Applying Quadratic Functions

Math Background

Previously, you

- Graphed and solved quadratic functions.
- Solved literal equations for a given variable.
- Found the inverse for a linear function.
- Verified by composition that two linear functions were inverses.
- Graphed piecewise functions with linear parts.
- Modeled real-world data with linear functions.
- Predicted response variables using linear models.

In this unit you will

- Find the inverse of a quadratic function.
- Verify inverses with composition.
- Graph a piecewise function with a quadratic piece.
- Use quadratic functions to model data, analyze and make predictions with technology.

You can use the skills in this unit to

- Describe how the domain of a quadratic function must be restricted so the function has an inverse.
- Show that two functions are inverses by composition.
- Solve a variety of different functions for a specific quantity.
- Rearrange a formula in context to answer a real-life problem.
- Interpret key features of functions in context of the real-world situation they model.
- Determine what constraints make sense in a variety of applied situations.

Vocabulary

- **Composite function** – The result of composing two functions together so that the output of the first becomes the input of the second**.**
- **Constraints** – A limitation usually imposed upon either the domain of a function or the range of a function.
- **Domain** – The input values of the function. For a quadratic function, the domain is all real numbers.
- **Invertible Function** – A function that has an inverse function.
- **Non-invertible Function** – A function that does not have an inverse function.
- **One-to-One Function** – A function whose inverse is a function. Both must pass the vertical and horizontal line tests**.**
- **Quadratic function** – A function where the highest exponent of the variable is a square.
- **Range** – The output values of a function.

Essential Questions

- How can we verify that two functions are inverses of each other? What is a composite function and why is it so important? Do all functions produce an inverse function?
- How do we model real-world situations when they cannot be described with a single function?
- Why is it useful to model real-world problems with equations and graphs?

Overall Big Ideas

Composite functions are common representations of real life situations and are used whenever a change in one quantity produces a change in another, which in turn produces a third quantity. By restricting the domain of a function, a one-to-one inverse can be created algebraically and graphically.

Piecewise functions have two or more parts, which may be any type of linear or non-linear function, and are used when a single function does not define a real-world situation well.

Equations and graphs can help to predict a future outcome of a real world problem or provide insight in to the problems past. Limitations, or constraints, create the region of viable options for real life solutions.

Skill

To find the inverse of a quadratic function and verify it by composition.

To graph a piecewise function that includes a quadratic portion.

To use quadratic functions to model data.

To use quadratic models to analyze and make predictions.

Related Standards

F.BF.B.4a-1

Solve an equation of the form $f(x)=c$ for simple linear and quadratic functions f that have an inverse and write an expression for the inverse. For example, $f(x)=2x^3$ or $f(x)=\frac{(x+1)}{(x-1)}$ for $x\neq 1$

F.BF.B.4b

Verify by composition that one function is the inverse of another.

F.BF.B.4d

Produce an invertible function from a non-invertible function by restricting the domain.

A.CED.A.4-2

Rearrange formulas of all types to highlight a quantity of interest, using the same reasoning as in solving equations. For example, rearrange Ohm's law V = IR to highlight resistance R. *(Modeling Standard)

F.IF.B.5-2

Relate the domain of any function to its graph and, where applicable, to the quantitative relationship it describes. For example, if the function *h(n)* gives the number of person-hours it takes to assemble *n* engines in a factory, then the positive integers would be an appropriate domain for the function. *(Modeling Standard)

F-IF.C.7b-1

Graph piecewise-defined functions, including step functions and absolute value functions. *(Modeling Standard)

A.CED.A.2-2

Create equations in two or more variables to represent relationships between quantities and graph equations on coordinate axes with labels and scales. Use all types of equations. *(Modeling Standard)

A.CED.A.3-2

Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or non-viable options in a modeling context. Use all types of equations. For example, represent inequalities describing nutritional and cost constraints on combinations of different foods. *(Modeling Standard)

S.ID.B.6a

Fit a linear function to the data; use functions fitted to data to solve problems in the context of the data. Use given functions or choose a function suggested by the context. Emphasize linear, quadratic, and exponential models. *(Modeling Standard)

S.ID.B.6b

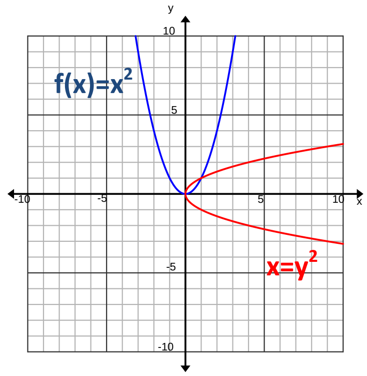
Informally assess the fit of a function by plotting and analyzing residuals. *(Modeling Standard)

