Compositions of linear functions

Math Background

Previously, you

- Found the inverse of a linear function and graphed the original function with its inverse.
- Performed operations with algebraic expressions and algebraic functions

In this unit you will

- Compose a function f(x) when the input x is also a function x(t)
- Interpret the composition of functions as applied to real world problems
- Calculate the inverse of a function
- Calculate $f(f^{-1}(x)) = f^{-1}(f(x)) = x$ to verify the composition of a function

You can use the skills in this unit to

- Combine two functions through composition
- Verify two functions are inverse functions with composition
- Solve real-world multi-step problems using composition of functions

Vocabulary

- **Composite function** – The result of composing two functions together so that the output of the first becomes the input of the second**.**
- **Composition** – The act of combining two mathematical functions.
- **Function** – A set of ordered pairs where no two ordered pairs have the same first element.
- **Horizontal line test** – The inverse of a function *f* is also a function if and only if no horizontal line intersects the graph of *f* more than once.
- **Inverse Functions –** Two functions are inverse functions if the domain of the original function matches the range of the second function.
- **One-to-one function** – A function whose inverse is a function. Both must pass the vertical and horizontal line tests.

Essential Questions

- What is a composite function and why is it so important?
- How can the composition of two functions be used to represent real life applications?
- How can we verify that two functions are inverses of each other?

Overall Big Ideas

Composite functions are common representation of real life situations and are used whenever a change in one quantity produces a change in another, which in turn produces a third quantity.

Skill

To perform compositions of linear functions.

To verify the linear inverse by composition.

Related Standards

F.BF.A.1c

Composite Functions. For example, if T(y) is the temperature in the atmosphere as a function of height, and h(t) is the height of a weather balloon as a function of time, then T(h(t)) is the temperature at the location of the weather balloon as a function of time. *(Modeling Standard)

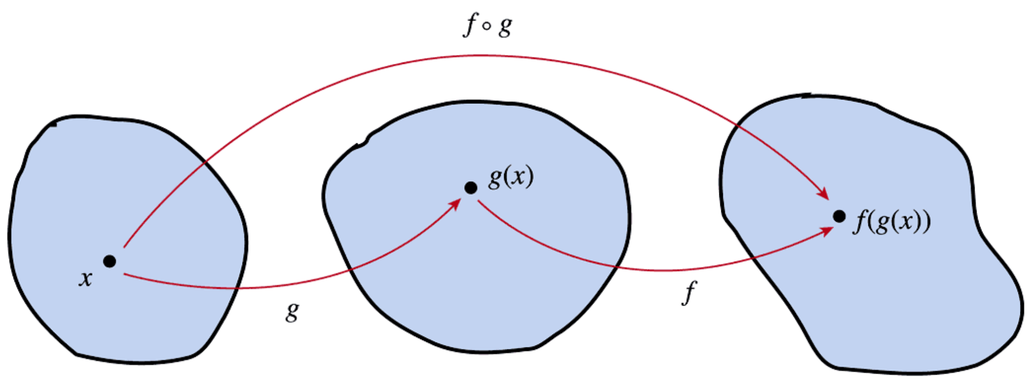
F-BF.B.4b

Verify by composition that one function is the inverse of another.

Notes, Examples, and Exam Questions

Composition of Functions: The composition of *f* of *g* uses the notations: $f \circ g = (f \circ g)(x) = f(g(x))$

- This is read "*f* of *g* of x"
- In the composition of f of *g*, the domain of f intersects the range of *g*.
- f is the outer function and *g* is the inner function.
- The domain of the composition functions consists of all x-values in the domain of *g* that are also *g(x)*-values in the domain of f.



x must be in the domain of *g* and *g*(*x*) must be in the domain of *f*

When we compose functions, we essentially nest one inside the other. Always remember to put the inner function in the outer function.

Ex 1

Find the composite function between $g(x) = 2x - 4$ and $h(x) = -4x + 3$. Find $(g \circ h)(x)$.

We plug our $h(x)$ into the position of x in $g(x)$, simplify and get the following composite function:

$(g \circ h)(x) = 2(-4x+3) - 4 = -8x + 6 - 4 = -8x + 2$ $g(h(x)) = -8x + 2$

Next, find $(h \circ g)(x)$.

