Syllabus Objective Note: Because this unit contains all prerequisite skills that were taught in courses prior to precalculus, there will not be any syllabus objectives listed. Teaching this unit within the school year is optional. A pretest may be given to students to check for mastery of prerequisite skills to determine if this unit should be taught. Another option would be to have students complete this unit as a summer assignment.

Real Numbers

Real Numbers $(\mathbb{R})$: Numbers that can be written as decimals. The braces { } are used to enclose the elements (objects) of a set.

The real number system contains several subsets:

- Natural Numbers: the set of counting numbers $(\mathbb{N})$ $\{1, 2, 3, \ldots\}$
- Whole Numbers: 0 and the set of the natural (counting) numbers (W) $\{0, 1, 2, 3, \ldots\}$
- Integers $(\mathbb{Z})$: the whole numbers and their opposites $\{\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\}$
- Rational Numbers $(\mathbb{Q})$: numbers that can be written as the ratio *a*/*b* of two integers, such that $b \neq 0$. We can use set builder notation to describe the rational numbers:

$$\left\{ \frac{a}{b} \middle| a, b \text{ are integers, and } b \neq 0 \right\}$$

- Irrational numbers: real numbers that are decimals that do not terminate and do not repeat.

Ex: Label each box with the five subsets above and place these elements (objects) in the appropriate boxes. Indicate the rational numbers that **t**erminate/**r**epeat:

$\sqrt{121}$ $\sqrt{120}$

-1 -3^2

$\frac{54}{6}$ $\frac{22}{7}$

3.14 $-\sqrt{2}$

$\sqrt{0}$ π $\frac{4}{9}$

Real

Rational: $\frac{22}{7}, \frac{4}{9}, 3.14$

Irrational: π, $\sqrt{120}$, $-\sqrt{2}$

Integers: $-1, -3^2$

Whole: $\sqrt{0}$

Natural: $\sqrt{121}, \frac{54}{6}$

If *a* and *b* are real numbers with $a < b$ then we can use the following notation:

Inequality Notation: $a < x \leq b$ Interval notation: $(a, b]$

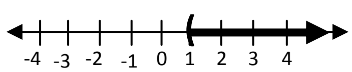
	Bounded	**Unbounded**
Closed	$[a, b]$	$[a, \infty)$ $(-\infty, b]$
Open	(a, b)	(a, ∞) $(-\infty, b)$
Half-open	$[a, b)$ $(a, b]$	Unbounded intervals are never considered half-open

We can describe inequalities by using words, graphs and interval notation.

For examples 2-4 also note whether the interval is bounded or unbounded and open, closed or half-open.

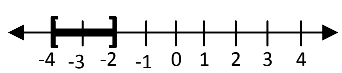
Ex.2 Describe in words and graph the interval.

a) $[\,1, \infty)$ *x* is greater than or equal to 1



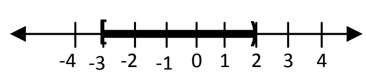
Unbounded, open

b) $-4 \leq x \leq -2$ all real numbers between and including –4 &–2



Bounded, closed

Ex.3 Use inequality and interval notation to describe the interval of real numbers.

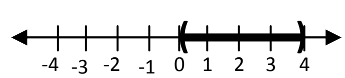


$-3 \leq x < 2$ $[-3, 2)$

Bounded, half-open

Ex.4 Use interval notation to describe the interval of real numbers and graph.

x is between 0 and 4. $(0, 4)$



Fill in the ? of the equation for each property: Solutions:

1.	Commutative	$x + y = ?$	$y + x$
2.	Associative	$(ab)c = ?$	$a(bc)$
3.	Identity	$m + ? = m$	0
		$n(?) = n$	1
4.	Inverse	$z + ? = 0$	$-z$
		$w(?) = 1,\ w \neq 0$	$\frac{1}{w}$
5.	Distributive	$? = rs - rt$	$r(s - t)$

What misconception might a student have about $-n$?

They might think it is negative when in fact, it represents the opposite of n and therefore is dependent on the value of n.

What is the difference between -4^2 and $(-4)^2$?

The bases are 4 and –4. So $-4^2 = -16$ and $(-4)^2 = 16$.

$10^3 = 1000.$

$10^2 = 100.$

$10^1 = 10.$

$10^0 = 1.$

$10^{-1} = .1 = \frac{1}{10} = \frac{1}{10^1}$

$10^{-2} = .01 = \frac{1}{100} = \frac{1}{10^2}$

$10^{-3} = .001 = \frac{1}{1000} = \frac{1}{10^3}$

We define $a^0 = 1$ & $a^{-n} = \frac{1}{a^n}$, $a \neq 0$. 0^0 is indeterminate.

