Topic B:  
Decimal Expansions of Numbers

8.NS.A.1, 8.NS.A.2, 8.EE.A.2

Focus Standard:  
8.NS.A.1 Know that numbers that are not rational are called irrational. Understand informally that every number has a decimal expansion; for rational numbers show that the decimal expansion repeats eventually, and convert a decimal expansion which repeats eventually into a rational number.

8.NS.A.2 Use rational approximations of irrational numbers to compare the size of irrational numbers, locate them approximately on a number line diagram, and estimate the value of expressions (e.g., \(\pi^2\)). For example, by truncating the decimal expansion of \(\sqrt{2}\), show that \(\sqrt{2}\) is between 1 and 2, then between 1.4 and 1.5, and explain how to continue to get better approximations.

8.EE.A.2 Use square root and cube root symbols to represent solutions to equations of the form \(x^2 = p\) and \(x^3 = p\), where \(p\) is a positive rational number. Evaluate square roots of small perfect squares and cube roots of small perfect cubes. Know that \(\sqrt{2}\) is irrational.

Instructional Days:  
9

Lesson 6: Finite and Infinite Decimals (P)\(^1\)
Lesson 7: Infinite Decimals (S)
Lesson 8: The Long Division Algorithm (E)
Lesson 9: Decimal Expansions of Fractions, Part 1 (P)
Lesson 10: Converting Repeating Decimals to Fractions (P)
Lesson 11: The Decimal Expansion of Some Irrational Numbers (S)
Lesson 12: Decimal Expansion of Fractions, Part 2 (S)
Lesson 13: Comparing Irrational Numbers (E)
Lesson 14: Decimal Expansion of \(\pi\) (S)

\(^1\) Lesson Structure Key: P-Problem Set Lesson, M-Modeling Cycle Lesson, E-Exploration Lesson, S-Socratic Lesson

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Throughout this topic the terms expanded form of a decimal and decimal expansion are used. The expanded form of a decimal refers to the value of a number written as a sum. For example, the expanded form of the decimal 0.125 is $\frac{1}{10} + \frac{2}{10^2} + \frac{5}{10^3}$, which is closely related to the notion of expanded form used at the elementary level. When students are asked to determine the decimal expansion of a number such as $\sqrt{2}$, we expect them to write the number in decimal form. For example, the decimal expansion of $\sqrt{2}$ begins with 1.4142. The examination of the decimal expansion leads to an understanding of irrational numbers.

Numbers with decimal expansions that are infinite, i.e., non-terminating, that do not have a repeating block are called irrational numbers. Numbers with finite, i.e., terminating, decimal expansions, as well as those numbers that are infinite with repeating blocks, are called rational numbers. Students spend significant time engaging with finite and infinite decimals before the notion of an irrational number is introduced in Lesson 11.

In Lesson 6, students learn that every number has a decimal expansion that is finite or infinite. Finite and infinite decimals are defined and students learn a strategy for writing a fraction as a finite decimal that focuses on the denominator and its factors. That is, a fraction can be written as a finite decimal if the denominator is a product of twos or fives. In Lesson 7, students learn that numbers that cannot be expressed as finite decimals are infinite decimals. Students write the expanded form of infinite decimals and show on the number line their decimal representation in terms of intervals of tenths, hundredths, thousandths, and so on. This work with infinite decimals prepares students for understanding how to approximate the decimal expansion of an irrational number. In Lesson 8, students use the long division algorithm to determine the decimal form of a number and can relate the work of the algorithm to why digits in a decimal expansion repeat. It is in these first few lessons of Topic B that students recognize that rational numbers have a decimal expansion that repeats eventually, either in zeros or a repeating block of digits. The discussion of infinite decimals continues with Lesson 9 where students learn how to use what they know about powers of ten and equivalent fractions to make sense of why the long division algorithm can be used to convert a fraction to a decimal. Students know that multiplying the numerator and denominator of a fraction by a power of ten is similar to putting zeros after the decimal point when doing long division.

In Lesson 10, students learn that a number with a decimal expansion that repeats can be expressed as a fraction. Students learn a strategy for writing repeating decimals as fractions that relies on their knowledge of multiplying by powers of 10 and solving linear equations. Lesson 11 introduces students to the method of rational approximation using a series of rational numbers to get closer and closer to a given number. Students write the approximate decimal expansion of irrational numbers in Lesson 11, and it is in this lesson that irrational numbers are defined as numbers that are not equal to rational numbers. Students realize that irrational numbers are different because they have infinite decimal expansions that do not repeat. Therefore, irrational numbers are those that are not equal to rational numbers. Rational approximation is used again in Lesson 12 to verify the decimal expansions of rational numbers. Students then compare the method of rational approximation to long division. In Lesson 13, students compare the value of rational and irrational numbers. Students use the method of rational approximation to determine the decimal expansion of an irrational number, and then compare that value to the decimal expansion of rational numbers in the form of a fraction, decimal, perfect square, or perfect cube. Students can now place irrational numbers on a number line with more accuracy than they did in Lesson 2. In Lesson 14, students approximate $\pi$ using the area of a quarter circle that is drawn on grid paper. Students estimate the area of the quarter circle using inner and outer boundaries. As with the method of rational approximation, students continue to refine their estimates of the area which improves their estimate of the value of $\pi$. Students then determine the approximate values of expressions involving $\pi$. 
Lesson 6: Finite and Infinite Decimals

Student Outcomes

- Students know that every number has a decimal expansion (i.e., is equal to a finite or infinite decimal).
- Students know that when a fraction has a denominator that is the product of 2’s and/or 5’s, it has a finite decimal expansion because the fraction can then be written in an equivalent form with a denominator that is a power of 10.

Lesson Notes

The terms expanded form of a decimal and decimal expansion are used throughout this topic. The expanded form of a decimal refers to the value of a number written as a sum. For example, the expanded form of the decimal 0.125 is $\frac{1}{10} + \frac{2}{100} + \frac{5}{1000}$. When students are asked to determine the decimal expansion of a number like $\sqrt{2}$ we expect them to write the decimal value of the number. For example, the decimal expansion of $\sqrt{2}$ is approximately 1.4142. The examination of the decimal expansion leads to an understanding of irrational numbers. Numbers with decimal expansions that are infinite (i.e., non-terminating) and do not have a repeat block are called irrational numbers. Numbers with finite (i.e., terminating) decimal expansions, as well as those numbers that are infinite with repeat blocks, are called rational numbers. Students will be exposed to the concepts of finite and infinite decimals here; however, the concept of irrational numbers will not be formally introduced until Lesson 11.

Classwork

Opening Exercises 1–5 (7 minutes)

Provide students time to work, then share their responses to Exercise 5 with the class.

Opening Exercises 1–5

1. Use long division to determine the decimal expansion of $\frac{54}{20}$

   The number $\frac{54}{20} = 2.7$.

2. Use long division to determine the decimal expansion of $\frac{7}{8}$

   The number $\frac{7}{8} = 0.875$.

3. Use long division to determine the decimal expansion of $\frac{8}{9}$

   The number $\frac{8}{9} = 0.8888 ...$

4. Use long division to determine the decimal expansion of $\frac{22}{7}$

   The number $\frac{22}{7} = 3.142857 ...$
5. What do you notice about the decimal expansions of Exercises 1 and 2 compared to the decimal expansions of Exercises 3 and 4?

The decimal expansions of Exercises 1 and 2 ended. That is, when I did the long division I was able to stop after a few steps. That was different than the work I had to do in Exercises 3 and 4. In Exercise 3, I noticed that the same number kept coming up in the steps of the division, but it kept going on. In Exercise 4, when I did the long division it did not end. I stopped dividing after I found a few decimal digits of the decimal expansion.

Discussion (5 minutes)

Use the discussion below to elicit a dialog about finite and infinite decimals that may not have come up in the debrief of the Opening Exercises and to prepare students for what is covered in this lesson in particular (i.e., writing fractions as finite decimals without using long division).

- Every number has a decimal expansion. That is, every number is equal to a decimal. For example, the numbers \( \sqrt{3} \) and \( \frac{17}{125} \) have decimal expansions. The decimal expansion of \( \sqrt{3} \) will be covered in a later lesson. For now, we will focus on the decimal expansion of a number like \( \frac{17}{125} \) and whether it can be expressed as a finite or infinite decimal.

- How would you classify the decimal expansions of Exercises 1–4?
  - Exercises 1 and 2 are finite decimals and Exercises 3 and 4 are infinite decimals.

- In the context of fractions, a decimal is, by definition, a fraction with a denominator equal to a power of 10. These decimals are known as finite decimals. The distinction must be made because we will soon be working with infinite decimals. Can you think of any numbers that are infinite decimals?
  - Decimals that repeat or a number like \( \pi \) are infinite decimals.

- Decimals that repeat, such as 0.8888888 \ldots \) or 0.4545454545 \ldots \), are infinite decimals and typically abbreviated as 0.8 and 0.\( \overline{45} \), respectively. The notation indicates that the digit 8 repeats indefinitely and that the two-digit block 45 repeats indefinitely. The number \( \pi \) is also a famous infinite decimal: 3.1415926535 \ldots \), which does not have a block of digits that repeats indefinitely.

- In Grade 7 you learned a general procedure for writing the decimal expansion of a fraction such as \( \frac{5}{14} \) using long division. In the next lesson, we will closely examine the long division algorithm and why the procedure makes sense.

- Today, we will learn a method for converting a fraction to a decimal that does not require long division. Each of the fractions in the Examples and Exercises in this lesson are simplified fractions. The method we will learn requires that we begin with a simplified fraction.

- Return to the Opening Exercise. We know that the decimals in Exercises 1 and 2 are finite, while the decimals in Exercises 3 and 4 are not. What do you notice about the denominators of these fractions that might explain this?
  - The denominators of the fractions in Exercises 1 and 2 are the products of 2’s and 5’s. For example, the denominator 20 = \( 2 \times 2 \times 5 \) and the denominator 8 = \( 2 \times 2 \times 2 \). The denominators of the fractions in Exercises 3 and 4 were not the product of 2’s and 5’s. For example, 9 = \( 3 \times 3 \) and 7 = \( 1 \times 7 \).
Certain fractions, those whose denominators are a product of 2’s or 5’s or both, are equal to finite decimals. Fractions like \( \frac{1}{4} \), \( \frac{6}{125} \), and \( \frac{9}{10} \) can be expressed as finite decimals because \( 4 = 2^2 \), \( 125 = 5^3 \), and \( 10 = 2 \times 5 \).

Other fractions like \( \frac{5}{14} \) cannot be expressed as a finite decimal because \( 14 = 2 \times 7 \). Therefore, \( \frac{5}{14} \) has an infinite decimal expansion.

**Example 1 (4 minutes)**

Consider the fraction \( \frac{5}{8} \). Is it equal to a finite decimal? How do you know?

Consider the fraction \( \frac{5}{8} \). Is it equal to a finite decimal? How do you know?

- The fraction \( \frac{5}{8} \) is equal to a finite decimal because the denominator 8 is a product of 2’s. Specifically, \( 8 = 2^3 \).

Since we know that the fraction \( \frac{5}{8} \) is equal to a finite decimal, then we can find a fraction \( \frac{k}{10^n} \) where \( k \) and \( n \) are positive integers, that will give us the decimal value that \( \frac{5}{8} \) is equal to.

We must find positive integers \( k \) and \( n \), so that \( \frac{5}{8} = \frac{k}{10^n} \).

Explain the meaning of \( k \) and \( 10^n \) in the equation above.

- The number \( k \) will be the numerator, a positive integer, of a fraction equivalent to \( \frac{5}{8} \) that has a denominator that is a power of 10, e.g., \( 10^2 \), \( 10^5 \), \( 10^n \).

Recall what we learned about the laws of exponents in Module 1: \( (ab)^n = a^n b^n \). We will now put that knowledge to use.

We know that \( 8 = 2^3 \) and \( 10^n = (2 \times 5)^n = 2^n \times 5^n \). Comparing the denominators of the fractions, \( 2^3 \times 5^n = 2^n \times 5^n = 10^n \).

What must \( n \) equal?

- \( n \) must be 3.