Notes, Examples, and Exam Questions

Finding the inverse of a quadratic function:

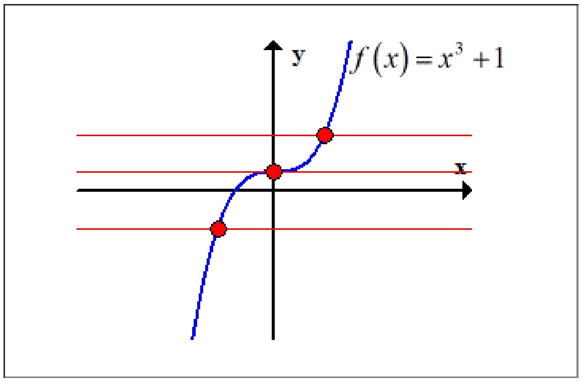
Not all functions have inverses that are functions. This happens in the case of quadratics if we do not restrict the domain. The inverse of a function *f* is a function *g* such that $g(f(x)) = x$. So, if you have the function $f(x) = ax^2 + bx + c$, then $g(f(x))$ must give you the original value *x*. You should already see the problem: there will be two functions, not one, since a function must provide a unique value in its range for each value in its domain and a quadratic maps two values to one (for example, $2^2 = (-2)^2 = 4$).

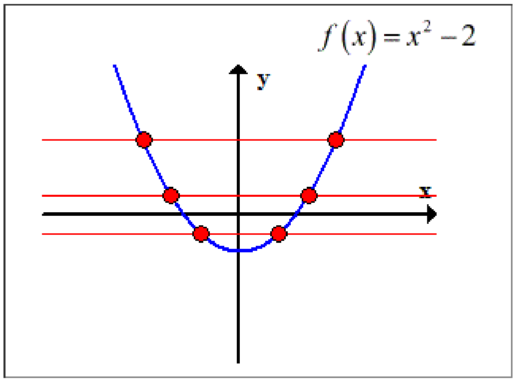
***Note that the graph in red, the inverse of *f(x)* is not a function as it does not pass the vertical line test.



However, we can carry on. If we restrict the domain to where the *x* values produce a graph that would pass the horizontal line test, then we will have an inverse function. The horizontal line test states that the inverse of a function *f* is also a function if and only if no horizontal line intersects the graph of *f* more than once.

Example of graph **with** an inverse	Example of graph **without** an inverse



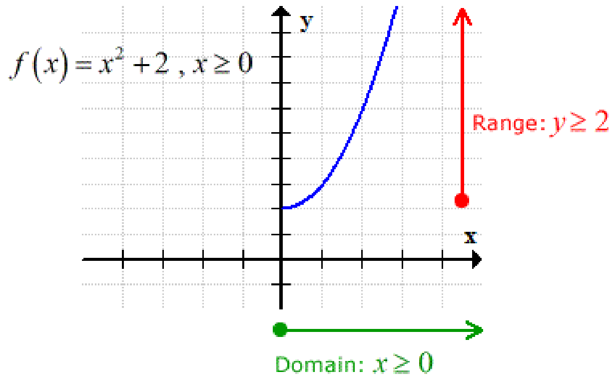


In the function above, if the domain of $f(x)=x^2$ is RESTRICTED to only nonnegative real numbers, then the inverse of *f* is a function. A function is **one-to-one** if each output of the function is paired with exactly one input. Only one-to-one functions have inverses that are also functions. We can restrict the domain to one-half of the parabola. Find the vertex and pick either the values greater than or less than your vertex's *x*-value to get an invertible function.

Steps:

1. Replace *f(x)* with *y*.
2. Interchange the *x* and *y* in the equation.
3. Solve for *y* in terms of *x*.
4. Replace *y* by $f^{-1}(x)$.
5. It is helpful to use the domain and range of the original function to identify the correct inverse function out of two possibilities. This happens when you get a "plus or minus" case in the end.

Ex 1



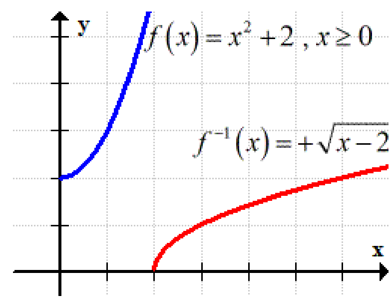
Find the inverse function $f(x)=x^2+2, x\geq 0$.

$y=x^2+2$	Step 1, replace f(x) with y.
$x=y^2+2$	Step 2, interchange x and y.
$x-2=y^2 \Rightarrow y=\pm\sqrt{x-2}$	Step 3, solve for y.
$f^{-1}(x)=\pm\sqrt{x-2}$	Step 4, replace the y with $f^{-1}(x)$.