Ex.5 Examine the properties of exponents:

$3^2 \cdot 3^4 =$ $3{\cdot}3 \cdot 3{\cdot}3{\cdot}3{\cdot}3 = 3^6$ 3^{2+4}

$(3^4)^2 =$ $3^4 \cdot 3^4 = 3{\cdot}3{\cdot}3{\cdot}3{\cdot}3{\cdot}3{\cdot}3{\cdot}3 = 3^8$ $3^{2(4)}$

$(2{\cdot}4)^3 =$ $(8)^3 = 8{\cdot}8{\cdot}8 = 512$

$2^3 \cdot 4^3 =$ $8 \cdot 64 = 512$

$\frac{3^5}{3^2} =$ $\frac{3\cdot3\cdot3\cdot3\cdot3}{3\cdot3} = 3{\cdot}3{\cdot}3 = 27 = 3^3$ 3^{5-2}

$\left(\frac{3}{5}\right)^2 =$ $\left(\frac{3}{5}\right)\cdot\left(\frac{3}{5}\right) = \frac{9}{25} = \frac{3^2}{5^2}$

Ex.6 $\left(\frac{-5x^{-3}y^2}{x^2y^{-1}}\right)\cdot\left(\frac{2x^2y}{x^3y}\right)^3$ $-5x^{-3-2}y^{2-(-1)}\cdot\left(2x^{2-3}y^{1-1}\right)^3$

$$-5x^{-5}y^3\cdot\left(2x^{-1}y^0\right)^3 = -5x^{-5}y^3\cdot 8x^{-3} = -40y^3x^{-8} = -\frac{40y^3}{x^8}$$

Ex.7 What is half of 2^{40}? What is one third of 3^{18}? Leave answers in exponential form.

$\frac{1}{2}\cdot 2^{40} = 2^{-1}\cdot 2^{40} = 2^{39}$ $\frac{1}{3}\cdot 3^{18} = 3^{-1}\cdot 3^{18} = 3^{17}$

Ex.8 A pile of gravel contains 10^{10} stones. Take ten stones from the original pile and throw nine onto a pile on the left and one onto a pile on the right. When the original pile is gone, how many stones are in each of the new piles?

$(9 + 1)(10)^9 = 9(10)^9$ and $1(10)^9$

<u>Scientific Notation</u>: ***one*** non-zero digit left of the decimal, multiplied by a power of 10

Ex.9 $\left(2.5\times10^4\right)\left(6\times10^2\right)\left(5\times10^{-1}\right)\div\left(3\times10^{-3}\right)$

$2.5(6)(5) \div 3 \times 10^{4+2+(-1)-(-3)} = 75 \div 3 \times 10^8 = 25 \times 10^8$

$= 2.5 \times 10^1 \times 10^8 = 2.5 \times 10^9$

You Try: Simplify using scientific notation. $\dfrac{(42,000,000)(6,000,000,000)}{12,000,000}$

QOD: Describe the four types of unbounded intervals and give examples of each.

Sample SAT Question(s): Taken from College Board online practice problems.

1. When 70,000 is written as 7.0×10^n, what is the value of *n*?
   (A) 1
   (B) 2
   (C) 3
   (D) 4
   (E) 5

2. If *P* and *Q* are two sets of numbers, and if every number in *P* is also in *Q*, Which of the following CANNOT be true?
   (A) 4 is in both *P* and *Q*.
   (B) 5 is in neither *P* nor *Q*.
   (C) 6 is in *P*, but not in *Q*.
   (D) 7 is in *Q*, but not in *P*.
   (E) if 8 is not in *Q*, then 8 is not in *P*.

3. For all numbers *a* and *b*, let $a\blacktriangle b$ be defined $a\blacktriangle b = ab + a + b$. For all numbers *x*, *y*, and *z*, which of the following must be true?
   I. $x\blacktriangle y = y\blacktriangle x$
   II. $(x-1)\blacktriangle(x+1) = (x\blacktriangle x) - 1$
   III. $x\blacktriangle(y+z) = (x\blacktriangle y) + (x\blacktriangle z)$
   (A) I only
   (B) II only
   (C) III only
   (D) I and II only
   (E) I, II, and III only

4. What is the result when 436,921 is rounded to the nearest thousand and then expressed in scientific notation?
   (A) 4.369×10^2
   (B) 4.369×10^4
   (C) 4.37×10^4
   (D) 4.37×10^4
   (E) 4.37×10^5

5. If n is an odd integer, which of the following must be an odd integer?
   (A) $n-1$
   (B) $n+1$
   (C) $2n$
   (D) $3n+1$
   (E) $4n+1$

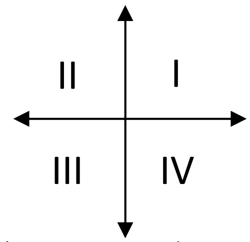
6. If n is an integer and if n^2 is a positive integer, which of the following must also be a positive integer?
   (A) n^2+n
   (B) $2n^2-n$
   (C) n^2-n^3
   (D) n^3+n
   (E) $2n^3+n$

Cartesian Coordinates

Coordinate (Cartesian) Plane: a plane in which an ordered pair can be located by reference to two perpendicular number lies, a horizontal (x-axis) and vertical (y-axis)

Origin: the intersection of the x- and y- axes

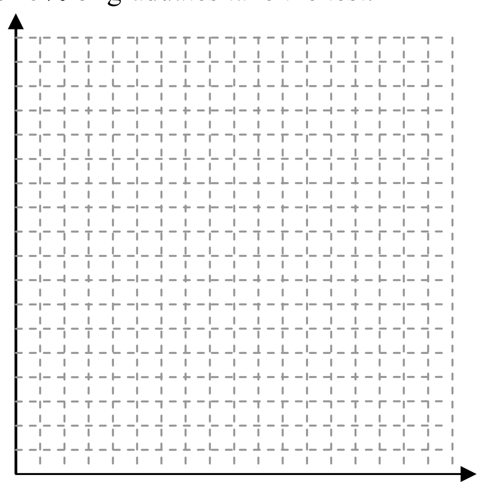
The coordinate axes separate the plane into four quadrants, I – IV.