To rewrite the fraction \( \frac{5}{8} \) so that it has a denominator of the form \( 10^n \), we must multiply \( 2^3 \) by \( 5^3 \). Based on what you know about equivalent fractions, by what must we multiply the numerator of \( \frac{5}{8} \)?

- To make an equivalent fraction we will need to multiply the numerator by \( 5^3 \) also.

By equivalent fractions:

\[
\frac{5}{8} = \frac{5 \times 5^3}{2^3 \times 5^3} = \frac{5^4}{(2 \times 5)^3} = \frac{625}{10^3}.
\]

where \( k = 625 \) and \( n = 3 \), both positive integers.

Using the fraction \( \frac{625}{10^3} \), we can write the decimal value of \( \frac{5}{8} \). What is it? Explain.

- \( \frac{5}{8} = 0.625 \) because \( \frac{625}{10^3} = \frac{625}{1000} \). Using what we know about place value we have six hundred twenty five thousandths, or 0.625.
Example 2 (4 minutes)

Example 2
Consider the fraction \( \frac{17}{125} \). Is it equal to a finite or infinite decimal? How do you know?

- Let’s consider the fraction \( \frac{17}{125} \) mentioned earlier. We want the decimal value of this number. Is it a finite or infinite decimal? How do you know?
  - We know that the fraction \( \frac{17}{125} \) is equal to a finite decimal because the denominator 125 is a product of 5’s. Specifically, \( 125 = 5^3 \).
- What will we need to multiply \( 5^3 \) by so that it is equal to \((2 \times 5)^n = 10^n\)?
  - We will need to multiply by \( 2^3 \) so that \( 2^3 \times 5^3 = (2 \times 5)^3 = 10^3 \).
- Begin with \( \frac{17}{125} = \frac{17}{5^3} \). Use what you know about equivalent fractions to rewrite \( \frac{17}{125} = \frac{k}{10^n} \), and then the decimal form of the fraction.
  - \( \frac{17}{125} = \frac{17}{5^3} = \frac{17 \times 2^3}{2^3 \times 5^3} = \frac{17 \times 8}{2 \times 5^3} = \frac{136}{10^3} = 0.136 \)

Exercises 6–10 (5 minutes)

Students complete Exercises 6–10 independently.

Exercises 6–10
Show your steps, but use a calculator for the multiplications.

6. Convert the fraction \( \frac{7}{8} \) to a decimal.
   a. Write the denominator as a product of 2’s or 5’s. Explain why this way of rewriting the denominator helps to find the decimal representation of \( \frac{7}{8} \).

   The denominator \( 8 = 2^3 \). It is helpful to know that \( 8 = 2^3 \) because it shows how many factors of 5 will be needed to multiply the numerator and denominator by so that an equivalent fraction is produced with a denominator that is a multiple of 10. When the denominator is a multiple of 10 the fraction can easily be written as a decimal using what I know about place value.

   b. Find the decimal representation of \( \frac{7}{8} \). Explain why your answer is reasonable.

   \[
   \frac{7}{8} = \frac{7}{2^3} = \frac{7 \times 5^3}{2^3 \times 5^3} = \frac{875}{10^3} = 0.875
   \]

   The answer is reasonable because the decimal value, 0.875 is less than one just like the fraction \( \frac{7}{8} \). Also, it is reasonable and correct because the fraction \( \frac{875}{1000} = \frac{7}{8} \), therefore, it has the decimal expansion 0.875.
7. Convert the fraction $\frac{43}{64}$ to a decimal.

The denominator $64 = 2^6$.

$$\frac{43}{64} = \frac{43 \times 5^6}{2^6 \times 5^6} = \frac{671875}{10^6} = 0.671875$$

8. Convert the fraction $\frac{29}{125}$ to a decimal.

The denominator $125 = 5^3$.

$$\frac{29}{125} = \frac{29 \times 2^3}{5^3 \times 2^3} = \frac{232}{10^3} = 0.232$$

9. Convert the fraction $\frac{19}{34}$ to a decimal.

Using long division, $\frac{19}{34} = 0.5588235...$

10. Identify the type of decimal expansion for each of the numbers in Exercises 6–9 as finite or infinite. Explain why their decimal expansion is such.

We know that the number $\frac{7}{8}$ had a finite decimal expansion because the denominator $8$ is a product of $2$’s. We know that the number $\frac{43}{64}$ had a finite decimal expansion because the denominator $64$ is a product of $2$’s. We know that the number $\frac{29}{125}$ had a finite decimal expansion because the denominator $125$ is a product of $5$’s. We know that the number $\frac{19}{34}$ had an infinite decimal expansion because the denominator was not a product of $2$’s or $5$’s, it had a factor of $17$.

Example 3 (4 minutes)

Example 3

Write $\frac{7}{80}$ as a decimal. Will it be finite or infinite? Explain.

- Let’s write $\frac{7}{80}$ as a decimal. Will it be finite or infinite? Explain.
  - We know that the fraction $\frac{7}{80}$ is equal to a finite decimal because the denominator $80$ is a product of $2$’s and $5$’s. Specifically, $80 = 2^4 \times 5$.
  - What will we need to multiply $2^4 \times 5$ by so that is it equal to $(2 \times 5)^4 = 10^4$?
    - We will need to multiply by $5^3$ so that $2^4 \times 5^4 = (2 \times 5)^4 = 10^4$.
  - Begin with $\frac{7}{80} = \frac{7}{2^4 \times 5}$, use what you know about equivalent fractions to rewrite $\frac{7}{80} = \frac{k}{10^n}$ and then the decimal form of the fraction.
    - $\frac{7}{80} = \frac{7 \times 5^3}{2^4 \times 5^4} = \frac{7 \times 125}{(2 \times 5)^4} = \frac{875}{10^4} = 0.0875$
Example 4 (4 minutes)

**Example 4**

Write $\frac{3}{160}$ as a decimal. Will it be finite or infinite? Explain.

- Let’s write $\frac{3}{160}$ as a decimal. Will it be finite or infinite? Explain.
  - We know that the fraction $\frac{3}{160}$ is equal to a finite decimal because the denominator 160 is a product of 2’s and 5’s. Specifically, $160 = 2^5 \times 5$.
- What will we need to multiply $2^5 \times 5$ by so that is it equal to $(2 \times 5)^n = 10^n$?
  - We will need to multiply by $5^4$ so that $2^5 \times 5^5 = (2 \times 5)^5 = 10^5$.
- Begin with $\frac{3}{160} = \frac{3}{2^5 \times 5}$, use what you know about equivalent fractions to rewrite $\frac{3}{160} = \frac{k}{10^n}$ and then the decimal form of the fraction.
  - $\frac{3}{160} = \frac{3}{2^5 \times 5} = \frac{3 \times 5^4}{2^5 \times 5 \times 5^4} = \frac{3 \times 625}{1875} = \frac{1875}{10^4} = 0.01875$

**Exercises 11–13 (5 minutes)**

Students complete Exercises 11–13 independently.

**Exercises 11–13**

Show your steps, but use a calculator for the multiplications.

11. Convert the fraction $\frac{37}{40}$ to a decimal.
   a. Write the denominator as a product of 2’s and/or 5’s. Explain why this way of rewriting the denominator helps to find the decimal representation of $\frac{37}{40}$.
      - The denominator $40 = 2^3 \times 5$. It is helpful to know that $40 = 2^3 \times 5$ because it shows by how many factors of 5 the numerator and denominator will need to be multiplied to produce an equivalent fraction with a denominator that is a multiple of 10. When the denominator is a multiple of 10, the fraction can easily be written as a decimal using what I know about place value.
   b. Find the decimal representation of $\frac{37}{40}$ Explain why your answer is reasonable.
      - $\frac{37}{40} = \frac{37}{2^3 \times 5} = \frac{37 \times 5^2}{2^3 \times 5 \times 5^2} = \frac{37 \times 5^2}{10^3} = 0.925$
      - The answer is reasonable because the decimal value, 0.925, is less than one just like the fraction $\frac{37}{40}$. Also, it is reasonable and correct because the fraction $\frac{925}{1000} = \frac{37}{40}$; therefore, it has the decimal expansion 0.925.
Lesson 6

Finite and Infinite Decimals

12. Convert the fraction \( \frac{3}{250} \) to a decimal.

The denominator \( 250 = 2 \times 5^3 \).

\[
\frac{3}{250} = \frac{3 \times 2^2}{2 \times 2^2 \times 5^3} = \frac{12}{10^3} = 0.012
\]

13. Convert the fraction \( \frac{7}{1,250} \) to a decimal.

The denominator \( 1,250 = 2 \times 5^4 \).

\[
\frac{7}{1,250} = \frac{7 \times 2^3}{2 \times 2^3 \times 5^4} = \frac{56}{10^4} = 0.0056
\]

Closing (3 minutes)

Summarize, or ask students to summarize, the main points from the lesson:

- We know that finite decimals are fractions with denominators that can be expressed as products of 2’s and 5’s.
- We know how to use equivalent fractions to convert a fraction to its decimal equivalent.
- We know that infinite decimals are those that repeat, like \( 0.\overline{3} \) or decimals that do not repeat, but do not terminate, such as \( \pi \).

Lesson Summary

Fractions with denominators that can be expressed as products of 2’s and/or 5’s have decimal expansions that are finite.

Example:

Does the fraction \( \frac{1}{8} \) have a finite or infinite decimal expansion?

Since \( 8 = 2^3 \), then the fraction has a finite decimal expansion. The decimal expansion is found by:

\[
\frac{1}{8} = \frac{1 \times 5^3}{2^3 \times 5^3} = \frac{125}{10^3} = 0.125
\]

When the denominator of a fraction cannot be expressed as a product of 2’s and/or 5’s then the decimal expansion of the number will be infinite.

When infinite decimals repeat, such as \( 0.888888 \ldots \) or \( 0.45454545 \ldots \), they are typically abbreviated using the notation \( 0.\overline{8} \) and \( 0.\overline{45} \), respectively. The notation indicates that the digit 8 repeats indefinitely and that the two-digit block 45 repeats indefinitely.

Exit Ticket (4 minutes)
Lesson 6: Finite and Infinite Decimals

Exit Ticket

Convert each fraction to a finite decimal. If the fraction cannot be written as a finite decimal, and then state how you know. Show your steps, but use a calculator for the multiplications.

1. \( \frac{9}{16} \)

2. \( \frac{8}{125} \)

3. \( \frac{4}{15} \)

4. \( \frac{1}{200} \)
Exit Ticket Sample Solutions

Convert each fraction to a finite decimal. If the fraction cannot be written as a finite decimal, and then state how you know. Show your steps, but use a calculator for the multiplications.

1. \( \frac{9}{16} \)
   
   The denominator \( 16 = 2^4 \).
   
   \[
   \frac{9}{16} = \frac{9 \times 5^4}{2^4 \times 5^4} = \frac{9 \times 625}{10^4} = \frac{5625}{10^4} = 0.5625
   \]

2. \( \frac{8}{125} \)
   
   The denominator \( 125 = 5^3 \).
   
   \[
   \frac{8}{125} = \frac{8 \times 2^3}{5^3 \times 2^3} = \frac{8 \times 8}{10^3} = \frac{64}{10^3} = 0.064
   \]

3. \( \frac{4}{15} \)
   
   The fraction \( \frac{4}{15} \) is not a finite decimal because the denominator \( 15 = 5 \times 3 \). Since the denominator cannot be expressed as a product of 2’s and 5’s, then \( \frac{4}{15} \) is not a finite decimal.

4. \( \frac{1}{200} \)
   
   The denominator \( 200 = 2^2 \times 5^2 \).
   
   \[
   \frac{1}{200} = \frac{1 \times 5}{2^2 \times 5^2 \times 5} = \frac{5}{2^2 \times 5^3} = \frac{5}{10^3} = 0.005
   \]

Problem Set Sample Solutions

Convert each fraction to a finite decimal. If the fraction cannot be written as a finite decimal, then state how you know. Show your steps, but use a calculator for the multiplications.

1. \( \frac{2}{32} \)
   
   The fraction \( \frac{2}{32} \) simplifies to \( \frac{1}{16} \).