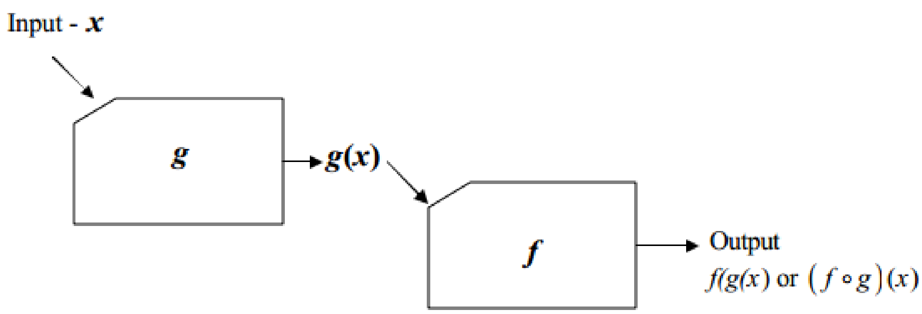
We plug our $g(x)$ into the position of x in $h(x)$, simplify and get the following composite function:

$(h \circ g)(x) = -4(2x-4) + 3 = -8x + 16 + 3 = -8x + 19$ $h(g(x) = -8x + 19$

***It is important to note from the above example that $g(h(x)) \neq h(g(x))$. These two compositions do not have to be equal.

The function $f(g(x))$ uses the *output* of the function g as the *input* to the function f. The function $g(f(x))$ uses the *output* of the function f as the *input* to the function g.

The composition of f and g is shown in the diagram.



Ex 2

Given $f(x)=3x+4$ and $g(x)=2x-7$ find $f(g(3))$.

We substitute the value, 3, in $g(x)$ and receive a value of -1. This value is then substituted into *f(x)*.

$g(3)=2(3)-7=-1$
$f(-1)=3(-1)+4=1$
$f(g(3))=1$

Ex 3

Use the graph of f and the table for g to evaluate the following:

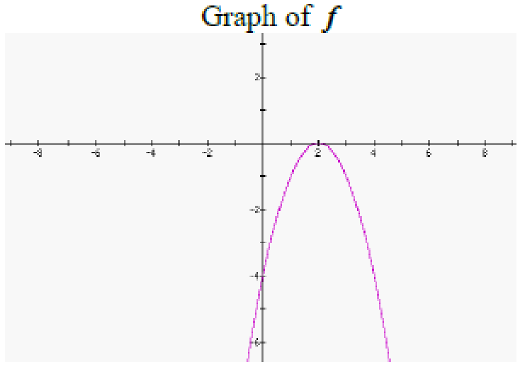
a) $f(g(4))$

$g(4)=2 \Rightarrow f(2)=0$
$f(g(4))=0$

b) $g(f(2))$

$f(2)=0 \Rightarrow g(0)=3$
$g(f(2))=3$

Graph of f



x	$g(x)$
0	3
2	0
3	4
4	2

c) $f(g(2))$

$g(2)=0 \Rightarrow f(0)=-4$
$f(g(2))=-4$

d) $g(g(2))$

$g(2)=0 \Rightarrow g(0)=3$
$g(g(2))=3$

Ex 4

Let $f(x) = x + 4$ and $g(x) = x - 4$. Find $f(g(x))$ and $g(f(x))$.

$f(g(x)) = (x-4)+4 = x$
$g(f(x)) = (x+4)-4 = x$

***Note that $f(g(x)) = g(f(x)) = x$

The example above illustrates a key concept with respect to composition of functions and inverses. You can use composition to verify that two functions are inverses of each other.

$$f \circ f^{-1}(x) = f\left(f^{-1}(x)\right) = x$$

AND

$$f^{-1} \circ f(x) = f^{-1}\left(f(x)\right) = x$$

This is a result of the most basic principle of function inverses. Think of a function as some sort of process that we put x through and it outputs some value. A function's inverse is simply the reverse process. So, if we put x through a process, f, then put it through the reverse process, f^{-1}, we end up with just x again.