There were two cases for the inverse due to the plus and minus. Pick the positive one as the answer since its domain and range are opposite of the original function.

$$f^{-1}(x)=+\sqrt{x-2}$$

Domain: $x\geq 2$, Range: $y\geq 0$



You can use composition to verify that two functions are inverses of each other.

$$f\circ f^{-1}(x)=f\left(f^{-1}(x)\right)=x$$

AND

$$f^{-1}\circ f(x)=f^{-1}\left(f(x)\right)=x$$

This is a result of the most basic principle of function inverses. Think of a function as some sort of process that we put *x* through and it outputs some value. A function's inverse is simply the reverse process. So, if we put *x* through a process, *f* , then put it through the reverse process, f^{-1}, we end up with just *x* again.

Ex 2

Find the inverse of the given function, $f(x)=-2x^2-5, x\leq 0$. Then, verify that your result and the original function are inverses.

$$y=-2x^2-5 \Rightarrow x=-2y^2-5 \Rightarrow x+5=-2y^2 \Rightarrow \frac{x+5}{-2}=y^2 \Rightarrow \pm\sqrt{\frac{-x-5}{2}}=y \text{ so } f^{-1}(x)=\pm\sqrt{\frac{-x-5}{2}}$$

The domain and range of the original function is: $D: x\leq 0, R: y\leq -5$. To pick the correct inverse function out of the two equations, find the domain and range of each possible answer. The correct inverse function should have a domain coming from the range of the original function and a range coming from the domain of the same function. The "negative" equation will give us the correct domain and range since our range has to be less than or equal to zero and the second answer gives us negative values. Answer: $f^{-1}(x)=-\sqrt{\frac{-x-5}{2}}$ with $D: x\leq -5, R: y\leq 0$

Verify they are inverses by composition:

Show that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$

$$f(f^{-1}(x)) = -2\left(-\sqrt{\frac{-x-5}{2}}\right)^2 - 5 = -2\left(\frac{-x-5}{2}\right) - 5 = x+5-5 = x \quad ✓$$

$$f^{-1}(f(x)) = -\sqrt{\frac{-\left(-2x^2-5\right)-5}{2}} = -\sqrt{\frac{2x^2+5-5}{2}} = -\sqrt{\frac{2x^2}{2}} = \left|-\sqrt{x^2}\right| = x \quad ✓$$

Ex 3

Find the inverse of the given function, $f(x) = (x+1)^2 - 4$. State whether the inverse is a function and find the domain and range. Then, verify that your result and the original function are inverses.

$$y = (x+1)^2 - 4 \Rightarrow x = (y+1)^2 - 4 \Rightarrow x+4 = (y+1)^2 \Rightarrow \pm\sqrt{x+4} = y+1 \Rightarrow \pm\sqrt{x+4} - 1 = y$$
so $f^{-1}(x) = \pm\sqrt{x+4} - 1$

The inverse is not a function. The domain and range of the original function are: D: all Reals, $R: y \geq -4$, so the domain and range of the inverse RELATION would be: $D: x \geq -4, R$: all Reals. To get an inverse function, we need to restrict the domain. Restricting the domain of *f(x)* to *x>0*, will give us an inverse function.

Inverse: $\boxed{f(x) = \sqrt{x+4} - 1}$ Domain: $x \geq -4$ and Range: $y > 0$.

<u>Verify by composition:</u>

Show that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$

$$f(f^{-1}(x)) = \left(\left(\sqrt{x+4} - 1\right) + 1\right)^2 - 4 = \left(\sqrt{x+4}\right)^2 - 4 = x+4-4 = x \quad ✓$$

$$f^{-1}(f(x)) = \sqrt{\left((x+1)^2 - 4\right) + 4} - 1 = \sqrt{(x+1)^2} - 1 = (x+1) - 1 = x \quad ✓$$

SAMPLE EXAM QUESTIONS

1. **The function $f(x) = x^2 - 2x - 3$ does not have an inverse unless the domain is restricted. Which restricted domain will allow $f(x)$ to have an inverse?**

A. $x \geq -4$

B. $x \geq -1$

C. $x \geq 0$

D. $x \geq 1$

Ans: D

2. Find the inverse of the function: $f(x)=x^2-4$. Is the inverse a function?

A $f^{-1}(x)=x^2+4$; yes it is a function.

B $f^{-1}(x)=\pm\sqrt{x+4}$; yes it is a function.

C $f^{-1}(x)=\pm\sqrt{x+4}$; no it is not a function.