   The denominator \( 16 = 2^4 \).
   
   \[
   \frac{1}{16} = \frac{1 \times 5^4}{2^4 \times 5^4} = \frac{625}{10^4} = 0.0625
   \]
2. \(\frac{99}{125}\)

a. Write the denominator as a product of 2’s and/or 5’s. Explain why this way of rewriting the denominator helps to find the decimal representation of \(\frac{99}{125}\).

The denominator 125 = 5^3. It is helpful to know that 125 = 5^3 because it shows how many factors of 2 the numerator and denominator will need to be multiplied to produce an equivalent fraction with a denominator that is a multiple of 10. When the denominator is a multiple of 10, the fraction can easily be written as a decimal using what I know about place value.

b. Find the decimal representation of \(\frac{99}{125}\). Explain why your answer is reasonable.

\[
\frac{99}{125} = \frac{2^3 \times 3^2}{2^3 \times 5^3} = \frac{99}{125} = 0.792
\]

The answer is reasonable because the decimal value, 0.792, is less than one just like the fraction \(\frac{99}{125}\). Also, it is reasonable and correct because the fraction \(\frac{792}{1000} = \frac{99}{125}\) therefore, it has the decimal expansion 0.792.

3. \(\frac{15}{128}\)

The denominator 128 = 2^7.

\[
\frac{15}{128} = \frac{15}{2^7} = \frac{15 \times 5^7}{10^7} = 0.1171875
\]

4. \(\frac{8}{15}\)

The fraction \(\frac{8}{15}\) is not a finite decimal because the denominator 15 = 3 \times 5. Since the denominator cannot be expressed as a product of 2’s and 5’s, then \(\frac{8}{15}\) is not a finite decimal.

5. \(\frac{3}{28}\)

The fraction \(\frac{3}{28}\) is not a finite decimal because the denominator 28 = 2^2 \times 7. Since the denominator cannot be expressed as a product of 2’s and 5’s, then \(\frac{3}{28}\) is not a finite decimal.

6. \(\frac{13}{400}\)

The denominator 400 = 2^4 \times 5^2.

\[
\frac{13}{400} = \frac{13}{2^4 \times 5^2} = \frac{13 \times 5^2}{10^4} = 0.0325
\]
8. \[
\frac{5}{64}
\]
   The denominator \(64 = 2^6\).
   \[
   \frac{5}{64} = \frac{5}{2^6} = \frac{5 \times 5^6}{2^6 \times 5^6} = \frac{78125}{10^6} = 0.078125
   \]

9. \[
\frac{15}{35}
\]
   The fraction \(\frac{15}{35}\) reduces to \(\frac{3}{7}\). The denominator 7 cannot be expressed as a product of 2’s and 5’s. Therefore, \(\frac{3}{7}\) is not a finite decimal.

10. \[
\frac{199}{250}
\]
    The denominator \(250 = 2 \times 5^3\).
    \[
    \frac{199}{250} = \frac{199}{2 \times 5^3} = \frac{199 \times 2^2}{2 \times 2^2 \times 5^3} = \frac{796}{10^3} = 0.796
    \]

11. \[
\frac{219}{625}
\]
    The denominator \(625 = 5^4\).
    \[
    \frac{219}{625} = \frac{219}{5^4} = \frac{219 \times 2^4}{2^4 \times 5^4} = \frac{3504}{10^4} = 0.3504
    \]
Lesson 7: Infinite Decimals

Student Outcomes

- Students know the intuitive meaning of an infinite decimal.
- Students will be able to explain why the infinite decimal 0.\(\overline{9}\) is equal to 1.

Lesson Notes

The purpose of this lesson is to show the connection between the various forms of a number, specifically the decimal expansion, the expanded form of a decimal, and a visual representation on the number line. Given the decimal expansion of a number, students use what they know about place value to write the expanded form of the number. That expanded form is then shown on the number line by looking at increasingly smaller intervals of 10, beginning with tenths, then hundredths, then thousandths, and so on. The strategy of examining increasingly smaller intervals of negative powers of 10 is how students will learn to write the decimal expansions of irrational numbers.

Classwork

Opening Exercises 1–4 (7 minutes)

Opening Exercises 1–4

1. Write the expanded form of the decimal 0.3765 using powers of 10.

\[
0.3765 = \frac{3}{10} + \frac{7}{10^2} + \frac{6}{10^3} + \frac{5}{10^4}
\]

2. Write the expanded form of the decimal 0.3333333 ... using powers of 10.

\[
0.333\overline{3} = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \frac{3}{10^4} + \frac{3}{10^5} + \frac{3}{10^6} ...
\]

3. What is an infinite decimal? Give an example.

An infinite decimal is a decimal with digits that do not end. They may repeat, but they never end. An example of an infinite decimal is 0.333333 ... .

4. Do you think it is acceptable to write that 1 = 0.99999 ... ? Why or why not?

Answers will vary. Have a brief discussion with students about this exercise. The answer will be revisited in the Discussion below.
Discussion (20 minutes)

Example 1

The number 0.253 on the number line:

- Each decimal digit is another division of a power of 10. Visually, the number 0.253 can be represented first as the segment from 0 to 1, divided into ten equal parts, noting the first division as 0.2. Then the segment from 0.2 to 0.3 is divided into 10 equal parts, noting the fifth division as 0.25. Then the segment from 0.25 to 0.26 is divided into 10 equal parts, noting the third division as 0.253.

- What we have done here is represented increasingly smaller increments of negative powers of 10: $\frac{2}{10^1}$, then $\frac{25}{10^2}$, and finally $\frac{253}{10^3}$.

- Now consider the expanded form of the decimal with denominators that are powers of 10, i.e., $\frac{1}{10^n}$ where $n$ is a whole number. The finite decimal can be represented in three steps:
  - The first decimal digit, $0.2 = \frac{2}{10}$.
  - The first two decimal digits, $0.25 = \frac{2}{10} + \frac{5}{10^2} = \frac{25}{10^2}$. 
Lesson 7

Infinite Decimals

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The first three decimal digits, $0.253 = \frac{2}{10} + \frac{5}{10^2} + \frac{3}{10^3} = \frac{253}{10^3}$.

- This number $0.253$ can be completely represented because there are a finite number of decimal digits. The value of the number $0.253$ can clearly be represented by the fraction $\frac{253}{10^3}$, i.e., $\frac{253}{1000} = 0.253$.

- Explain how $0.253$, the number lines, and the expanded form of the number are related.
  - The number $0.253$ is equal to the sum of the following fractions: $\frac{2}{10} + \frac{5}{10^2} + \frac{3}{10^3}$. Then, $0.253 = \frac{2}{10} + \frac{5}{10^2} + \frac{3}{10^3}$. The first number line above shows the first term of the sum, $\frac{2}{10}$. When the interval from $0.2$ to $0.3$ is examined in hundredths, we can locate the second term of the sum, $\frac{5}{10^2}$, and specifically the sum of the first two terms $\frac{2}{10} + \frac{5}{10^2} = \frac{25}{100}$. Then the interval between $0.25$ and $0.26$ is examined in thousandths. We can then locate the third term of the sum, $\frac{3}{10^3}$, and specifically the entire sum of the expanded form of $0.253$, which is $\frac{253}{1000}$.

- What do you think the sequence would look like for an infinite decimal?
  - The sequence for an infinite decimal would never end; it would go on infinitely.

Example 2

The number $\frac{5}{6} = 0.8\overline{3}$ on the number line:

- Now consider the equality $\frac{5}{6} = 0.8\overline{3}$. Notice that at the second step, the work begins to repeat, which coincides with the fact that the decimal digit of $3$ repeats.
- What is the expanded form of the decimal 0.833333 ...?
  - $0.833333 ... = \frac{8}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \frac{3}{10^4} + \frac{3}{10^5} + \frac{3}{10^6} + \cdots$

- Each step can be represented by increasing powers of 10 in the denominator: $\frac{8}{10}, \frac{83}{10^2}, \frac{833}{10^3}, \frac{8333}{10^4}, \frac{83333}{10^5}, \frac{833333}{10^6},$ and so on. When will it end? Explain.
  - It will never end because the decimal is infinite.

- Notice that in the last few steps, the value of the number being represented gets increasingly smaller. For example, in the sixth step, we have included $\frac{3}{10^6}$ more of the value of the number. That is 0.000003. As the steps increase, we are dealing with incrementally smaller numbers that approach a value of 0.

- Consider the 20th step, we would be adding $\frac{3}{10^{20}}$ to the value of the number, which is 0.00000000000000000003. It should be clear that $\frac{3}{10^{20}}$ is a very small number and is fairly close to a value of 0.

- At this point in our learning we know how to convert a fraction to a decimal, even if it is infinite. How do we do that?
  - We use long division when the fraction is equal to an infinite decimal.

- We will soon learn how to write an infinite decimal as a fraction; in other words, we will learn how to convert a number in the form of 0.83 to a fraction, $\frac{5}{6}$.

- Now back to Exercise 4. Is it acceptable to write that $1 = 0.9999999 \ldots$? With an increased understanding of infinite decimals, have you changed your mind about whether or not this is an acceptable statement?
Have a discussion with students about Exercise 4. If students have changed their minds, ask them to explain why.

- When you consider the infinite steps that represent the decimal 0.9999999..., it is clear that the value we add with each step is an increasingly smaller value so it makes sense to write that 0.9 = 1.
- A concern may be that the left side is not really equal to one; it only gets closer and closer to 1. However, such a statement confuses the process of representing a finite decimal with an infinite decimal. That is, as we increase the steps, we are adding smaller and smaller values to the number. It is so small, that the amount we add is practically zero. That means with each step, we are showing that the number 0.9 is getting closer and closer to 1. Since the process is infinite, it is acceptable to write 0.9 = 1.

Provide students time to convince a partner that 0.9 = 1. Encourage students to be as critical as possible. Select a student to share his or her argument with the class.

- In many (but not all) situations, we often treat infinite decimals as finite decimals. We do this for the sake of computation. Imagine multiplying the infinite decimal 0.83333333... by any other number or even another infinite decimal. To do this work precisely, you would never finish writing one of the infinite decimals, let alone perform the multiplication. For this reason, we often shorten the infinite decimal using the repeat block as our guide for performing operations.

- Every finite decimal is the sum of a whole number (which could be zero) and a finite decimal that is less than 1. Show that this is true for the number 3.141592.
  - The number 3.141592 is equal to the whole number 3 plus the finite decimal 0.141592:
    \[ 3.141592 = 3 + 0.141592 \]

- By definition of a finite decimal (one whose denominators can be expressed as a product of 2’s and 5’s), the number 3.141592 is equivalent to
  \[
  \frac{3141592}{10^6} = \frac{(3 \times 10^6) + 141592}{10^6} = \frac{3 \times 10^6}{10^6} + \frac{141592}{10^6} = 3 + \frac{141592}{10^6} = 3 + 0.141592
  \]

- We will soon claim that every infinite decimal is the sum of a whole number and an infinite decimal that is less than 1. Consider the infinite decimal 3.141592...
  \[ 3.141592 ... = 3 + 0.141592 ... \]

This fact will help us to write an infinite decimal as a fraction in Lesson 10.

**Exercises 5–10 (8 minutes)**

Students complete Exercises 5–10 independently or in pairs.

**Exercises 5–10**

5. a. Write the expanded form of the decimal 0.125 using powers of 10.

\[
0.125 = \frac{1}{10} + \frac{2}{10^2} + \frac{5}{10^3}
\]
Lesson 7 8.7

b. Show on the number line the representation of the decimal 0.125.

```
0 0.1
0.1 0.12
0.12 0.125
```

c. Is the decimal finite or infinite? How do you know?

*The decimal 0.125 is finite because it can be completely represented by a finite number of steps.*

6. a. Write the expanded form of the decimal 0.3875 using powers of 10.

\[0.3875 = \frac{3}{10} + \frac{8}{10^2} + \frac{7}{10^3} + \frac{5}{10^4}\]

b. Show on the number line the representation of the decimal 0.3875.

```
0 0.3
0.3 0.38
0.38 0.387
0.387 0.3875
```

c. Is the decimal finite or infinite? How do you know?