Ex.1 The table lists the percent of graduates taking the SAT and their average Math score.

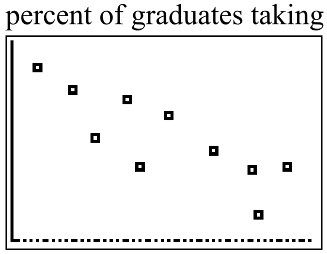
a. State the independent variable.
b. Make a scatter plot for the data.
c. Draw a trend line for the data.
d. What type of association is there?
e. Predict the mean SAT score for a state where 25% of graduates take the test.

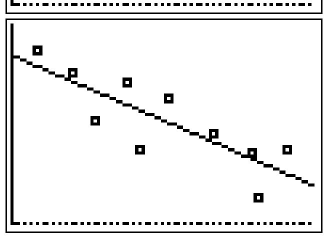
Arizona	22%	520
California	44%	484
Colorado	29%	511
Idaho	16%	501
Nevada	24%	486
New Mexico	12%	524
Oregon	50%	486
Texas	45%	462
Utah	6%	536
Washington	37%	494



Solutions:

a. percent of graduates taking the SAT

b. 

c. 

d. negative correlation
e. 505

Absolute Value: $|a| = \begin{cases} a, & a \ge 0 \\ -a & a < 0 \end{cases}$

Recall: the absolute value of a real number is its distance from the origin on the number line.

Ex.2: Rewrite without an absolute value symbol.

a. $\left|4-\sqrt{7}\right|$ $\quad 4-\sqrt{7} > 0$, so by the definition, $\left|4-\sqrt{7}\right| = \boxed{4-\sqrt{7}}$

b. $\left|4\pi - 17\right|$ $\quad 4\pi - 17 < 0$, so by the definition, $\left|4\pi-17\right| = \boxed{-4\pi+17}$

Number Line Formulas: Distance between *a* and *b*: $|a-b|$ Midpoint: $\dfrac{a+b}{2}$

Ex.3: Find the distance between the points and the midpoint of the line segment that is formed by them.

a. $-3, 5.4$ Distance $= \left|-3-5.4\right| = \left|-8.4\right| = \boxed{8.4}$ Midpoint $= \dfrac{-3+5.4}{2} = \dfrac{2.4}{2} = \boxed{1.2}$

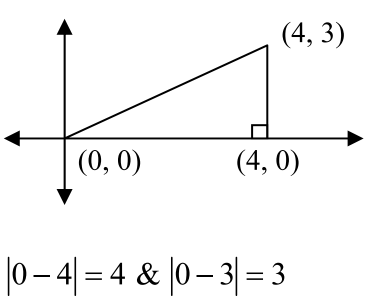
b. $\dfrac{1}{6}, \dfrac{3}{4}$ Distance $= \left|\dfrac{1}{6}-\dfrac{3}{4}\right| = \left|\dfrac{-7}{12}\right| = \boxed{\dfrac{7}{12}}$ Midpoint $= \dfrac{\frac{1}{6}+\frac{3}{4}}{2} = \dfrac{\frac{11}{12}}{2} = \boxed{\dfrac{11}{24}}$

Ex. 4: Write the statement using absolute value notation: The distance between *x* and −5 is less than 4.

$\left|x-(-5)\right| < 4$ or $\boxed{|x+5| < 4}$

Coordinate Plane Formulas:

Distance Formula (derived from the Pythagorean Theorem)

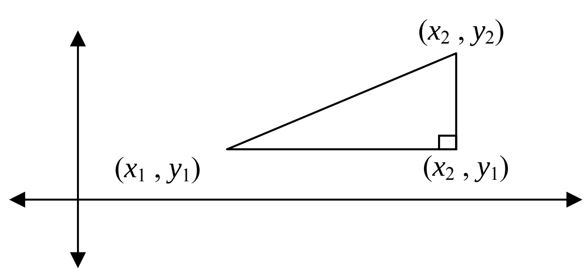


$|0-4| = 4$ & $|0-3| = 3$

$4^2 + 3^2 = c^2$

$c = \sqrt{4^2+3^2}$

$= \sqrt{16+9} = \sqrt{25} = \boxed{5}$



$|x_1 - x_2|$ & $|y_1 - y_2|$

$|x_1 - x_2|^2 + |y_1 - y_2|^2 = d^2$

$\boxed{d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}}$

Ex.5: Determine if the points form an isosceles triangle: $(-2,-3)$, $(2,0)$, $(-5,1)$

Use the distance formula to find the lengths of the three sides:

$$\sqrt{(-2-2)^2+(-3-0)^2}=\sqrt{(-4)^2+(-3)^2}=\sqrt{16+9}=\sqrt{25}=5$$

$$\sqrt{(-2-(-5))^2+(-3-1)^2}=\sqrt{(3)^2+(-4)^2}=\sqrt{9+16}=\sqrt{25}=5$$

$$\sqrt{(-5-2)^2+(1-0)^2}=\sqrt{(-7)^2+(1)^2}=\sqrt{49+1}=\sqrt{50}=5\sqrt{2}$$

Yes – the triangle has two congruent sides, therefore it is isosceles.