D $f^{-1}(x)=x^2+4$; no it is not a function

Ans: C

3. Find the inverse of the function: $f(x)=(x-2)^2+3$. State the domain and range of the inverse.

A $f^{-1}(x)=\pm\sqrt{x-3}+2$
Domain: $\{x \mid x\in\Re\}$ Range: $\{y \mid y\geq 3\}$

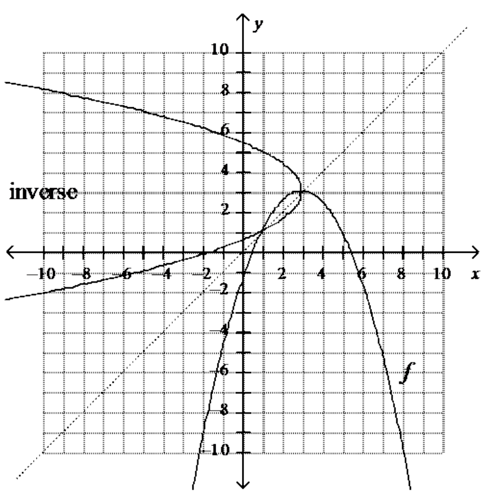
B $f^{-1}(x)=\pm\sqrt{x-3}+2$
Domain: $\{x \mid x\geq 3\}$ Range: $\{y \mid y\in\Re\}$

C $f^{-1}(x)=\pm\sqrt{x+3}-2$
Domain: $\{x \mid x\in\Re\}$ Range: $\{y \mid y\geq 3\}$

D $f^{-1}(x)=\pm\sqrt{x+3}-2$
Domain: $\{x \mid x\geq 3\}$ Range: $\{y \mid y\in\Re\}$

Ans: B

4. Consider the graph of $f(x)=-\frac{1}{2}(x-3)^2+3$ and its inverse. Notice that its inverse is not a function.



Describe two different domain restrictions for $f(x)=-\frac{1}{2}(x-3)^2+3$ that would make it invertible, without changing its range.

a. $x\geq 10$ or $x\leq 10$

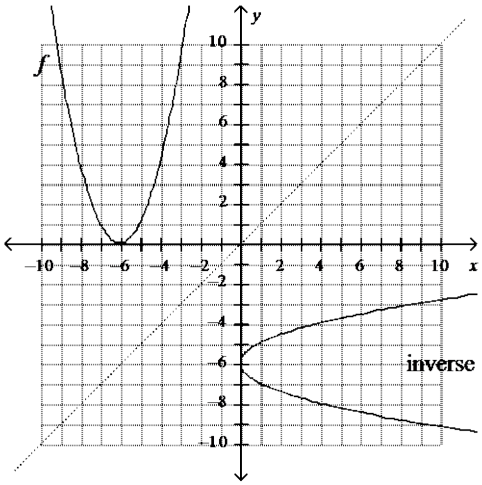
b. $x\geq 4$ or $x\leq 4$

c. $x\geq 3$ or $x\leq 3$

d. $x\geq 2$ or $x\leq 2$

Ans: C

5. The graph of $f(x) = (x+6)^2$ and its inverse are shown. Notice that its inverse is not a function.



Describe two different domain restrictions for $f(x) = (x+6)^2$ that would make it invertible, without changing its range.

a. $x \geq -6$ or $x \leq -6$
c. $x \geq -5$ or $x \leq -5$
b. $x \geq 10$ or $x \leq 10$
d. $x \geq -7$ or $x \leq -7$

Ans: A

PIECEWISE FUNCTIONS

Evaluating a Piecewise Function with a quadratic portion:

Ex 4 Find $f(-3)$ and $f(2)$ for the function $f(x) = \begin{cases} x^2 - 1, & x < 0 \\ 4x + 1 & x \geq 0 \end{cases}$.

Step One: Determine which equation to use based upon the value of *x*.

$f(-3)$: $x = -3 < 0$, so we will use the first equation.

$f(2)$: $x = 2 \geq 0$, so we will use the second equation.

Step Two: Substitute the value of *x* into the appropriate equation.

$f(-3) = (-3)^2 - 1 = 9 - 1 = 8$

$f(2) = 4(2) + 1 = 9$

<u>Graphing piecewise functions:</u> Since piecewise functions are defined in pieces, then you have to graph them in pieces, too. It may be helpful to do T-Charts for each portion of the function. The break between the parts of the functions (the point at which the function changes rules) is where your T-Charts will break.

Ex 5 Graph the function $f(x)=\begin{cases} x^2-2, & x<1 \\ -2x+4 & x\geq 1 \end{cases}$.