*The decimal 0.3875 is finite because it can be completely represented by a finite number of steps.*

7. a. Write the expanded form of the decimal 0.777777 ... using powers of 10.

\[0.777777 ... = \frac{7}{10} + \frac{7}{10^2} + \frac{7}{10^3} + \frac{7}{10^4} + \frac{7}{10^5} + \frac{7}{10^6} + \cdots\]

*and so on.*
b. Show on the number line the representation of the decimal $0.777777 \ldots$.

![Number line diagram](image)

8. a. Write the expanded form of the decimal $0.45$ using powers of 10.

$$0.45 = \frac{4}{10} + \frac{5}{10^2} + \frac{4}{10^3} + \frac{5}{10^4} + \frac{4}{10^5} + \frac{5}{10^6} + \ldots$$

and so on.

b. Show on the number line the representation of the decimal $0.45$.

![Number line diagram](image)
c. Is the decimal finite or infinite? How do you know?

The decimal 0.45 is infinite because it cannot be represented by a finite number of steps. Because the digits 4 and 5 continue to repeat, there will be an infinite number of steps in the sequence.

9. Order the following numbers from least to greatest: 2.121212, 2.1, 2, 2, and 2.12.

2.1, 2.121212, 2.12, 2

10. Explain how you knew which order to put the numbers in.

Each number is the sum of the whole number 2 and a decimal. When you write each number in this manner you get

\[
\begin{align*}
2.121212 &= 2 + \frac{1}{10} + \frac{2}{10^2} + \frac{1}{10^3} + \frac{2}{10^4} + \frac{1}{10^5} + \frac{2}{10^6} \\
2.1 &= 2 + \frac{1}{10} \\
2.2 &= 2 + \frac{2}{10} \\
2.12 &= 2 + \frac{1}{10} + \frac{2}{10^2} + \frac{1}{10^3} + \frac{2}{10^4} + \frac{1}{10^5} + \frac{2}{10^6} + \frac{1}{10^7} + \frac{2}{10^8} + \cdots
\end{align*}
\]

In this form it is clear that 2.1 is the least of the four numbers, followed by the finite decimal 2.1212, then the infinite decimal 2.12, and finally 2.2.

Closing (5 minutes)

Summarize, or ask students to summarize, the main points from the lesson:

- We know that an infinite decimal is a decimal whose expanded form and number line representation is infinite.
- We know that each step in the sequence of an infinite decimal adds an increasingly smaller value to the number, so small that the amount approaches zero.
- We know that the infinite decimal $0.\overline{9} = 1$ and can explain why this is true.
Lesson Summary

An infinite decimal is a decimal whose expanded form and number line representation are infinite.

Example:
The expanded form of the decimal $0.8\overline{3}$ is $0.8 + \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \cdots$

The number is represented on the number line shown below. Each new line is a magnification of the interval shown above it. For example, the first line is the unit from 0 to 1 divided into 10 equal parts, or tenths. The second line is the interval from 0.8 to 0.9 divided into ten equal parts, or hundredths. The third line is the interval from 0.83 to 0.84 divided into ten equal parts, or thousandths, and so on.

With each new line we are representing an increasingly smaller value of the number, so small that the amount approaches a value of 0. Consider the 20th line of the picture above. We would be adding $\frac{3}{10^{20}}$ to the value of the number, which is $0.00000000000000000003$. It should be clear that $\frac{3}{10^{20}}$ is a very small number and is fairly close to a value of 0.

This reasoning is what we use to explain why the value of the infinite decimal $0.\overline{9}$ is 1.

Exit Ticket (5 minutes)

There are three items as part of the Exit Ticket, but it may be necessary to only use the first two to assess students’ understanding.
Lesson 7: Infinite Decimals

Exit Ticket

1. a. Write the expanded form of the decimal 0.829 using powers of 10.

b. Show on the number line the representation of the decimal 0.829.

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2. a. Write the expanded form of the decimal 0.55555 ... using powers of 10.

b. Show on the number line the representation of the decimal 0.555555 ... 

```
0                           1
```

 c. Is the decimal finite or infinite? How do you know?
3. a. Write the expanded form of the decimal $0.\overline{573}$ using powers of 10.

b. Show on the number line the representation of the decimal $0.\overline{573}$.

c. Is the decimal finite or infinite? How do you know?
Exit Ticket Sample Solutions

1. a. Write the expanded form of the decimal 0.829 using powers of 10.

\[ 0.829 = \frac{8}{10} + \frac{2}{10^2} + \frac{9}{10^3} \]

b. Show on the number line the representation of the decimal 0.829.

![Number line representation of 0.829](image)

c. Is the decimal finite or infinite? How do you know?

The decimal 0.829 is finite because it can be completely represented by a finite number of steps.

2. a. Write the expanded form of the decimal 0.5555... using powers of 10.

\[ 0.5555... = \frac{5}{10} + \frac{5}{10^2} + \frac{5}{10^3} + \frac{5}{10^4} + \frac{5}{10^5} + \frac{5}{10^6} + \cdots \]

and so on.

b. Show on the number line the representation of the decimal 0.55555...

![Number line representation of 0.55555...](image)

c. Is the decimal finite or infinite? How do you know?

The decimal 0.55555... is infinite because it cannot be represented by a finite number of steps. Because the number 5 continues to repeat, there will be an infinite number of steps.
3.  a. Write the expanded form of the decimal 0.573 using powers of 10.

\[ 0.573 = \frac{5}{10} + \frac{7}{10^2} + \frac{3}{10^3} + \frac{5}{10^4} + \frac{7}{10^5} + \frac{3}{10^6} + \cdots \]

and so on.

b. Describe the sequence that would represent the decimal 0.573.

\[ 0 \rightarrow 0.5 \rightarrow 0.57 \rightarrow 0.573 \rightarrow 0.5735 \rightarrow 0.57357 \rightarrow 0.573573 \rightarrow \cdots \]

c. Is the decimal finite or infinite? How do you know?

The decimal 0.573 is infinite because it cannot be represented by a finite number of steps. Because the digits 5, 7, and 3 continue to repeat, there will be an infinite number of steps.

Problem Set Sample Solutions

1.  a. Write the expanded form of the decimal 0.625 using powers of 10.

\[ 0.625 = \frac{6}{10} + \frac{2}{10^2} + \frac{5}{10^3} \]

b. Show on the number line the representation of the decimal 0.625.

\[ 0 \rightarrow 0.6 \rightarrow 0.62 \rightarrow 0.625 \]

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c. Is the decimal finite or infinite? How do you know?

The decimal 0.625 is finite because it can be completely represented by a finite number of steps in the sequence.

2. a. Write the expanded form of the decimal 0.370 using powers of 10.

\[0.370 = \frac{3}{10} + \frac{7}{10^2} + \frac{0}{10^3} + \frac{3}{10^4} + \frac{7}{10^5} + \frac{0}{10^6} + \cdots\]

and so on.

b. Show on the number line the representation of the decimal 0.370370 ... 

\[0 \ldots 0.3 \ldots 0.37 \ldots 0.370 \ldots \]

\[0.370370 \ldots 0.3704 \]

The number \(\frac{2}{3}\) is more accurately represented by the decimal 0.6 compared to 0.6666. The long division algorithm with \(\frac{2}{3}\) shows that the digit 6 repeats. Then the expanded form of the decimal 0.6 = \(\frac{6}{10} + \frac{6}{10^2} + \frac{6}{10^3} + \frac{6}{10^4} + \frac{6}{10^5} + \cdots\) and so on, where the number 0.6666 = \(\frac{6}{10} + \frac{6}{10^2} + \frac{6}{10^3} + \frac{6}{10^4} + \cdots\). For this reason, 0.6 is more accurate.

For computations, I would prefer to use 0.6666. My answer would be less precise, but at least I'd be able to compute with it. When attempting to compute with an infinite number, you would never finish writing it, thus you could never compute with it.
4. Explain why we shorten infinite decimals to finite decimals to perform operations. Explain the effect of shortening an infinite decimal on our answers.

We often shorten infinite decimals to finite decimals to perform operations because it would be impossible to represent an infinite decimal precisely because the sequence that describes infinite decimals has an infinite number of steps. Our answers are less precise; however, they are not that much less precise because with each additional digit in the sequence we include, we are adding a very small amount to the value of the number. The more decimals we include, the closer the value we add approaches zero. Therefore, it does not make that much of a difference with respect to our answer.

5. A classmate missed the discussion about why $0.\overline{9} = 1$. Convince your classmate that this equality is true.

When you consider the infinite sequence of steps that represents the decimal $0.999999 \ldots$, it is clear that the value we add with each step is an increasingly smaller value, so it makes sense to write that $0.\overline{9} = 1$. As we increase the number of steps in the sequence, we are adding smaller and smaller values to the number. Consider the $12^{th}$ step: $0.999999999999$. The value added to the number is just $0.000000000009$, which is a very small amount. The more steps that we include, the closer that value is to zero. Which means that with each new step, the number $0.\overline{9}$ gets closer and closer to $1$. Since this process is infinite, the number $0.\overline{9} = 1$.

6. Explain why $0.3333 < 0.33333$.

The number $0.3333 = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \frac{3}{10^4}$ and the number $0.33333 = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \frac{3}{10^4} + \frac{3}{10^5}$. That means that $0.33333$ is exactly $\frac{3}{10^5}$ larger than $0.3333$. If we examined the numbers on the number line, $0.33333$ is to the right of $0.3333$ meaning that it is larger than $0.3333$. 

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Lesson 8: The Long Division Algorithm

Student Outcomes

- Students know that the long division algorithm is the basic skill to get division-with-remainder and the decimal expansion of a number in general.
- Students know why digits repeat in terms of the algorithm.
- Students know that every rational number has a decimal expansion that repeats eventually.

Lesson Notes

In this lesson we move towards being able to define an irrational number by formalizing the definition of a rational number.

Classwork

Example 1 (5 minutes)

Example 1

Show that the decimal expansion of \( \frac{26}{4} \) is 6.5.

Use the Example with students so they have a model to complete Exercises 1–5.

- Show that the decimal expansion of \( \frac{26}{4} \) is 6.5.
  - Students will most likely use the long division algorithm.
- Division is really just another form of multiplication. Here is a demonstration of that fact: Let’s consider the fraction \( \frac{26}{4} \) in terms of multiplication. We want to know the greatest number of groups of 4 that are in 26. How many are there?
  - There are 6 groups of 4 in 26.
- Is there anything leftover, a remainder?
  - Yes, there are 2 leftover.
- Symbolically, we can express the number 26 as:
  \[ 26 = 6 \times 4 + 2 \]
With respect to the fraction $\frac{26}{4}$ we can represent the division as
\[
\begin{align*}
\frac{26}{4} &= 6 \times 4 + 2 \\
\frac{26}{4} &= 6 \times 4 + \frac{2}{4} \\
\frac{26}{4} &= 6 + \frac{2}{4} \\
\frac{26}{4} &= 6 + \frac{2}{4} = 6 + \frac{1}{2} \\
\frac{26}{4} &= 6 + \frac{1}{2} \\
\frac{26}{4} &= 6 + \frac{1}{2} = 6 \frac{1}{2}.
\end{align*}
\]

The fraction $\frac{26}{4}$ is equal to the finite decimal 6.5. When the fraction is not equal to a finite decimal, then we need to use the long division algorithm to determine the decimal expansion of the number.

Exploratory Challenge

Exercises 1–5 (15 minutes)

Students complete Exercises 1–5 independently or in pairs. The discussion that follows is related to the concepts in the Exercises.

Exercises 1–5

1. Use long division to determine the decimal expansion of $\frac{142}{2}$.

\[
\begin{array}{r}
2 \overline{142.0} \\
7 1.0 \\
\hline
14 2 \\
14 2 \\
\hline
0
\end{array}
\]

a. Fill in the blanks to show another way to determine the decimal expansion of $\frac{142}{2}$.

\[
\begin{align*}
142 &= \frac{71 \times 2 + 0}{2} \\
142 &= \frac{71 \times 2 + 0}{2} \\
142 &= \frac{71 \times 2 + 0}{2} \\
142 &= \frac{71}{2} + \frac{0}{2} \\
142 &= \frac{71}{2} + \frac{0}{2} \\
142 &= \frac{71}{2} + \frac{0}{2} = 71.0
\end{align*}
\]

b. Does the number $\frac{142}{2}$ have a finite or infinite decimal expansion? Explain how you know.