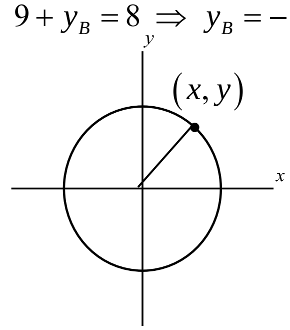
Midpoint Formula: average of the *x*-coordinates, average of the *y*-coordinates

Midpoint of the line segment connecting (x_1,y_1) & (x_2,y_2): $M=\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)$

Ex. 6: The midpoint of $\overline{AB}$ is $M(2,4)$, and the coordinates of *A* are $(-1,9)$. Find *B*.

$$\frac{-1+x_B}{2}=2 \Rightarrow -1+x_B=4 \Rightarrow x_B=5 \qquad \frac{9+y_B}{2}=4 \Rightarrow 9+y_B=8 \Rightarrow y_B=-1$$

$$B=\boxed{(5,-1)}$$



A circle is the locus of points (x,y) equidistant (*r*) from a given point.

Use the distance formula to find the equation of a circle centered at the origin. Then find the equation of a circle centered at (h,k).

$$r=\sqrt{(x-0)^2+(y-0)^2}$$

$$r^2=\left(\sqrt{(x)^2+(y)^2}\right)^2$$

$$x^2+y^2=r^2$$

$$r=\sqrt{(x-h)^2+(y-k)^2}$$

$$r^2=\left(\sqrt{(x-h)^2+(y-k)^2}\right)^2$$

$$(x-h)^2+(y-k)^2=r^2$$

Ex.7: Find the equation of a circle that has $(-1,-2)$ and $(5,3)$ as the endpoints of one of its diameters.

The center will be at the midpoint of the diameter: $(h,k)=\left(\frac{-1+5}{2},\frac{-2+3}{2}\right)=\left(2,\frac{1}{2}\right)$

The radius is equal to half the length of the diameter:

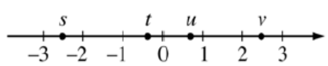
$$d=\sqrt{(-1-5)^2+(-2-3)^2}=\sqrt{36+25}=\sqrt{61} \qquad \text{radius: } \frac{1}{2}d=\frac{\sqrt{61}}{2}$$

Equation of the circle: $(x-2)^2+\left(y-\frac{1}{2}\right)^2=\left(\frac{\sqrt{61}}{2}\right)^2 \Rightarrow \boxed{(x-2)^2+\left(y-\frac{1}{2}\right)^2=\frac{61}{4}}$

<u>You Try</u>: Find the center and **diameter** of the circle. $(x-9)^2+y^2=24$

<u>QOD</u>: Write the standard equation of a circle, label the variables, and describe how it is derived.

<u>Sample SAT Question(s):</u> Taken from College Board online practice problems.



1. If *s*, *t*, *u*, and *v* are the coordinates of the indication points on the number line above, which of the following is greatest?
   (A) $|s+t|$
   (B) $|s+v|$
   (C) $|s-t|$
   (D) $|s-v|$
   (E) $|s+u|$

2. At a snack bar, a customer who orders a small soda gets a cup containing c ounces of soda, where c is at least 12 but no more than $12\frac{1}{2}$. Which of the following describes all possible values of c?

   (A) $\left|12\frac{1}{2}-c\right| \le \frac{1}{2}$

   (B) $\left|c-12\right| \le \frac{1}{2}$

   (C) $\left|c-12\right| \le \frac{1}{4}$

   (D) $\left|c-12\frac{1}{4}\right| \le \frac{1}{2}$

   (E) $\left|c-12\frac{1}{4}\right| \le \frac{1}{4}$

Linear Equations and Inequalities

Linear Equation: an equation that can be written in the form $ax+b=0,\ a\neq 0$

Equations are equivalent if they have the same solutions.

Ex.1: Show that $z=2$ is **not** a solution of $2(4z-3)-3(z+1)=3z-1$. Then find the solution.

Substitute $z=2$ into the equation:

$$2(4(2)-3)-3((2)+1)\stackrel{?}{=}3(2)-1$$

$$2(5)-3(3)\stackrel{?}{=}5$$

$$10-9\neq 5 \Rightarrow 1\neq 5$$

Therefore, $z=2$ is **not** a solution.

Solve for z: $8z-6-3z-3=3z-1 \Rightarrow 5z-9=3z-1 \Rightarrow 2z=8 \Rightarrow \boxed{z=4}$

Ex.2: Solve the equation for *t*. $\dfrac{t-1}{4}+3(t+5)=\dfrac{1}{6}$

Wipe out the fractions by multiplying by the least common denominator, in this case, 12.

$$12\cdot\frac{t-1}{4}+12\cdot 3(t+5)=12\cdot\frac{1}{6} \Rightarrow 3t-3+36t+180=2 \Rightarrow 39t=-175 \Rightarrow \boxed{t=-\frac{175}{39}}$$

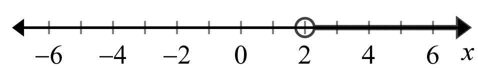
Solving a Linear Inequality:

Ex.3: Solve and graph the solution on the number line. Express the solution in interval notation.
$-5x+8<2(x-3)$

$-5x+8<2x-6 \Rightarrow -7x<-14 \Rightarrow x>2$ or $-5x+8<2x-6 \Rightarrow 14<7x \Rightarrow 2>x$

Note: When multiplying or dividing by a negative number, you must flip the inequality sign.