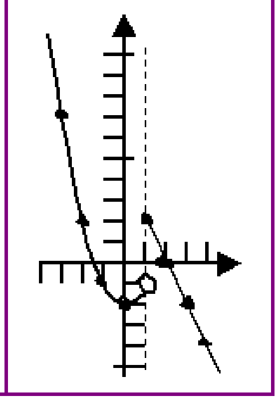
T-chart 1

x	$y=x^2-2$
-4	$16-2=14$
-3	$9-2=7$
-2	$4-2=2$
-1	$1-2=-1$
0	$0-2=-2$
(1)	$(1-2=-1)$

T-chart 2

x	$y=-2x+4$
1	$-2+4=2$
2	$-4+4=0$
3	$-6+4=-2$
4	$-8+4=-4$

Graph

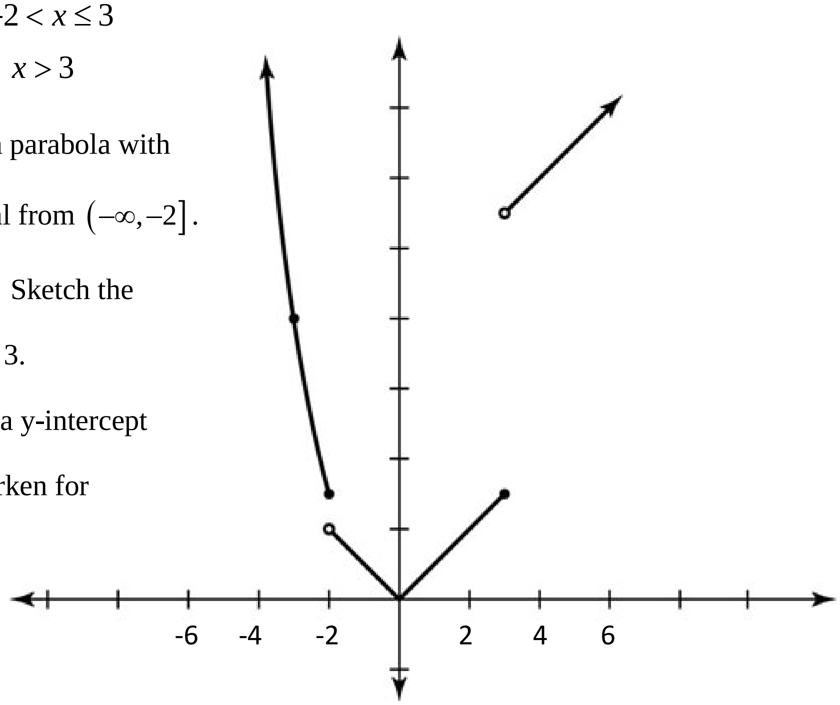


Ex 6 Graph the function $f(x)=\begin{cases} x^2-1, & x\leq -2 \\ |x| & -2<x\leq 3 \\ x+8 & x>3 \end{cases}$

The first piece is quadratic. Lightly sketch out a parabola with vertex at (0, -1) and then darken only the interval from $(-\infty,-2]$.

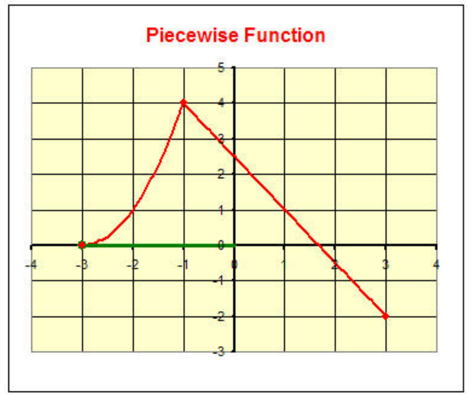
The second piece is the absolute value function. Sketch the absolute-value graph for the x-values from -2 to 3.

The third piece is linear with a slope of one and a y-intercept of eight. Lightly draw the line and then only darken for values of x bigger than 3.



***Note the open and closed dots on the graph

Ex 7 Given the graph at the right, find the corresponding function.



There is a quadratic function for values of x from -3 to -1.

There is a linear function for values of x from -1 to 3.