The decimal expansion of $\frac{142}{2}$ is 71.0 and is finite because the denominator of the fraction, 2, can be expressed as a product of 2's.

2. Use long division to determine the decimal expansion of $\frac{142}{4}$.

\[
\begin{array}{r}
4 \overline{35.5} \\
3 5.5 \\
\hline
1 4 2 \\
1 4 2 \\
\hline
0
\end{array}
\]
Lesson 8

The Long Division Algorithm

Date: 1/31/14

NYS COMMON CORE MATHEMATICS CURRICULUM

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a. Fill in the blanks to show another way to determine the decimal expansion of \( \frac{142}{4} \).

\[
\begin{align*}
142 &= 35 \times 4 + 2 \\
\frac{142}{4} &= 35 + \frac{2}{4} \\
142 &= 35 \times 4 + 2 \\
\frac{142}{4} &= 35 + \frac{2}{4} \\
142 &= 35 \times 4 + 2 \\
\frac{142}{4} &= 35 + \frac{2}{4} \\
142 &= 35\frac{2}{4} = 35.5
\end{align*}
\]

b. Does the number \( \frac{142}{4} \) have a finite or infinite decimal expansion? Explain how you know.

The decimal expansion of \( \frac{142}{4} \) is 35.5 and is finite because the denominator of the fraction, 4, can be expressed as a product of 2's.

3. Use long division to determine the decimal expansion of \( \frac{142}{6} \).

\[
\begin{array}{c|c}
6 & 142.000 \\
\hline
23 & 12 \\
22 & 18 \\
40 & 36 \\
36 & 40 \\
36 & 40 \\
4 & 36 \\
\end{array}
\]

a. Fill in the blanks to show another way to determine the decimal expansion of \( \frac{142}{6} \).

\[
\begin{align*}
142 &= 23 \times 6 + 4 \\
\frac{142}{6} &= 23 \times 6 + 4 \\
142 &= 23 \times 6 + 4 \\
\frac{142}{6} &= 23 + \frac{4}{6} \\
142 &= 23 \frac{4}{6} = 23.666...
\end{align*}
\]

b. Does the number \( \frac{142}{6} \) have a finite or infinite decimal expansion? Explain how you know.

The decimal expansion of \( \frac{142}{6} \) is 23.666... and is infinite because the denominator of the fraction, 6, cannot be expressed as a product of 2's and/or 5's.
4. Use long division to determine the decimal expansion of \( \frac{142}{11} \).

\[
\begin{array}{c|c}
\text{12.90909} & \\
\hline
11 & 142.00000 \\
\hline
11 & 132 \\
\hline
11 & 30 \\
\hline
11 & 9 \\
\hline
11 & 0 \\
\hline
11 & 9 \\
\hline
11 & 0 \\
\hline
11 & \\
\hline
11 & \\
\hline
11 & \\
\hline
\end{array}
\]

a. Fill in the blanks to show another way to determine the decimal expansion of \( \frac{142}{11} \).

\[
\begin{align*}
142 & = 12 \times 11 + 10 \\
142 & = 12 \times 11 + 10 \\
142 & = 12 \times 11 + \frac{10}{11} \\
142 & = 12 + \frac{10}{11} \\
142 & = 12 \frac{10}{11} = 12.90909 \ldots
\end{align*}
\]

b. Does the number \( \frac{142}{11} \) have a finite or infinite decimal expansion? Explain how you know.

"The decimal expansion of \( \frac{142}{11} \) is 12.90909 \ldots and is infinite because the denominator of the fraction, 6, cannot be expressed as a product of 2's and/or 5's."

5. Which fractions produced an infinite decimal expansion? Why do you think that is?

"The fractions that required the long division algorithm to determine the decimal expansion were \( \frac{142}{6} \) and \( \frac{142}{11} \). The fact that these numbers had an infinite decimal expansion is due to the fact that the divisor was not a product of 2's and/or 5's compared to the first two fractions where the divisor was a product of 2's and/or 5's. In general, the decimal expansion of a number will be finite when the divisor, i.e., the denominator of the fraction, can be expressed as a product of 2's and/or 5's. Similarly, the decimal expansion will be infinite when the divisor cannot be expressed as a product of 2's and/or 5's."
Discussion (10 minutes)

- What is the decimal expansion of $\frac{142}{2}$?
  
  If students respond “71”, ask them what decimal digits they could include without changing the value of the number.
  
  - The fraction $\frac{142}{2}$ is equal to the decimal 71.00000 ...
  
  - Did you need to use the long division algorithm to determine your answer? Why or why not?
    - No, the long division algorithm was not necessary because there was a whole number of 2’s in 142.
  
- What is the decimal expansion of $\frac{142}{4}$?
  
  - The fraction $\frac{142}{4}$ is equal to the decimal 35.5.
  
- What decimal digits could we include to the right of the 0.5 without changing the value?
    - We could write the decimal as 35.500000 ...
  
- Did you need to use the long division algorithm to determine your answer? Why or why not?
    - No, the long division algorithm was not necessary because $\frac{142}{4} = 35 + \frac{2}{4}$ and $\frac{2}{4}$ is a finite decimal. We could use what we learned in the last lesson to write $\frac{2}{4}$ as 0.5.
  
- What is the decimal expansion of $\frac{142}{6}$?
  
  - The fraction $\frac{142}{6}$ is equal to the decimal 23.66666 ...
  
- Did you need to use the long division algorithm to determine your answer? Why or why not?
    - Yes, the long division algorithm was necessary because $\frac{142}{6} = 23 + \frac{2}{3}$ and $\frac{2}{3}$ is not a finite decimal.
      
      Note: Some students may have recognized the fraction $\frac{2}{3}$ as 0.6666 ... and not used the long division algorithm to determine the decimal expansion.
  
- How did you know when you could stop dividing?
    - I knew to stop dividing because the remainder kept repeating. Specifically, when I used the long division algorithm, the number 40 kept appearing, and there are 6 groups of 6 in 40, leaving 4 as a remainder each time, which became 40 when I brought down another 0.
  
- We represent the decimal expansion of $\frac{142}{6}$ as 23.6$, where the line above the 6 is the “repeating block”; that is, the digit 6 repeats as we saw in the long division algorithm.
  
- What is the decimal expansion of $\frac{142}{11}$?
  
  - The fraction $\frac{142}{11}$ is equal to the decimal 12.90909090 ...
  
- Did you need to use the long division algorithm to determine your answer? Why or why not?
    - Yes, the long division algorithm was necessary because $\frac{142}{11} = 12 + \frac{10}{11}$ and $\frac{10}{11}$ is not a finite decimal.
  
- How did you know when you could stop dividing?
    - I knew to stop dividing because the remainder kept repeating. Specifically, when I used the long division algorithm, I kept getting the number 10, which is not divisible by 11, so I had to bring down another 0 making the number 100. This kept happening, so I knew to stop once I noticed the work I was doing was the same.
Lesson 8: The Long Division Algorithm

- Which block of digits kept repeating?
  - The block of digits that kept repeating was 90.
- How do we represent the decimal expansion of $\frac{142}{11}$?
  - The decimal expansion of $\frac{142}{11}$ is $12.\overline{90}$.
- In general, we say that every rational number has a decimal expansion that repeats eventually. It is obvious by the repeat blocks that $\frac{142}{6}$ and $\frac{142}{11}$ are rational numbers. Are the numbers $\frac{142}{2}$ and $\frac{142}{4}$ rational? If so, what is their repeat block?

Provide students a minute or two to discuss in small groups what the repeat blocks for $\frac{142}{2}$ and $\frac{142}{4}$ are.

- The decimal expansion of $\frac{142}{2}$ is $71.0000 \ldots$ where the repeat block is 0. The decimal expansion of $\frac{142}{4}$ is $35.5000 \ldots$ where the repeat block is 0. Since the numbers $\frac{142}{2}$ and $\frac{142}{4}$ have decimal expansions that repeat, then the numbers are rational.

Exercises 6–10 (5 minutes)

Students complete Exercises 6–10 independently.

### Exercises 6–10

6. Does the number $\frac{65}{13}$ have a finite or infinite decimal expansion? Based on our definition of rational numbers having a decimal expansion that repeats eventually, is the number rational? Explain.

   $\frac{65}{13} = \frac{5 \times 13}{13} = 5$ so it is a finite decimal. The decimal expansion of $\frac{65}{13}$ is $5.0000 \ldots$ where the repeat block is 0. Therefore, the number is rational.

7. Does the number $\frac{17}{11}$ have a finite or infinite decimal expansion? Based on our definition of rational numbers having a decimal expansion that repeats eventually, is the number rational? Explain.

   $\frac{17}{11} = \frac{1 \times 11}{11} + \frac{6}{11}$

   \[
   \begin{array}{c|c}
   \hline
   11 & 1.5454 \ldots \\
   \hline
   11 & 17.0000 \\
   \hline
   60 & \\
   55 & \\
   44 & \\
   44 & \\
   60 & \\
   \hline
   \end{array}
   \]

   The number $\frac{17}{11}$ has an infinite decimal expansion, 1.54. The block of digits 54 repeats. In doing the long division, I realized that the remainder of 6 and remainder of 5 kept reappearing in my work. Since the number has a repeat block, it is rational.
8. Does the number $\pi = 3.1415926535897...$ have a finite or infinite decimal expansion? Based on our definition of rational numbers having a decimal expansion that repeats eventually, is the number rational? Explain.

_The number has an infinite decimal expansion. However, it does not have decimal digits that repeat in a block. For that reason, the number is not rational._

9. Does the number $\frac{860}{999} = 0.860860860...$ have a finite or infinite decimal expansion? Based on our definition of rational numbers having a decimal expansion that repeats eventually, is the number rational? Explain.

_The number has an infinite decimal expansion. However, the decimal expansion has a repeat block of 860. Because every rational number has a block that repeats, the number is rational._

10. Does the number $\sqrt{2} = 1.41421356237...$ have a finite or infinite decimal expansion? Based on our definition of rational numbers having a decimal expansion that repeats eventually, is the number rational? Explain.

_The number has an infinite decimal expansion. However, it does not have decimal digits that repeat in a block. For that reason, the number is not rational._

**Closing (5 minutes)**

Summarize, or ask students to summarize, the main points from the lesson:

- We know that the long division algorithm is a procedure that allows us to write the decimal expansion for infinite decimals.
- We know that every rational number has a decimal expansion that repeats eventually.

**Lesson Summary**

The long division algorithm is a procedure that can be used to determine the decimal expansion of infinite decimals. Every rational number has a decimal expansion that repeats eventually. For example, the number $\frac{32}{10}$ is rational because it has a repeat block of the digit 0 in its decimal expansion, 0.32. The number $\frac{1}{3}$ is rational because it has a repeat block of the digit 3 in its decimal expansion, 0.333... The number 0.454545... is rational because it has a repeat block of the digits 45 in its decimal expansion, 0.454545...
Lesson 8: The Long Division Algorithm

Exit Ticket

1. Write the decimal expansion of $\frac{125}{8}$. Based on our definition of rational numbers having a decimal expansion that repeats eventually, is the number rational? Explain.

2. Write the decimal expansion of $\frac{13}{7}$. Based on our definition of rational numbers having a decimal expansion that repeats eventually, is the number rational? Explain.
Exit Ticket Sample Solutions

1. Write the decimal expansion of \( \frac{125}{8} \). Based on our definition of rational numbers having a decimal expansion that repeats eventually, is the number rational? Explain.

\[
\frac{125}{8} = 15 \times \frac{8}{8} + \frac{5}{8} = \frac{125}{8} \quad \frac{15.625}{8 | \underline{125.000}}
\]

The decimal expansion of \( \frac{125}{8} \) is 15.625. The number is rational because it is a finite decimal with a repeating block of 0.

2. Write the decimal expansion of \( \frac{13}{7} \). Based on our definition of rational numbers having a decimal expansion that repeats eventually, is the number rational? Explain.

\[
\frac{13}{7} = 1 \times \frac{7}{7} + \frac{6}{7} = \frac{13}{7} \quad \frac{1.857142857142}{7 | \underline{13.00000000000000}}
\]

The decimal expansion of \( \frac{13}{7} \) is 1.857142. The number is rational because there is a repeating block of 857142.