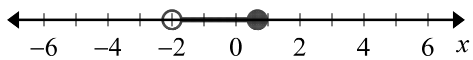
Interval notation: $\boxed{(2,\infty)}$



Solving a Compound Inequality:

Ex.4: Solve and graph. Express in interval notation. $3 \le -\frac{3}{2}x + 4 < 7$

$-1 \le -\frac{3}{2}x < 3 \Rightarrow \frac{2}{3} \ge x > -2 \Rightarrow -2 < x \le \frac{2}{3}$ Interval notation: $\boxed{\left(-2, \frac{2}{3}\right]}$



Ex.5: Find the volume of material that makes up the Earth's crust, which is ten miles thick.

(Earth's radius ≈ 3960 miles) Volume of a sphere: $V = \frac{4}{3}\pi r^3$

Volume of the crust = volume of the earth minus volume of the earth not including the crust

$$V = \frac{4}{3}\pi(3960)^3 - \frac{4}{3}\pi(3950)^3 \approx \boxed{1{,}965{,}635{,}880 \text{ cubic miles}}$$

You Try: Solve the inequality and graph the solution on a number line. $\frac{x}{2} - 1 > \frac{2}{3}x - 3$

QOD: Explain why the inequality sign must be flipped when multiplying or dividing by a negative number.

Sample SAT Question(s): Taken from College Board online practice problems.

1. If $x + k = 12$ and $p(x+k) = 36$, what is the value of *p*?

   (A) 3
   (B) 4
   (C) 6
   (D) 9
   (E) 12

2. If $3 < 3t - 6 < 18$, which of the following must be true?

   (A) $t = 5$
   (B) $5 < t < 6$
   (C) $4 < t + 1 < 9$
   (D) $12 < 3t < 24$
   (E) $t < 3$ or $t > 8$

Lines in the Plane

Slope: rate of change; steepness of a line

$$m = \frac{\text{rise}}{\text{run}} = \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x} = \frac{y_1 - y_2}{x_1 - x_2} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{dy}{dx}$$

Ex.1: Find *y* so that $m = 2$ on a line through the points $(5,-8)$ and $(3, y)$.

$$\frac{y-(-8)}{3-5} = 2 \Rightarrow \frac{y+8}{-2} = 2 \Rightarrow y+8 = -4 \Rightarrow \boxed{y = -12}$$

Equations of a Line

- Point-Slope Form — A line with slope *m* that passes through the point (x_1, y_1)

  $y - y_1 = m(x - x_1)$
- Slope-Intercept Form — A line with slope *m* and *y*-intercept *b*

  $y = mx + b$
- Standard Form — *A* & *B* are not both 0; *x*-intercept = $\frac{C}{A}$, *y*-intercept = $\frac{C}{B}$

  $Ax + By = C$

Ex.2: Write an equation of the line that passes through the points $(-1,3)$ and $(-5,4)$ in standard form.

Find the slope: $m = \frac{3-4}{-1-(-5)} = \frac{-1}{4}$

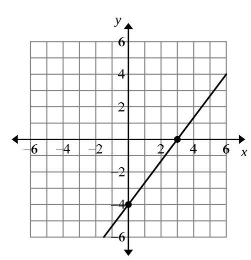
Use the slope and one of the given points to write the equation in point-slope form: $y - 3 = -\frac{1}{4}(x+1)$

Multiply by the LCD and write in standard form:

$$-4(y-3) = x+1 \Rightarrow -4y+12 = x+1 \Rightarrow -x-4y = -11 \Rightarrow \boxed{x+4y=11}$$

Ex.3: Graph the line $4x - 3y = 12$.

Find the *x*- and *y*-intercepts: *x*-int. $4x = 12$, $x = 3$ $(3,0)$ *y*-int. $-3y = 12$, $y = -4$ $(0,-4)$



<u>Slope of Horizontal Lines</u>: $m = 0$ <u>Slope of Vertical Lines</u>: m is undefined

Ex.4: Write an equation of the line that passes through the points $(2,4)$ & $(2,1)$.

$m = \frac{4-1}{2-2} = \frac{3}{0}$ undefined Equation of the vertical line: $\boxed{x = 2}$

<u>Parallel Lines</u> $(\|)$: lines in the same plane that do not intersect; Parallel lines have the **same** slope.

<u>Perpendicular Lines</u> $(\perp)$: lines in the same plane that intersect at a right angle; Perpendicular lines have **opposite reciprocal** slopes.

Ex.5: Write an equation of the line parallel to $4x + 3y = 24$ that passes through the point $(0,-2)$.

Find the slope of the given line: $3y = -4x + 24 \Rightarrow y = -\frac{4}{3}x + 8$ $m = -\frac{4}{3}$

Parallel lines have the same slope, so use this slope and the given point to write the equation in point-slope form.

$$y + 2 = -\frac{4}{3}(x - 0) \Rightarrow y = -\frac{4}{3}x - 2 \Rightarrow \boxed{4x + 3y = -6}$$

<u>Rate of Change</u>

Ex.6: A 5-minute phone call costs \$0.38, and a 9-minute call costs \$0.62. What is the rate of change? Find a linear function to represent the total cost (*C*) of a call to the duration in minutes (*m*). Then use this function to find the cost of a call that lasts 1 hour and 15 minutes.

$(m, C): (5, 0.38) \& (9, 0.62)$ Rate of Change $= \frac{0.62 - 0.38}{9 - 5} = \frac{0.24}{4} = \boxed{\$0.06 \text{ per minute}}$

Linear function: $C - 0.38 = 0.06(m - 5)$ or $\boxed{C = 0.06m + 0.08}$

1 hour and 15 minutes: $m = 75$ $C(75) = 0.06(75) + 0.08 = \boxed{\$4.58}$

<u>You Try</u>: Write an equation of the line perpendicular to the line that passes through the points $(-5,2)$ & $(3,-1)$ in point-slope form.