Parabola: Vertex: (-3,0), opens up , point: (-1 4)

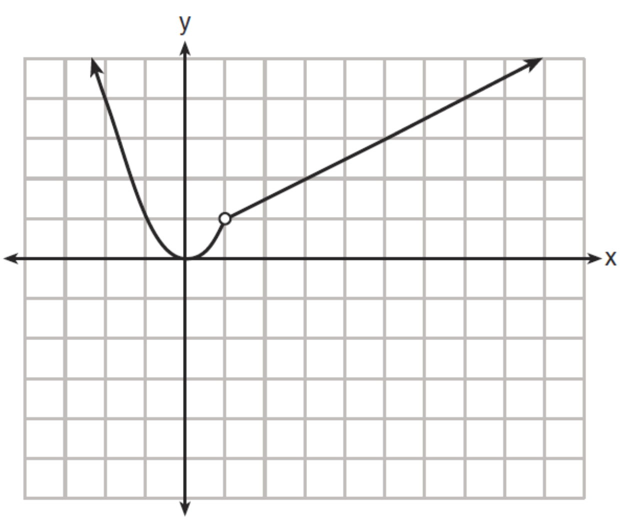
$$4 = a((-1)-(-3))^2 + 0$$
$$4 = 4a \qquad y = (x+3)^2$$
$$a = 1$$

Line: Slope = $-\frac{3}{2}$, y-int = $2\frac{1}{2}$

$$f(x) = \begin{cases} (x+3)^2 & -3 \le x \le -1 \\ -\frac{3}{2}x + \frac{5}{2} & -1 \le x \le 3 \end{cases}$$

SAMPLE EXAM QUESTIONS

1. A function is graphed on the set of axes below.



Which function is related to the graph?

(1) $f(x) = \begin{cases} x^2, & x < 1 \\ x - 2, & x > 1 \end{cases}$ (3) $f(x) = \begin{cases} x^2, & x < 1 \\ 2x - 7, & x > 1 \end{cases}$

(2) $f(x) = \begin{cases} x^2, & x < 1 \\ \frac{1}{2}x + \frac{1}{2}, & x > 1 \end{cases}$ (4) $f(x) = \begin{cases} x^2, & x < 1 \\ \frac{3}{2}x - \frac{9}{2}, & x > 1 \end{cases}$

Ans: 2

MODELING QUADRATIC FUNCTIONS

<u>Identifying Quadratic Data</u>

Ex 8 Determine whether the data set could represent a quadratic function. Explain.

x	0	2	4	6	8
y	12	10	9	9	10

Find the first and second differences.

Check that the x values are equally spaced.

x	0	2	4	6	8
y	12	10	9	9	10

1st: –2 –1 0 1

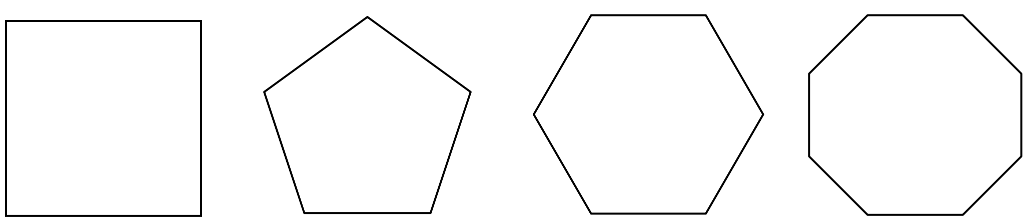
2nd: 1 1 1

Quadratic function; **second** differences are constant for equally spaced x values. Linear functions have first differences that are constant for equally spaced x values.

<u>Modeling a Function Using Data</u>

Ex 9 Exploration Activity

Create a table that relates the number of diagonals to the number of sides of a polygon. Use the graphing calculator to graph a scatter plot and find at least two regression equations and their r and r^2 values.



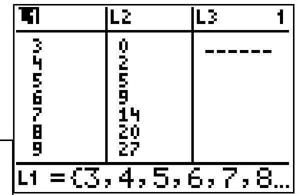
n	3	4	5	6	7	8	9	10
d	0	2	5	9	14	20	27	35

Note by the second differences that this is quadratic!

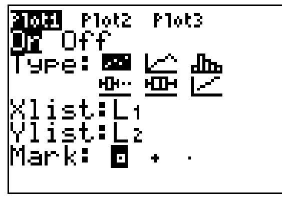
1^{st} diff: 2, 3, 4, 5, 6, 7, 8 2^{nd}: diff: 1, 1, 1, 1, 1, 1, 1

Directions for the TI-84:

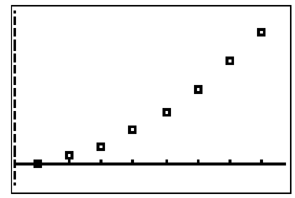
and enter the data for "n" in L_1 and the data for "d" in L_2.



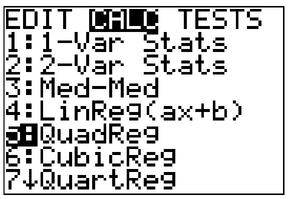
and enter into the first STAT PLOT

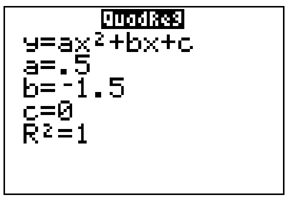


Zoom 9 will produce the graph.