Ex 5

Find the inverse of the given function, $f(x) = 2x - 1$. Then, verify that your result and the original function are inverses.

$y = 2x - 1 \Rightarrow x = 2y - 1 \Rightarrow x + 1 = 2y \Rightarrow \frac{x+1}{2} = y$ so $f^{-1}(x) = \frac{x+1}{2}$

Show that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$

$$f(f^{-1}(x)) = 2\left(\frac{x+1}{2}\right) - 1 = x + 1 - 1 = x$$ ☑

$$f^{-1}(f(x)) = \frac{(2x-1)+1}{2} = \frac{2x}{2} = x$$ ☑

The composition of functions is an important topic. It is often helpful to think of a function as a rule. The composition of functions consists of applying one rule, getting a result, and then applying the second rule to what you obtained from the first rule. Here is an example.

Ex 6

In the mail, you receive a coupon for \$5 off of a pair of jeans. When you arrive at the store, you find that all jeans are 25% off. Let *x* represent the original cost of the jeans.

1. Write a function, f(x), that represents the effect of your original coupon. $f(x) = x - 5$

2. Write a function, g(x), that represents the effect of the 25% discount at the store. $g(x) = x - 0.25x = 0.75x$

3. Write a function, h(x), that represents how much you would pay if you use the mail coupon first followed by applying the discount from the store. $h(x) = g(f(x)) = 0.75(x-5) = 0.75x - 3.75$

4. Write a function, j(x), that represents how much you would pay if you use the store discount first, followed by the mail coupon. $j(x) = f(g(x)) = 0.75x - 5$

5. You find a pair of jeans for \$36. How much would you pay for it using both functions h(x) and j(x).

$$h(x) = 0.75x - 3.75 = 0.75(36) - 3.75 = \$23.25$$
$$j(x) = 0.75x - 5 = 0.75(36) - 5 = \$22.00$$

Ex 7

The formula $K(C) = C + 273$ converts Celsius temperature to Kelvin. The formula $C(F) = \frac{5}{9}(F - 32)$ converts Fahrenheit temperature to Celsius. Write a composite function that will convert Fahrenheit temperature to Kelvin and convert the boiling point of water $(212^\circ F)$ and the freezing point of water $(32^\circ F)$ to Kelvin.

$$K(F) = \frac{5}{9}(F - 32) + 273$$
$$K(212) = \frac{5}{9}(212 - 32) + 273 = 373 \text{ K}$$
$$K(32) = \frac{5}{9}(32 - 32) + 273 = 273 \text{ K}$$

SAMPLE EXAM QUESTIONS

1. Let $f(x) = 3x + 1$ and $g(x) = x + 2$. Which expression is equal to $f(g(x))$?

A. $3x + 3$

B. $3x + 7$

C. $4x + 3$

D. $3x^2 + 7x + 2$

Ans: B

2. Which statement must be true if f and g are inverses of one another?

A. $(f \circ g)(x) + (g \circ f)(x) = x$

B. $(f \circ g)(x) = f(x) \bullet g(x) = (g \circ f)(x) = g(x) \bullet f(x) = x$

C. $(f \circ g)(x) = f(g(x)) = (g \circ f)(x) = g(f(x)) = x$

D. $(f \circ g)(x) = \dfrac{1}{(g \circ f)(x)} = x$

Ans: C

3. If $f(x) = \frac{1}{2}x - 3$ and $g(x) = 2x + 5$, what is the value of $g(f(4))$?

A. -13

B. 3.5

C. 3

D. 6

Ans: C

4. The accompanying tables define functions *f* and *g* . What is $(g \circ f)(3)$?

x	1	2	3	4	5
f(x)	3	4	5	6	7

x	3	4	5	6	7
g(x)	4	6	8	10	12

A. 6

B. 2

C. 4

D. 8

Ans: D

5. If $f(x) = 3x - 5$ and $g(x) = x - 9$, which expression is equivalent to $(f \circ g)(x)$?

A. $3x - 32$

B. $3x - 14$

C. $4x - 14$

D. $3x^2 - 32x + 45$

Ans: A