Rational numbers have decimal expansions that repeat; therefore, \( \frac{13}{7} \) is a rational number.
Problem Set Sample Solutions

1. Write the decimal expansion of $\frac{7000}{9}$. Based on our definition of rational numbers having a decimal expansion that repeats eventually, is the number rational? Explain.

$$\frac{7000}{9} = \frac{777 \times 9}{9} + \frac{7}{9}$$

$$= 777\frac{7}{9}$$

$$= 777.\overline{7}$$

$\frac{7000}{9} = 777.\overline{7}$

The decimal expansion of $\frac{7000}{9}$ is $777.\overline{7}$. The number is rational because it has the repeating digit $7$. Rational numbers have decimal expansions that repeat; therefore, $\frac{7000}{9}$ is a rational number.

2. Write the decimal expansion of $\frac{655555}{3}$. Based on our definition of rational numbers having a decimal expansion that repeats eventually, is the number rational? Explain.

$$\frac{655555}{3} = \frac{2185185 \times 3}{3} = \frac{0}{3}$$

$$= 2,185,185$$

The decimal expansion of $\frac{655555}{3}$ is 2,185,185. The number is rational because we can write the repeating digit of 0 following the whole number. Rational numbers have decimal expansions that repeat; therefore, $\frac{655555}{3}$ is a rational number.
3. Write the decimal expansion of \(\frac{35,000}{11}\). Based on our definition of rational numbers having a decimal expansion that repeats eventually, is the number rational? Explain.

\[
\frac{35,000}{11} = \frac{31,818 \times 11}{11} + \frac{2}{11} = 31,818 \frac{2}{11}
\]

\[
\frac{31818.18}{11} = \frac{315,006.60}{33} \]

\[
\frac{315,006.60}{33} = 90 \]

\[
\frac{90}{11} = 8 \]

\[
\frac{8}{20} = 0.4
\]

The decimal expansion of \(\frac{35,000}{11}\) is 31818.18. The number is rational because there is a repeating block of 18.

Rational numbers have decimal expansions that repeat; therefore, \(\frac{35,000}{11}\) is a rational number.

4. Write the decimal expansion of \(\frac{12,000,000}{37}\). Based on our definition of rational numbers having a decimal expansion that repeats eventually, is the number rational? Explain.

\[
\frac{12,000,000}{37} = \frac{324,324 \times 37}{37} + \frac{12}{37} = 324,324 \frac{12}{37}
\]

\[
\frac{324,324.324}{37} = \frac{32,000,000.000}{111}
\]

\[
\frac{32,000,000.000}{111} = 90 \]

\[
\frac{90}{74} = 1.22373
\]

\[
\frac{1.22373}{160} = 0.0053
\]

\[
\frac{0.0053}{148} = 0.00036
\]

\[
\frac{0.00036}{120} = 0.000003
\]

\[
\frac{0.000003}{111} = 0.00000027
\]

\[
\frac{0.00000027}{90} = 0.000000003
\]

\[
\frac{0.000000003}{74} = 0.0000000000435
\]

\[
\frac{0.0000000000435}{160} = 0.000000000000272
\]

\[
\frac{0.000000000000272}{148} = 0.0000000000000183
\]

\[
\frac{0.0000000000000183}{120} = 0.000000000000001525
\]

\[
\frac{0.000000000000001525}{111} = 0.00000000000000001387
\]

\[
\frac{0.00000000000000001387}{90} = 0.000000000000000001541
\]

\[
\frac{0.000000000000000001541}{74} = 0.0000000000000000002078
\]

\[
\frac{0.0000000000000000002078}{160} = 0.00000000000000000001311
\]

\[
\frac{0.00000000000000000001311}{148} = 0.00000000000000000000009
\]

\[
\frac{0.00000000000000000000009}{120} = 0.0000000000000000000000075
\]

\[
\frac{0.0000000000000000000000075}{111} = 0.00000000000000000000000068
\]

\[
\frac{0.00000000000000000000000068}{90} = 0.0000000000000000000000000075
\]

\[
\frac{0.0000000000000000000000000075}{74} = 0.000000000000000000000000000101
\]

\[
\frac{0.000000000000000000000000000101}{160} = 0.0000000000000000000000000000063
\]

\[
\frac{0.0000000000000000000000000000063}{148} = 0.00000000000000000000000000000043
\]

\[
\frac{0.00000000000000000000000000000043}{120} = 0.000000000000000000000000000000033
\]

\[
\frac{0.000000000000000000000000000000033}{111} = 0.000000000000000000000000000000003
\]

\[
\frac{0.000000000000000000000000000000003}{90} = 0.00000000000000000000000000000000033
\]

\[
\frac{0.00000000000000000000000000000000033}{74} = 0.000000000000000000000000000000000043
\]

\[
\frac{0.000000000000000000000000000000000043}{160} = 0.00000000000000000000000000000000000063
\]

The decimal expansion of \(\frac{12,000,000}{37}\) is 324.324. The number is rational because there is a repeating block of 324. Rational numbers have decimal expansions that repeat; therefore, \(\frac{12,000,000}{37}\) is a rational number.
5. Someone notices that the long division of 2,222,222 by 6 has a quotient of 370,370 and remainder 2 and wonders why there is a repeating block of digits in the quotient, namely 370. Explain to the person why this happens.

\[
\begin{align*}
  2222222 \div 6 &= 370,370 \times \frac{6}{6} + \frac{2}{6} \\
  &= 370,370 \times \frac{2}{6} \\
  &= \frac{370370}{6} 2222222 \\
  &= \frac{18}{42} \\
  &= 022
\end{align*}
\]

The reason that the block of digits 370 keeps repeating is because the long division algorithm leads us to perform the same division over and over again. In the algorithm shown above, we see that there are 3 groups of 6 in 22, leaving a remainder of 4. When we bring down the next 2, we see that there are exactly 7 groups of 6 in 42. When we bring down the next 2, we see that there are 0 groups of 6 in 2, leaving a remainder of 2. It is then that the process starts over because the next step is to bring down another 2, giving us 22, which is what we started with. Since the division repeats, then the digits in the quotient will repeat.

6. Is the number \( \frac{9}{11} = 0.81818181 \ldots \) rational? Explain.

The number appears to be rational because the decimal expansion has a repeat block of 81. Because every rational number has a block that repeats, the number is rational.

7. Is the number \( \sqrt{3} = 1.73205080 \ldots \) rational? Explain.

The number appears to have a decimal expansion that does not have decimal digits that repeat in a block. For that reason, this is not a rational number.

8. Is the number \( \frac{41}{333} = 0.1231231231 \ldots \) rational? Explain.

The number appears to be rational because the decimal expansion has a repeat block of 123. Because every rational number has a block that repeats, the number is rational.
Lesson 9: Decimal Expansions of Fractions, Part 1

Student Outcomes

- Students apply knowledge of equivalent fractions, long division, and the distributive property to write the decimal expansion of fractions.

Classwork

Opening Exercises 1–2 (5 minutes)

1. a. We know that the fraction $\frac{5}{8}$ can be written as a finite decimal because its denominator is a product of 2’s.

   Which power of 10 will allow us to easily write the fraction as a decimal? Explain.

   Since $8 = 2^3$ we will multiply the numerator and denominator by $5^3$, which means that $2^3 \times 5^3 = 10^3$ will be the power of 10 that allows us to easily write the fraction as a decimal.

   b. Write the equivalent fraction using the power of 10.

   $$\frac{5}{8} \times \frac{5^3}{5^3} = \frac{625}{1000}$$

2. a. We know that the fraction $\frac{17}{125}$ can be written as a finite decimal because its denominator is a product of 5’s.

   Which power of 10 will allow us to easily write the fraction as a decimal? Explain.

   Since $125 = 5^3$ we will multiply the numerator and denominator by $2^3$, which means that $5^3 \times 2^3 = 10^3$ will be the power of 10 that allows us to easily write the fraction as a decimal.

   b. Write the equivalent fraction using the power of 10.

   $$\frac{17}{125} \times \frac{2^3}{2^3} = \frac{136}{1000}$$

Example 1 (5 minutes)

Example 1

Write the decimal expansion of the fraction $\frac{5}{8}$

- Based on our previous work with finite decimals, we already know how to convert $\frac{5}{8}$ to a decimal. We will use this example to learn a strategy using equivalent fractions that can be applied to converting any fraction to a decimal.
What is true about these fractions and why?

\[
\frac{5}{8}, \quad \frac{10}{16}, \quad \frac{50}{80}
\]

- The fractions are equivalent. In all cases, when the numerator and denominator of \( \frac{5}{8} \) are multiplied by the same factor it produces one of the other fractions. For example, \( \frac{5 \times 2}{8 \times 2} = \frac{10}{16} \) and \( \frac{5 \times 10}{8 \times 10} = \frac{50}{80} \).

What would happen if we chose \( 10^3 \) as this factor? We will still produce an equivalent fraction, but note how we use the factor of \( 10^3 \) in writing the decimal expansion of the fraction.

\[
\frac{5}{8} = 5 \times \frac{10^3}{8} \times \frac{1}{10^3}
\]

\[
= \frac{5000}{8} \times \frac{1}{10^3}
\]

Now we use what we know about division with remainders for \( \frac{5000}{8} \):

\[
= 625 \times 8 + 0 \times \frac{1}{10^3}
\]

\[
= \left( 625 + \frac{0}{8} \right) \times \frac{1}{10^3}
\]

\[
= 625 \times \frac{1}{10^3}
\]

\[
= 0.625
\]

Because of our work with Opening Exercise 1, we knew ahead of time that using \( 10^3 \) will help us achieve our goal. However, any power of 10 would achieve the same result. Assume we used \( 10^5 \) instead. Do you think our answer would be the same?

- Yes, it should be the same, but I would have to do the work to check it.

Let's verify that our result would be the same if we used \( 10^5 \).

\[
\frac{5}{8} = 5 \times \frac{10^5}{8} \times \frac{1}{10^5}
\]

\[
= \frac{500000}{8} \times \frac{1}{10^5}
\]

\[
= \left( 62500 \times 8 + 0 \right) \times \frac{1}{10^5}
\]

\[
= \left( 62500 + \frac{0}{8} \right) \times \frac{1}{10^5}
\]

\[
= 62500 \times \frac{1}{10^5}
\]

\[
= \frac{62500}{10^5}
\]

\[
= 0.62500
\]

\[
= 0.625
\]

Using \( 10^5 \) resulted in the same answer. Now we know that we can use any power of 10 with the method of converting a fraction to a decimal.
• This process of selecting a power of 10 to use is similar to putting zeroes after the decimal point when we do the long division. You do not quite know how many zeroes you will need, and if you put extra that’s ok! Using lower powers of 10 can make things more complicated. It is similar to not including enough zeroes when doing the long division. For that reason, it is better to use a higher power of 10 because we know the extra zeroes will not change the value of the fraction nor its decimal expansion.

Example 2 (5 minutes)

Example 2
Write the decimal expansion of the fraction \(\frac{17}{125}\).

• We go through the same process to convert \(\frac{17}{125}\) to a finite decimal. We know from Opening Exercise 2 that we need to use \(10^3\) to write \(\frac{17}{125}\) as a finite decimal, but from the last example we know that any power of 10 will work: \[
\frac{17}{125} = \frac{17 \times 10^3}{125} \times \frac{1}{10^3}
\]

• What do we do next?
  - Since \(17 \times 10^3 = 17,000\), we need to do division with remainder for \(\frac{17,000}{125}\).

• Do the division and write the next step.
  - \(\frac{17,000}{125} = 136\), then \(\frac{17}{125} = \frac{136 \times 125 + 0}{125} \times \frac{1}{10^3}\)

Check to make sure all students have the equation above; then instruct them to finish the work and write \(\frac{17}{125}\) as a finite decimal.

\[
= 136 \times \frac{1}{10^3}
= 136
= 0.136
\]

Verify that students have the correct decimal; then work on Example 3.

Example 3 (7 minutes)

Example 3
Write the decimal expansion of the fraction \(\frac{35}{11}\).
Now we apply this strategy to a fraction, \( \frac{35}{11} \), that is not a finite decimal. How do you know it’s not a finite decimal?