To perform the quadratic regression on the data, press STAT, arrow over to CALC and number 5 is the Quadratic Regression.





The quadratic function that models this data is: $f(x) = .5x^2 - 1.5x$

<u>Coefficient of Determination (r^2):</u> The closer the value is to 1, the better the curve fits the data. Since our coefficient of determination is equal to one, it is a perfect fit.

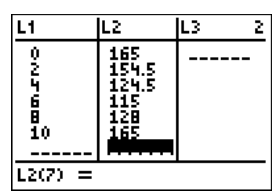
Teacher Note: If a student does not see r^2 values on their calculator, use the command DiagnosticsOn in the Catalog. Also – After the command QuadReg, type in Y_1 (in the VARS menu) to have the function automatically appear in Y_1 to graph.

<u>Graphing Calculator Activity: Using a quadratic model to represent data</u>.

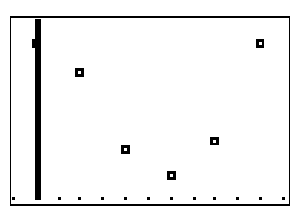
Ex 10 The table shows the average sale price *p* of a house for various years *t* since 1988. Use a quadratic regression on the graphing calculator to write a quadratic model for the data.

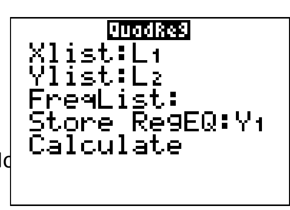
Years Since 1988, *t*	0	2	4	6	8	10
Average Sale Price (thousands of dollars), *p*	165	154.5	124.5	115	128	165

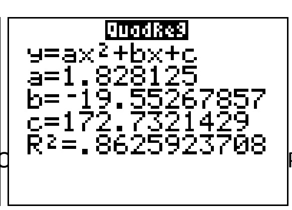
Enter the data from the table into the Lists. Enter *t* values into L_1 and *p* values into L_2. (Use STAT – Edit to enter data into the Lists.)

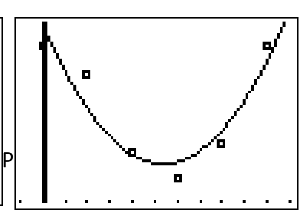


Graph the scatterplot and notice that the data points follow a parabolic shape.
On the home screen, use the QuadReg to find the quadratic regression.
Take a look at the graph of the quadratic model with the scatterplot of the data.









Ex 11

The table shows the sizes and prices of decorative square patio tiles.

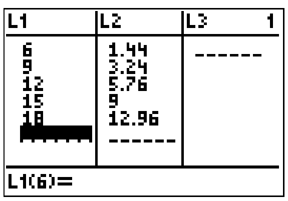
Side Length (in.)	6	9	12	15	18
Price Each ($)	1.44	3.24	5.76	9.00	12.96

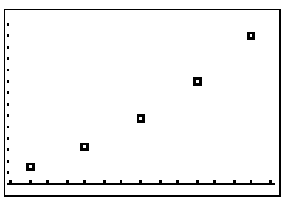
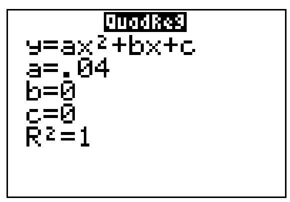
Determine whether the data set could represent a quadratic function. Then, find the appropriate function to model the data.

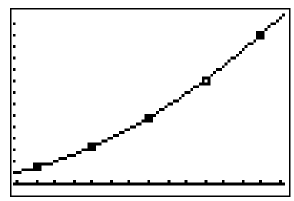
Looking at the differences, the first differences are: 1.8, 2.52, 3.24, and 3.96, so it is not a linear function.

The second differences have a constant value of 0.72 verifying that this data represents a quadratic function.

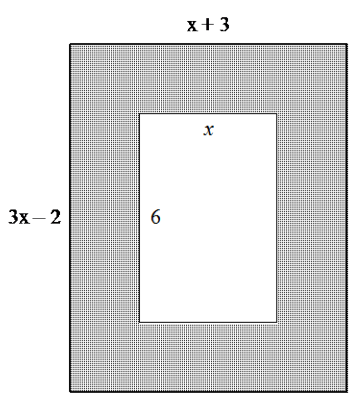
Using the TI-84, plot the data, run a Quadratic Regression, find the model and graph the model.





