc^2 \\
(-3bc)x &= -2abc-2bc^2 \\
\frac{2(a+c)}{3} x &= \frac{2b}{a-c}
\end{align*}
\]

Now we want to see if \( \overline{AF} \) and \( \overline{BC} \) intersect at a midpoint of \( \overline{BC} \). Then all three lines will be medians meeting at a common point \( F \).

Find the slope of \( \overline{AF} \)

\[
A = (0, 0) \quad F = \left( \frac{2(a+c)}{3}, \frac{2b}{3} \right)
\]

\[
m = \frac{2b}{3} - 0 = \frac{2(a+c)}{3} - 0
\]

\[
m = \frac{2b}{2(a+c)}
\]

\[
m = \frac{b}{a+c}
\]

Find the equation of \( \overline{AF} \) using point slope form

\[
y - 0 = \frac{b}{a+c}(x - 0)
\]

\[
y = \frac{b}{a+c} x
\]

Find the slope of \( \overline{BC} \)

\[m = \frac{2b - 0}{2a - 2c} = \frac{2b}{2a - 2c} = \frac{b}{a-c}\]

\[y = \frac{4ab + 4bc}{3(2a-c)} - \frac{(3)2bc}{3(2a-c)}
\]

\[y = \frac{4ab + 4bc - 6bc}{3(2a-c)}
\]

\[y = \frac{4ab - 2bc}{3(2a-c)} \quad y = \frac{2b(2a-c)}{3(2a-c)}
\]

\[y = \frac{2b}{3}
\]

Find the equation of \( \overline{BC} \) using point slope form

\[
y - 0 = \frac{b}{a-c}(x - 2c)
\]

\[
y = \frac{b}{a-c} x - \frac{2bc}{a-c}
\]
Solve the system of equations between these two equations.

\[
\begin{align*}
\frac{b}{a+c}x &= b - \frac{2bc}{a-c} \\
\left( \frac{b}{a+c} - \frac{b}{a-c} \right)x &= -\frac{2bc}{a-c} \\
(b(a-c) - b(a+c))x &= -2bc(a+c) \\
(ab - bc - ab - bc)x &= -2bc(a+c) \\
(-2bc)x &= -2bc(a+c) \\
x &= a + c
\end{align*}
\]

The point of intersection is found to be at \((a+c, b)\).

Let us see if this is the midpoint of \(BC\)

\[
\left( \frac{2a + 2c}{2}, \frac{2b + 0}{2} \right) = (a + c, b)
\]

Therefore the three medians are indeed concurrent!!

Notice our points of the triangle were \(A(0,0), B(2a, 2b)\) and \(C(2c, 0)\) and our intersection of the medians (the centroid) was \(F = \left( \frac{2(a+c)}{3}, \frac{2b}{3} \right)\).

If we had used points \(A(0,0), B(a, b)\) and \(C(c, 0)\) then our intersection of the medians (the centroid) would be \(F = \left( \frac{(a+c)}{3}, \frac{b}{3} \right)\).

**FUTURE CONNECTIONS** (What will they use these skills for later?)

- These triangle relationships will be used throughout the entire year as foundational concepts to understand more complex relationships. Specifically, triangle relationships are essential to establish quadrilateral relationships.
- The proof of the medians meeting at a point is probably best handled after parallelograms properties have been established or through coordinate proof. It is an involved proof. It seems to be placed here because the midsegment theorem is used in one of its proofs.

**ADDITIONAL EXTENSIONS OR EXPLANATIONS** (What needs greater explanation?)

- It seems that while the following items are not explicitly mentioned in the standard that this would be where the following should be taught and connected:
  - To classify triangles by their sides and angles.
  - To prove and then apply the exterior angle theorem.
  - To prove and then apply the Triangle Inequality Theorem
  - To prove acute angles in a right triangle are complementary.
  - To prove the converse of the Isosceles Triangle Theorem

**HELPFUL RESOURCES/LINKS**

- Proofs of the medians meeting at a point
  - [http://jwilson.coe.uga.edu/EMAT6680Fa05/Hawkins/Assignment%204/triangleproofs.html](http://jwilson.coe.uga.edu/EMAT6680Fa05/Hawkins/Assignment%204/triangleproofs.html)
• Proof of the midsegment theorem

**ASSESSMENT ITEMS**  (What questions would evaluate these skills?)

**MULTIPLE CHOICE**
(From CCSD Geometry Honors Semester 1 Practice 2012-2013)  (First 3 questions)

1. In \( \triangle ABC \), \( M \) is the midpoint of \( AB \) and \( N \) is the midpoint of \( AC \). For which type of triangle is \( MN = \frac{1}{2} BC \)?
   - (A) equilateral only
   - (B) isosceles only
   - (C) scalene only
   - (D) any triangle

For questions 2–3, consider \( \triangle ABC \) where \( AB = BC \) and \( m \angle A = 40^\circ \).