- We know that the fraction will not be a finite decimal because the denominator is not a product of 2’s and/or 5’s.

What do you think the difference will be in our work?

- When we do the division with remainder, we will likely get a remainder, where the first two examples had a remainder of 0.

Let’s use \( 10^6 \) to make sure we get enough decimal digits in order to get a good idea of what the infinite decimal is:

\[
\frac{35}{11} = 35 \times 10^6 \div 11 \times \frac{1}{10^6}
\]

What do we do next?

- Since \( 35 \times 10^6 = 35,000,000 \), we need to do division with remainder for \( \frac{35,000,000}{11} \).

We need to determine what numbers make the following statement true:

\[
35,000,000 = \_ \times 11 + \_.
\]

- 3,181,818 and 2 would give us \( 35,000,000 = 3,181,818 \times 11 + 2 \).

With this information, we can continue the process:

\[
\frac{35}{11} = \frac{3181818 \times 11 + 2}{11} \times \frac{1}{10^6}
\]

At this point we have a fairly good estimation of the decimal expansion of \( \frac{35}{11} \) as 3.181818. But we need to consider the value of \( \frac{2}{11} \times \frac{1}{10^6} \). We know that \( \frac{2}{11} < 1 \).

By the Basic Inequality, we know that

\[
\frac{2}{11} \times \frac{1}{10^6} < 1 \times \frac{1}{10^6}
\]

\[
\frac{2}{11} \times \frac{1}{10^6} < \frac{1}{10^6}
\]

Which means that the value of \( \frac{2}{11} \times \frac{1}{10^6} \) is less than 0.000001, and we have confirmed that 3.181818 is a good estimation of the infinite decimal that is equal to \( \frac{35}{11} \).
Example 4 (8 minutes)

Example 4

Write the decimal expansion of the fraction \( \frac{6}{7} \).

- Let’s write the decimal expansion of \( \frac{6}{7} \). Will it be a finite or infinite decimal? How do you know?
  - We know that the fraction will not be a finite decimal because the denominator is not a product of 2’s and/or 5’s.
- We want to make sure we get enough decimal digits in order to get a good idea of what the infinite decimal is. What power of 10 should we use?
  - Accept any power of 10 students give. Since we know it’s an infinite decimal, \( 10^6 \) should be sufficient to make a good estimate of the value of \( \frac{35}{11} \), but any power of 10 greater than 6 will work too. The work below uses \( 10^6 \).
- Using \( 10^6 \) we have
  \[
  \frac{6}{7} = \frac{6 \times 10^6}{7} \times \frac{1}{10^6}
  \]
  What do we do next?
  - Since \( 6 \times 10^6 = 6,000,000 \), we need to do division with remainder for \( \frac{6,000,000}{7} \).
- Determine which numbers make the following statement true:
  \[
  6,000,000 = \underline{857,142} \times 7 + \underline{6}
  \]
  - 857,142 and 6 would give us \( 6,000,000 = 857,142 \times 7 + 6 \)
- Now we know that
  \[
  \frac{6}{7} = \frac{857142 \times 7 + 6}{7} \times \frac{1}{10^6}
  \]
  Finish the work to write the decimal expansion of \( \frac{6}{7} \).
  - Sample response:
    \[
    \frac{6}{7} = \left( \frac{857142 \times 7 + 6}{7} \right) \times \frac{1}{10^6}
    = \left( 857142 + \frac{6}{7} \right) \times \frac{1}{10^6}
    = 857142 \times \frac{1}{10^6} + \left( \frac{6}{7} \times \frac{1}{10^6} \right)
    = \frac{857142}{10^6} + \left( \frac{6}{7} \times \frac{1}{10^6} \right)
    = 0.857142 + \left( \frac{6}{7} \times \frac{1}{10^6} \right)
    \]
Again we can verify how good our estimate is using the Basic Inequality:

\[
\frac{6}{7} < 1 \\
\frac{6}{7} \times \frac{1}{10^6} < 1 \times \frac{1}{10^6} \\
\frac{6}{7} \times \frac{1}{10^6} < \frac{1}{10^6}
\]

Therefore, \( \frac{6}{7} \times \frac{1}{10^6} < 0.000001 \) and stating that \( \frac{6}{7} = 0.857142 \) is a good estimate.

**Exercises 3–5 (5 minutes)**

Students complete Exercises 3–5 independently or in pairs. Allow students to use a calculator to check their work.

**Exercises 3–5**

3. a. Choose a power of ten to use to convert this fraction to a decimal: \( \frac{4}{13} \). Explain your choice.

   Choices will vary. The work shown below uses the factor \( 10^6 \). Students should choose a factor of at least \( 10^4 \) in order to get an approximate decimal expansion and a small remainder that will not greatly affect the value of the number.

   b. Determine the decimal expansion of \( \frac{4}{13} \) and verify you are correct using a calculator.

\[
\frac{4}{13} = 4 \times \frac{10^6}{13} \times \frac{1}{10^6} \\
= \frac{4,000,000}{13} \times \frac{1}{10^6} \\
4,000,000 = 307,692 \times 13 + 4
\]

The decimal expansion of \( \frac{4}{13} \) is approximately 0.307692.

4. Write the decimal expansion of \( \frac{1}{11} \). Verify you are correct using a calculator.

\[
\frac{1}{11} = 1 \times \frac{10^6}{11} \times \frac{1}{10^6} \\
= \frac{1,000,000}{11} \times \frac{1}{10^6} \\
1,000,000 = 90,909 \times 11 + 1
\]

The decimal expansion of \( \frac{1}{11} \) is approximately 0.0909090.
### Closing (5 minutes)

Summarize, or ask students to summarize, the main points from the lesson:

- We know how to write the decimal expansion for any fraction.
- Using what we know about equivalent fractions, we can multiply a fraction by a power of 10 large enough to give us enough decimal digits to estimate the decimal expansion of a fraction.
- We know that the amount we do not include in the decimal expansion is a very small amount that will not change the value of the number in any meaningful way.
Lesson Summary

Multiplying a fraction’s numerator and denominator by the same power of 10 to determine its decimal expansion is similar to including extra zeroes to the right of a decimal when using the long division algorithm. The method of multiplying by a power of 10 reduces the work to whole number division.

Example: We know that the fraction \( \frac{5}{3} \) has an infinite decimal expansion because the denominator is not a product of 2's and/or 5's. Its decimal expansion is found by the following procedure:

\[
\frac{5}{3} = \frac{5 \times 10^2}{3 \times 10^2} \times \frac{1}{10^2}
\]

Multiply numerator and denominator by \( 10^2 \)

\[
= \frac{166 \times 3 + 2}{3 \times 10^2} \times \frac{1}{10^2}
\]

Rewrite the numerator as a product of a number multiplied by the denominator

\[
= \frac{166 \times 3 + 2 \times 1}{3 \times 10^2}
\]

Rewrite the first term as a sum of fractions with the same denominator

\[
= \frac{166}{3 \times 10^2} + \frac{2}{3} \times \frac{1}{10^2}
\]

Simplify

\[
= \frac{166}{3 \times 10^2} + \frac{2}{3} \times \frac{1}{10^2}
\]

Use the Distributive Property

\[
= \frac{166}{3 \times 10^2} + \frac{2 \times 1}{3 \times 10^2}
\]

Simplify

\[
= 166 \times \frac{1}{3 \times 10^2} + \frac{2 \times 1}{3 \times 10^2}
\]

Simplify the first term using what you know about place value

Notice that the value of the remainder, \( \frac{2 \times 1}{3 \times 10^2} \), is quite small and does not add much value to the number. Therefore, 1.66 is a good estimate of the value of the infinite decimal for the fraction \( \frac{5}{3} \).

Exit Ticket (5 minutes)
Lesson 9: Decimal Expansions of Fractions, Part 1

Exit Ticket

1. Write the decimal expansion of \( \frac{823}{40} \).

2. Write the decimal expansion of \( \frac{48}{21} \).
Exit Ticket Sample Solutions

1. Write the decimal expansion of $\frac{823}{40}$.

\[
\frac{823}{40} = \frac{823 \times 10^3}{40 \times 10^3} = \frac{823000}{40} = \frac{20575 \times 40 + 1}{40} = 20575 + \frac{1}{40} = 20575.0025
\]

The decimal expansion of $\frac{823}{40}$ is approximately 20.575.

2. Write the decimal expansion of $\frac{48}{21}$.

\[
\frac{48}{21} = \frac{48 \times 10^6}{21 \times 10^6} = \frac{48000000}{21} = \frac{2285714 \times 21 + 6}{21} = 2285714 + \frac{6}{21} = 2285714.2857
\]

The decimal expansion of $\frac{48}{21}$ is approximately 2.285714.

Problem Set Sample Solutions

1. a. Choose a power of ten to convert this fraction to a decimal: $\frac{4}{11}$. Explain your choice.

Choices will vary. The work shown below uses the factor $10^5$. Students should choose a factor of at least $10^4$ in order to get an approximate decimal expansion and notice that the decimal expansion repeats.
b. Determine the decimal expansion of \( \frac{4}{11} \) and verify you are correct using a calculator.

\[
\frac{4}{11} = \frac{4 \times 10^6}{11} \times \frac{1}{10^6} = \frac{4000000}{11} \times \frac{1}{10^6}
\]

\[
4,000,000 = 363,636 \times 11 + 4
\]

The decimal expansion of \( \frac{4}{11} \) is approximately 0.363636.

2. Write the decimal expansion of \( \frac{5}{13} \). Verify you are correct using a calculator.

\[
\frac{5}{13} = \frac{5 \times 10^6}{13} \times \frac{1}{10^6} = \frac{5000000}{13} \times \frac{1}{10^6}
\]

\[
5,000,000 = 384,615 \times 13 + 5
\]

The decimal expansion of \( \frac{5}{13} \) is approximately 0.384615.

3. Write the decimal expansion of \( \frac{23}{39} \). Verify you are correct using a calculator.

\[
\frac{23}{39} = \frac{23 \times 10^6}{39} \times \frac{1}{10^6} = \frac{2300000}{39} \times \frac{1}{10^6}
\]

\[
23,000,000 = 589,743 \times 39 + 23
\]

The decimal expansion of \( \frac{23}{39} \) is approximately 0.589743.
4. Tamer wrote the decimal expansion of \( \frac{3}{7} \) as 0.418571, but when he checked it on a calculator it was 0.428571. Identify his error and explain what he did wrong.

\[
\frac{3}{7} = \frac{3 \times 10^6}{7} \times \frac{1}{10^6} = \frac{300000}{7} \times \frac{1}{10^6}
\]

\[
3,000,000 = 418,571 \times 7 + 3
\]

Tamer did the division with remainder incorrectly. He wrote that \(3,000,000 = 418,571 \times 7 + 3\) when actually \(3,000,000 = 428,571 \times 7 + 3\). This error led to his decimal expansion being incorrect.

5. Given that \( \frac{6}{7} = 0.857142 + \left( \frac{6}{7} \times \frac{1}{10^6} \right) \)

Explain why 0.857142 is a good estimate of \( \frac{6}{7} \).

When you consider the value of \( \frac{6}{7} \times \frac{1}{10^6} \), then it is clear that 0.857142 is a good estimate of \( \frac{6}{7} \). We know that \( \frac{6}{7} < 1 \). By the Basic Inequality, we also know that \( \frac{6}{7} \times \frac{1}{10^6} < 1 \times \frac{1}{10^6} \) which means that \( \frac{6}{7} \times \frac{1}{10^6} < 0.000001 \).

That is such a small value that it will not affect the estimate of \( \frac{6}{7} \) in any real way.
Lesson 10: Converting Repeating Decimals to Fractions

Student Outcomes

- Students know the intuitive reason why every repeating decimal is equal to a fraction. Students convert a decimal expansion that eventually repeats into a fraction.
- Students know that the decimal expansions of rational numbers repeat eventually.
- Students understand that irrational numbers are numbers that are not rational. Irrational numbers cannot be represented as a fraction and have infinite decimals that never repeat.