SAMPLE EXAM QUESTIONS



1. Write a polynomial to represent the area of the shaded region. Then solve for *x* given that the area of the shaded region is 24 square units.

a. $3x^2 + 7x - 6; x = 5$

c. $3x^2 + x - 6; x = 4$

b. $3x^2 + x - 6; x = 3$

d. $3x^2 + 7x - 6; x = 2$

Ans: B

2. During a halftime show, a baton twirler releases her baton from a point 4 feet above the ground with an initial vertical velocity of 25 feet per second.

Part A: Use the vertical motion model to write a function for the height h (in feet) of the baton after t seconds.

Part B: Graph the function in ***Part A***. Label the vertex of the graph.

Part C: How high does the baton go? Round your answer to the nearest tenth.

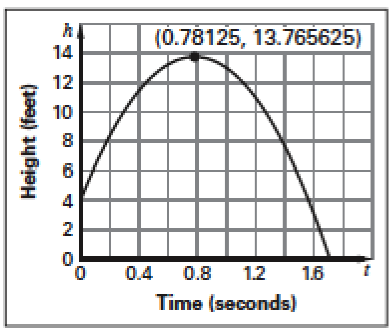
Part D: How long after the baton is released does it reach its maximum height?

Part E: At what moments is the baton at a height of 10 feet? Round your answer to the nearest hundredth.

Part F: How much time does the twirler have if she plans to catch the baton on its way down at a height of 5 feet? Round your answer to the nearest hundredth.

Ans: ***Part A:*** $h = -16t^2 + 25t + 4$

Part B:



Part C: 13.8 ft
Part D: 0.78125 sec
Part E: 0.30 sec and 1.27 sec after release
Part F: 1.52 sec

3. Several values of the quadratic function $f(x)$ are given in the table.

x	$f(x)$
–4	–96
–2	–24
0	0
4	–96
9	–486

The function $g(x)$ is given by $g(x) = -(x-2)^2 + 3$. Which function has the greater maximum for which value of x?

(A) $f(x)$; for $x = 0$ (B) $f(x)$; for $x = -6$

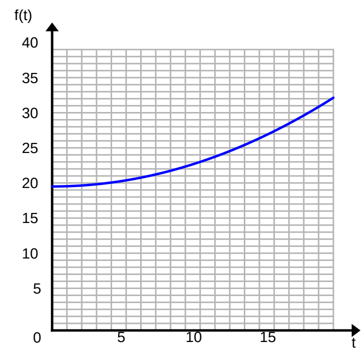
(C) $g(x)$; for $x = 2$ (D) $g(x)$; for $x = 3$

Ans: C

4. The amount of fuel F (in billions of gallons) used by trucks from 1990 through 2009 can be approximated by the function $F = f(t) = 20.5 + 0.035t^2$ where $t = 0$ represents 1990.

a) Describe the transformation of the common function $f(t) = t^2$. Then sketch the graph over the interval $0 \le t \le 19$.

Vertical shrink by a factor of 0.035 and a vertical shift of 20.5 units up.



b) Find and interpret $\frac{f(19) - f(0)}{19 - 0}$.

$$\frac{f(19) - f(0)}{19 - 0} = \frac{33.135 - 20.5}{19} = \frac{12.635}{19} = 0.665$$

On average 0.665 billion(665 million) of gallons of fuel is used per year by trucks from 1990 to 2009.

c) Rewrite the function so that $t = 0$ represents 2000. Explain how you got your answer.

Move $f(t)$ ten units to the left. Let's call the new function *w(t).* $w(t) = f(t + 10) = 0.035(t + 10)^2 + 20.5$
In f(t), the year 2000 is represented by t=10. If you want the year 2000 to be located at t=0, f(t) has to be moved horizontally 10 units to the left.

d) Use the model from part (c) to predict the amount of fuel used by trucks in 2015. Does your answer seem reasonable? Explain.

$w(15) = 0.035(15 + 10)^2 + 20.5 = 42.375$

So, 42.375 billions of gallons of fuel will be used in the year 2015. This is reasonable because fuel consumption is increasing.

5. Which statement best describes these two functions? $f(x) = x^2 - x + 4$
$g(x) = -3x^2 + 3x + 7$

(A) The maximum of $f(x)$ is less than the minimum of $g(x)$.

(B) The minimum of $f(x)$ is less than the maximum of $g(x)$.

(C) The maximum of $f(x)$ is greater than the minimum of $g(x)$.

(D) The minimum of $f(x)$ is greater than the maximum of $g(x)$.

Ans: B

6. The equation $h = -16t^2 + 40t + 5$ gives the height *h*, in feet, of a baseball as a function of time *t*, in seconds, after it is hit. How long does it take for the baseball to hit the ground?

Ans: about 2.6 s