2. \( m \angle B + m \angle C = 140^\circ \)
   - (A) True  (B) False

3. \( m \angle C = 100^\circ \)
   - (A) True  (B) False

4. Which of the following is **not** true about the medians of a triangle?
   - A) Regardless of the triangle's shape, the three medians always meet at a single point
   - B) Three medians divide a triangle into six smaller triangles, all with the same area
   - C) The spot where the medians intersect is called a centroid
   - D) An isosceles triangle has only one median.

5. A segment connects the midpoints on two sides of a triangle. What is true about this segment?
   - A) It is always horizontal and half the length of each side
   - B) It is always perpendicular to the two sides it joins and forms an isosceles triangle with the portions of the sides above it
   - C) It is always parallel to the third side and half as long as the third side
   - D) It is always half the length of those two sides and parallel to the third

**SHORT ANSWER**

1. **Given:**

   Give a step by step explanation (a proof) of why the following statement is true.
   **Prove:** \( m \angle 1 + m \angle 2 = m \angle 4 \)
   - Hint: you know a relationship about \( \angle 1, \angle 2 \) & \( \angle 3 \) and you should also know a relationship about \( \angle 3 \) & \( \angle 4 \).

2. **Given:** \( \overline{BD} \) is the perpendicular bisector of \( \overline{AC} \)

   Give a step by step explanation (a proof) of why the following statement is true.
   **Prove:** \( \triangle ABC \) is an isosceles triangle.
3. Given:
\[ R_m(A) \to (A') \]
\[ R_m(B) \to (A') \]
\[ R_m(C) \to (C) \]

Prove: \( \triangle ABC \cong \triangle A'B'C \)

ANSWERS

1) Given:

\[ \text{Prove: } m\angle 1 + m\angle 2 = m\angle 4 \]

\[ m\angle 1 + m\angle 2 + m\angle 3 = 180^\circ \quad \text{Angle Sum of } \triangle \]

\[ m\angle 3 + m\angle 4 = 180^\circ \quad \text{Linear Pair} \]

\[ m\angle 1 + m\angle 2 + m\angle 3 = m\angle 3 + m\angle 4 \quad \text{Substitution} \]

\[ m\angle 1 + m\angle 2 = m\angle 4 \quad \text{Subtraction Property} \]

2) Given: \( \overline{BD} \) is the perpendicular bisector of \( AC \).

\[ \text{Prove: } \triangle ABC \text{ is an isosceles triangle.} \]

Method #1 (Transformational)

\[ R_{BD}(A) \to (C) \quad \text{because } \overline{BD} \text{ is the perpendicular bisector of } AC. \]

\[ R_{BD}(B) \to (B) \]

Thus \( AB \cong CB \) because Reflection is an isometry and if two sides are equal then the triangle is an isosceles triangle.

Method #2 (Congruence Criteria)

\[ AD \cong CD \text{ because } \overline{BD} \text{ is the bisector of } AC \]

\[ \angle ADC \cong \angle CDB \quad & \quad m\angle ADC = m\angle CDB = 90^\circ \quad \text{because } \overline{BD} \text{ is } \perp \text{ to } AC \]

\[ \overline{BD} \cong \overline{BD} \text{ because it is common (reflexive).} \]

Thus \( \triangle ADC \cong \triangle CDB \) and \( AB \cong CB \) because of congruent triangles.

Thus \( \triangle ABC \) is an isosceles triangle.

Method #3 (Perpendicular Bisector)

\( AB \cong CB \) because all points on \( \perp \) bisector are equidistant.

Thus \( \triangle ABC \) is an isosceles triangle.

3) Given:

\[ \text{Prove: } \triangle ABC \cong \triangle A'B'C \]

\( \triangle ABC \cong \triangle A'B'C \) because reflection preserves angles and distances between points (isometry).