<u>QOD</u>: Explain why the slope of a horizontal line is zero and the slope of a vertical line is undefined graphically and using the definition of slope.

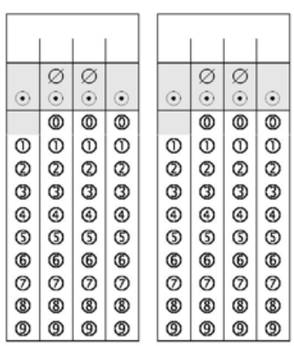
Sample SAT Question(s): Taken from College Board online practice problems.

x	y
1	7.5
2	13.0
3	18.5
4	24.0

1. Which of the following equations expresses y in terms of x for each of the four pairs of values shown in the table above?
   (A) $y = 5x + 7.5$
   (B) $y = 5.5x + 2$
   (C) $y = 5.5x + 7.5$
   (D) $y = 7.5x$
   (E) $y = 7.5x + 5.5$

2. In the xy-coordinate plane, how many points are a distance of 4 units from the origin?
   (A) One
   (B) Two
   (C) Three
   (D) Four
   (E) More than four

Grid-Ins



1. The price of a certain item was \$10 in 1990 and it has gone up by \$2 per year since 1990. If this trend continues, in what year will the price be \$100?

$$tx + 12y = -3$$

2. The equation above is the equation of a line in the xy-plane, and t is a constant. If the slope of the line is -10, what is the value of t?

Solving Equations

Solving Absolute Value Equations

Ex.1: Find the solution set for the equation. $3|2x+1|-8=16$

Isolate the absolute value expression. $3|2x+1|=24 \Rightarrow |2x+1|=8$

There are two values that have an absolute value of 8, 8 and −8. So $2x+1=8$ or $2x+1=-8$.

Solve each equation for *x*. $2x+1=8$, $x=\frac{7}{2}$ $\quad$ $2x+1=-8$, $x=-\frac{9}{2}$ $\quad$ Solutions: $\boxed{\left\{-\frac{9}{2},\frac{7}{2}\right\}}$

Quadratic Equation: an equation of the form $ax^2+bx+c=0$, with $a\neq 0$

Methods for Solving a Quadratic Equation

- Square Roots

  Ex.2: Solve the equation $2(x+1)^2=5$ by extracting square roots.

Isolate the expression that is squared: $(x+1)^2=\frac{5}{2}$

Square root both sides of the equation: $\sqrt{(x+1)^2}=\sqrt{\frac{5}{2}} \Rightarrow |x+1|=\sqrt{\frac{5}{2}}$ $\quad$ **Reminder**: $\sqrt{x^2}=|x|$

Solve for *x*:

$$x+1=\sqrt{\frac{5}{2}},\ \ x+1=-\sqrt{\frac{5}{2}}$$

$$x=-1+\sqrt{\frac{5}{2}},\ -1-\sqrt{\frac{5}{2}}$$

Set Notation: $\left\{-1-\sqrt{\frac{5}{2}},-1+\sqrt{\frac{5}{2}}\right\}$ or $\boxed{\left\{-1-\frac{\sqrt{10}}{2},-1+\frac{\sqrt{10}}{2}\right\}}$ (rationalized)

- Completing the Square Note: *a* must equal 1 in order to complete the square.

  Ex.3: Solve the equation $x^2-8x+1=0$ by completing the square.

Move *c* to the other side of the equation: $x^2-8x\quad=-1$

Complete the square by adding $\boxed{\left(\frac{b}{2}\right)^2}$ to both sides: $x^2-8x+\boxed{\left(\frac{-8}{2}\right)^2}=-1+\boxed{\left(\frac{-8}{2}\right)^2}$

$$x^2-8x+16=-1+16$$

Rewrite the perfect square trinomial as a binomial squared: $(x-4)^2=15$

Solve by square roots:

$$\sqrt{(x-4)^2}=\sqrt{15} \Rightarrow |x-4|=\sqrt{15}$$

$$x-4=\sqrt{15},\ x-4=-\sqrt{15}$$

$x=4\pm\sqrt{15}$ or $\boxed{\left\{4-\sqrt{15},4+\sqrt{15}\right\}}$

- <u>Quadratic Formula</u> $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
  - Discriminant: $b^2 - 4ac$
    - $b^2 - 4ac > 0 \Leftrightarrow$ two real solutions
    - $b^2 - 4ac < 0 \Leftrightarrow$ no real solutions (2 complex solutions)
    - $b^2 - 4ac = 0 \Leftrightarrow$ one real solution

Ex.4: Solve the equation using the quadratic formula. $x^2 = 8x - 6$

Rewrite in standard form and determine *a*, *b*, and *c*. $x^2 - 8x + 6 = 0$ $a = 1,\ b = -8, c = 6$

Find the discriminant to determine the number of solutions.

$b^2 - 4ac = (-8)^2 - 4(1)(6) = 64 - 24 = 40 > 0$ two real solutions

Use the quadratic formula to solve. $x = \dfrac{-(-8) \pm \sqrt{40}}{2(1)} = \dfrac{8 \pm 2\sqrt{10}}{2} = 4 \pm \sqrt{10}$

Solution set: $\boxed{\left\{4 - \sqrt{10},\ 4 + \sqrt{10}\right\}}$

- <u>Factoring</u>: use the zero product property

Ex.5: Solve the equation by factoring. $3x^2 - 20x = 7$

Set the equation equal to zero. $3x^2 - 20x - 7 = 0$

Factor the quadratic. (*ac* method is shown) $ac = -21 = -21 \cdot 1$ $b = -21 + 1 = -20$

Split the middle term: $3x^2 - 21x + x - 7 = 0$

Factor by grouping:

$3x(x - 7) + 1(x - 7) = 0$

$(3x + 1)(x - 7) = 0$

$3x + 1 = 0$ $x - 7 = 0$

Set each factor equal to zero and solve: $\boxed{x = -\frac{1}{3}, 7}$

Solving Rational Equations

Ex.5: Find the value(s) of x that make the equation true. $\frac{3}{x-7}+1=\frac{8}{x^2-9x+14}$

Factor and find the LCD. $\frac{3}{x-7}+1=\frac{8}{(x-7)(x-2)}$ LCD: $(x-7)(x-2)$

Multiply each term by the LCD.

$$\cancel{(x-7)}(x-2)\cdot\frac{3}{\cancel{(x-7)}}+(x-7)(x-2)\cdot 1=\cancel{(x-7)}\cancel{(x-2)}\cdot\frac{8}{\cancel{(x-7)}\cancel{(x-2)}}$$

$$3(x-2)+(x-7)(x-2)=8$$

Simplify and solve the resulting equation.

$3x-6+x^2-9x+14=8 \Rightarrow x^2-6x=0$

$x(x-6)=0 \Rightarrow \boxed{x=0,6}$

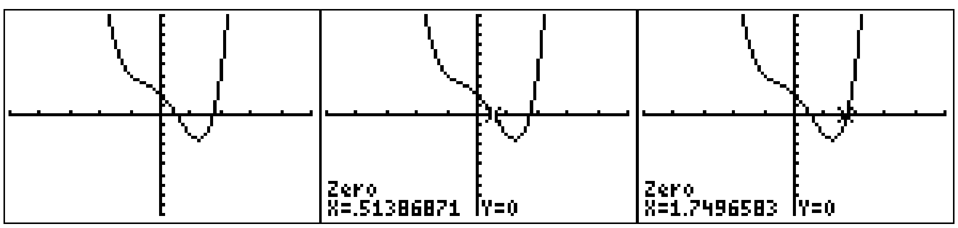
Solving an Equation Graphically

Method 1: Find the zeros (roots).

Ex.6: Solve the equation graphically. $x^4-2x^2=3x-2$

Set the equation equal to zero. $x^4-2x^2-3x+2=0$

Graph the function and find the zero(s). $y=x^4-2x^2-3x+2$

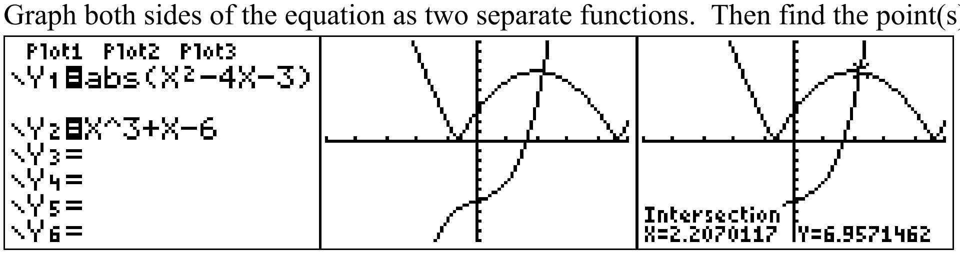


Solutions: $\boxed{\{0.513, 1.750\}}$

Method 2: Finding the point(s) of intersection.

Ex.7: Solve the equation graphically. $|x^2-4x-3|=x^3+x-6$

Graph both sides of the equation as two separate functions. Then find the point(s) of intersection.



Solution: $\boxed{x\approx 2.207}$

Think About It: What does the Y-value represent in the point of intersection?

You Try: Solve for x in the equation $ax^2 + bx + c = 0$ by completing the square.

QOD: True or False: An absolute value equation always has two solutions. Explain your answer.

Sample SAT Question(s): Taken from College Board online practice problems.

1. If $a \neq 0$ and $\frac{5}{x} = \frac{5+a}{x+a}$, what is the value of x?
   (A) −5
   (B) −1
   (C) 1
   (D) 2
   (E) 5

2. If $x^2 = x + 6$, which of the following must be true?
   (A) $x = 6$
   (B) $x < 3$
   (C) $x > 0$
   (D) $x^2 < x$
   (E) $x^2 > x$

Solving Inequalities

Solving an Absolute Value Inequality

If $|x| < a$, then $-a < x < a$. Or *x* is between –*a* and *a*.

If $|x| > a$, then $x < -a$ or $x > a$.

Teacher Note: Remind students that an absolute value is a distance from the origin. This should make the inequalities above make sense.

Ex.1: Solve the inequality. $|2x-4| < 12$

Since the absolute value is less than 12, the value of the expression inside the absolute value must be between -12 and 12. $-12 < 2x - 4 < 12$

Solve the compound inequality.

$$-8 < 2x < 16$$

$$\boxed{-4 < x < 8}$$

*Have students pick a value within the interval to verify that it is a solution to the original inequality.

Ex.2: Solve the inequality and verify your solution graphically. $|2x-3| \geq 7$

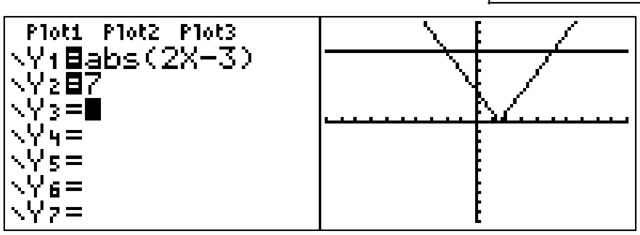
Since the absolute value is greater than or equal to 7, the value of the expression inside the absolute value must be less than or equal to -7 or greater than or equal to 7. $2x-3 \leq -7$ or $2x-3 \geq 7$

Solve the compound inequality.

$$2x-3 \leq -7 \quad \text{or} \quad 2x-3 \geq 7$$

$$2x \leq -4 \qquad 2x \geq 10$$

$$\boxed{x \leq -2 \quad \text{or} \quad x \geq 5}$$



We can see that the graph of $y = |2x-3|$ is greater than or equal to the graph of $y = 7$ when $x \leq 2$ or $x \geq 5$.

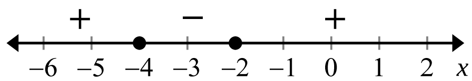
<u>Solving Quadratic Inequalities</u>

- Algebraic Method (sign chart)

  Ex.3: Solve the inequality. $x^2+6x+8>0$

Solve the equation to find the zeros. $x^2+6x+8=0 \Rightarrow (x+4)(x+2)=0 \Rightarrow x=-4,-2$

Make a sign chart using test values between and outside of the zeros.

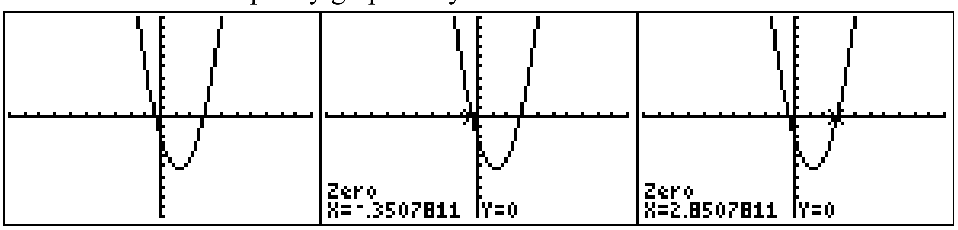


The solutions to the inequality are the *x*-values that make the expression positive (greater than zero).

Set notation: $\boxed{\{x\in\mathbb{R}\,|\,x\le -4 \text{ or } x\ge -2\}}$

- Graphing Method

Ex.4: Solve the inequality graphically. $2x^2-5x-2<0$



Because the graph is below the *x*-axis $(y<0)$ between $x=-0.351$ and $x=2.851$, the solution, written in interval notation is: $\boxed{(-0.351, 2.851)}$

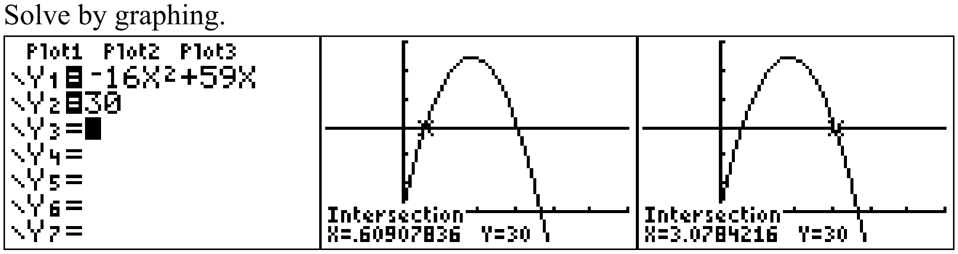
<u>Projectile Motion:</u> When an object is launched vertically from an initial height of s_0 feet and an initial velocity of v_0 feet per second, then the vertical position *s* of the object *t* seconds after it is launched is $\boxed{s=-16t^2+v_0t+s_0}$.

Ex.5: A ball is thrown straight up from ground level with an initial velocity of 59 ft/sec. When will the ball's height above the ground be more than 30 ft?

Write the equation of the height of the ball. $s=-16t^2+59t$

Write an inequality to model the question. $-16t^2+59t>30$

Solve by graphing.



The ball's height will be more than 30 ft when $0.609<t<3.078$ seconds.

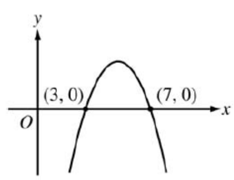
<u>You Try:</u> Solve the quadratic inequality algebraically. Then verify your answer graphically.

$2x^2 - 7x - 30 \geq 0$

<u>QOD:</u> Can a quadratic and/or absolute value inequality have no solutions or one solution? Explain your answer with an example.

<u>Sample SAT Question(s):</u> Taken from College Board online practice problems.

1. If $y = 2x + 3$ and $x < 2$, which of the following represents all the possible values for *y*?
   (A) $y < 7$
   (B) $y > 7$
   (C) $y < 5$
   (D) $y > 5$
   (E) $5 < y < 7$



2. The figure above shows the graph of a quadratic function in the *xy*-plane. Of all the points (x, y) on the graph, for what value of *x* is the value of *y* greatest?

Grid-In