Classwork

Discussion (4 minutes)

- We have just seen that every fraction (therefore every rational number) is equal to a repeating decimal, and we have learned strategies for determining the decimal expansion of fractions. Now we must learn how to write a repeating decimal as a fraction.
- We begin by noting a simple fact about finite decimals: Given a finite decimal, such as 1.2345678, if we multiply the decimal by $10^5$ we get 123,456.78. That is, when we multiply by a power of 10, in this case $10^5$, the decimal point is moved 5 places to the right, i.e.,
  \[ 1.2345678 \times 10^5 = 123,456.78 \]
  This is true because of what we know about the Laws of Exponents:
  \[ 10^5 \times 1.2345678 = 10^5 \times (12,345,678 \times 10^{-7}) \]
  \[ = 12,345,678 \times 10^{-2} \]
  \[ = 123,456.78 \]
- We have discussed in previous lessons that we treat infinite decimals as finite decimals in order to compute with them. For that reason, we will now apply the same basic fact we observed about finite decimals to infinite decimals. That is,\n  \[ 1.2345678 \ldots \times 10^5 = 123,456.78 \ldots \]
  We will use this fact to help us write infinite decimals as fractions.

Example 1 (10 minutes)

Example 1

Find the fraction that is equal to the infinite decimal $0.\overline{81}$.

- We want to find the fraction that is equal to the infinite decimal $0.\overline{81}$.
- We let $x = 0.\overline{81}$. 
Allow students time to work in pairs or small groups to write the fraction equal to 0.81\(\overline{2}\). Students should recognize that the preceding discussion has something to do with this process and should be an entry point for finding the solution. They should also recognize that since we let \(x = 0.81\overline{2}\), an equation of some form will lead them to the fraction. Give them time to make sense of the problem. Make a plan for finding the fraction, and then attempt to figure it out.

- Since \(x = 0.81\overline{2}\), we will multiply both sides of the equation by \(10^2\) and then solve for \(x\). We will multiply by \(10^2\) because there are two decimal digits that repeat immediately following the decimal point.

\[
\begin{align*}
x &= 0.81\overline{2} \\
x &= 0.818181\ldots \\
10^2x &= (10^2)0.818181\ldots \\
100x &= 81.818181\ldots
\end{align*}
\]

Ordinarily we would finish solving for \(x\) by dividing both sides of the equation by 100. Do you see why that is not a good plan for this problem?

- If we divide both sides by 100, we would get \(x = \frac{81.8181\ldots}{100}\), which does not really show us that the repeating decimal is equal to a fraction (rational number) because the repeating decimal is still in the numerator.

- We know that \(81.8181\ldots\) is the same as \(81 + 0.8181\ldots\). Then by substitution, we have \(100x = 81 + 0.8181\ldots\).

How can we rewrite \(100x = 81 + 0.8181\ldots\) in a useful way using the fact that \(x = 0.81\overline{2}\)?

- We can rewrite \(100x = 81 + 0.8181\ldots\) as \(100x = 81 + x\) because \(x\) represents the repeating decimal block \(0.8181\ldots\).

Now we can solve for \(x\) to find the fraction that represents the repeating decimal \(0.81\overline{2}\):

\[
\begin{align*}
100x &= 81 + x \\
100x - x &= 81 + x - x \\
(100 - 1)x &= 81 \\
99x &= 81 \\
99x &= 81 \\
x &= \frac{81}{99} \\
x &= \frac{9}{11}
\end{align*}
\]

Therefore, the repeating decimal \(0.81\overline{2}\) = \(\frac{9}{11}\).

Have students verify that we are correct using a calculator.

**Exercises 1–2 (5 minutes)**

Students complete Exercises 1–2 in pairs. Allow students to use a calculator to check their work.

**Exercises 1–2**

1. a. Let \(x = 0.123\overline{4}\). Explain why multiplying both sides of this equation by \(10^3\) will help us determine the fractional representation of \(x\).

   *When we multiply both sides of the equation by \(10^3\), on the right side we will have 123.123123\ldots. This is helpful because we will be able to subtract the repeating decimal from both sides by subtracting \(x\).*
b. After multiplying both sides of the equation by $10^3$, rewrite the resulting equation by making a substitution that will help determine the fractional value of $x$. Explain how you were able to make the substitution.

$x = 0.123$

$10^3x = (10^3)0.123$

$1,000x = 123.123$...

$1,000x = 123 + x$

*Since we let $x = 0.123$, we can substitute the repeating decimal $0.123123...$ with $x.*

c. Solve the equation to determine the value of $x$.

$1,000x - x = 123 + x - x$

$999x = 123$

$999x = 123$

$x = \frac{123}{999}$

$x = \frac{41}{333}$

d. Is your answer reasonable? Check your answer using a calculator.

*Yes, my answer is reasonable and correct. It is reasonable because the denominator cannot be expressed as a product of 2’s and 5’s; therefore, I know that the fraction must represent an infinite decimal. It is also reasonable because the decimal value is closer to 0 than to 0.5, and the fraction $\frac{41}{333}$ is also closer to 0 than to $\frac{1}{2}$. It is correct because the division of $\frac{41}{333}$ using a calculator is 0.123123...*.

2. Find the fraction equal to 0.4. Check that you are correct using a calculator.

*Let $x = 0.4$*

$x = 0.4$

$10x = (10)0.4$

$10x = 4.4$

$10x = 4 + x$

$10x - x = 4 + x - x$

$9x = 4$

$x = \frac{4}{9}$

$x = \frac{4}{9}$

Example 2 (6 minutes)

**Example 2**

Find the fraction that is equal to the infinite decimal 2.138.

*We want to find the fraction that is equal to the infinite decimal 2.138. Notice that this time there is just one digit that repeats, but it is three places to the right of the decimal point. If we let $x = 2.138$, by what power of 10 should we multiply? Explain.*

*The goal is to multiply by a power of 10 so that the only remaining decimal digits are those that repeat. For that reason, we should multiply by $10^3$.***
• We let $x = 2.13\overline{8}$, and multiply both sides of the equation by $10^2$.

$$x = 2.13\overline{8}$$
$$10^2x = (10^2)2.13\overline{8}$$
$$100x = 213.\overline{8}$$
$$100x = 213 + 0.\overline{8}$$

This time, we cannot simply subtract $x$ from each side. Explain why.

- Subtracting $x$ in previous problems allowed us to completely remove the repeating decimal. This time, $x = 2.13\overline{8}$, not just $0.\overline{8}$.

• What we will do now is treat $0.\overline{8}$ as a separate, mini-problem. Determine the fraction that is equal to $0.\overline{8}$.

- Let $y = 0.\overline{8}$.

$$y = 0.\overline{8}$$
$$10y = 8.\overline{8}$$
$$10y = 8 + 0.\overline{8}$$
$$10y = 8 + y$$
$$10y - y = 8 + y - y$$
$$9y = 8$$
$$y = \frac{8}{9}$$

Now that we know that $0.\overline{8} = \frac{8}{9}$, we will go back to our original problem:

$$100x = 213 + 0.\overline{8}$$
$$100x = 213 + \frac{8}{9}$$
$$100x = \frac{213 \times 9}{9} + \frac{8}{9}$$
$$100x = \frac{213 \times 9 + 8}{9}$$
$$100x = \frac{1925}{9}$$

$$\frac{1}{100}(100x) = \frac{1925}{9} \times \frac{1}{100}$$

$$x = \frac{1925}{900}$$
$$x = \frac{77}{36}$$
Exercises 3–4 (6 minutes)

Students complete Exercises 3–4 independently or in pairs. Allow students to use a calculator to check their work.

### Exercises 3–4

#### 3. Find the fraction equal to $\frac{1}{1.623}$. Check that you are correct using a calculator.

Let $x = 1.623$

\[
x = 1.623
\]

\[
10x = (10)1.623
\]

\[
10x = 16.23
\]

Let $y = 0.23$

\[
y = 0.23
\]

\[
10^2y = (10^2)0.23
\]

\[
100y = 23.23
\]

\[
100y + y = 23 + y
\]

\[
100y - y = 23 + y - y
\]

\[
99y = 23
\]

\[
\frac{99y}{99} = \frac{23}{99}
\]

\[
y = \frac{23}{99}
\]

\[
\frac{1}{10}(10x) = \frac{1}{10}\left(\frac{1.607}{99}\right)
\]

\[
x = \frac{1.607}{990}
\]

Therefore, $\frac{1}{1.623} = \frac{1.607}{990}$

#### 4. Find the fraction equal to $\frac{2}{2.960}$. Check that you are correct using a calculator.

Let $x = 2.960$

\[
x = 2.960
\]

\[
10x = (10)2.960
\]

\[
10x = 29.60
\]

Let $y = 0.60$

\[
y = 0.60
\]

\[
10^2y = (10^2)0.60
\]

\[
100y = 60.60
\]

\[
100y + y = 60 + y
\]

\[
100y - y = 60 + y - y
\]

\[
99y = 60
\]

\[
\frac{99y}{99} = \frac{60}{99}
\]

\[
y = \frac{60}{99}
\]

\[
\frac{1}{10}(10x) = \frac{1}{10}\left(\frac{977}{33}\right)
\]

\[
x = \frac{977}{330}
\]

Therefore, $\frac{2}{2.960} = \frac{977}{330}$
Discussion (4 minutes)

- What we have observed so far is that when an infinite decimal repeats, it can be written as a fraction, which means that it is a rational number. Do you think infinite decimals that do not repeat are rational as well? Explain.

Provide students time to discuss with a partner before sharing their thoughts with the class.

- Considering the work from this lesson, it does not seem reasonable that an infinite decimal that does not repeat can be expressed as a fraction. We would not have a value that we could set for $x$ and use to compute in order to find the fraction. For those reasons, we do not believe that an infinite decimal that does not repeat is a rational number.

- Infinite decimals that do not repeat are irrational numbers, that is, when a number is not equal to a rational number, it is irrational. What we will learn next is how to use rational approximation to determine the approximate decimal expansion of an irrational number.

Closing (5 minutes)

Summarize, or ask students to summarize, the main points from the lesson:

- We know that to work with infinite decimals we must treat them as finite decimals.
- We know how to use our knowledge of powers of 10 and linear equations to write an infinite decimal that repeats as a fraction.
- We know that every decimal that eventually repeats is a rational number.

Lesson Summary

Numbers with decimal expansions that repeat are rational numbers and can be converted to fractions using a linear equation.

Example: Find the fraction that is equal to the number $0.\overline{567}$.

Let $x$ represent the infinite decimal $0.\overline{567}$.

\[
\begin{align*}
x &= \overline{567} \\
10^3x &= 10^3(0.\overline{567}) \\
1000x &= 567.\overline{567} \\
1000x &= 567 + x \\
1000x - x &= 567 + x - x \\
999x &= 567 \\
999x &= 567 \\
999x &= 999 \\
x &= \frac{567}{999} \\
\end{align*}
\]

Multiply by $10^3$ because there are 3 digits that repeat

Simplify

By addition

Subtraction Property of Equality

Simplify

Division Property of Equality

Simplify

This process may need to be used more than once when the repeating digits do not begin immediately after the decimal. For numbers such as $1.\overline{26}$, for example.

Irrational numbers are numbers that are not rational. They have infinite decimals that do not repeat and cannot be represented as a fraction.

Exit Ticket (5 minutes)
Lesson 10: Converting Repeating Decimals to Fractions

Exit Ticket

1. Find the fraction equal to 0.5\overline{34}.

2. Find the fraction equal to 3.0\overline{15}.
Exit Ticket Sample Solutions

1. Find the fraction equal to 0.5\overline{34}.
   Let \(x = 0.\overline{534}\).
   \[x = 0.\overline{534}\]
   \[10^2x = (10^2)0.\overline{534}\]
   \[1,000x = 534.\overline{534}\]
   \[1,000x = 534 + x\]
   \[1,000x - x = 534 + x - x\]
   \[999x = 534\]
   \[999x = 534\]
   \[999 - 999\]
   \[x = 534\]
   \[999\]
   \[x = \frac{178}{333}\]

   \[0.\overline{534} = \frac{178}{333}\]

2. Find the fraction equal to 3.0\overline{15}.
   \[\text{Let } x = 3.0\overline{15}\]
   \[\text{Let } y = 0.\overline{15}\]
   \[x = 3.0\overline{15}\]
   \[y = 0.\overline{15}\]
   \[10x = (10)3.0\overline{15}\]
   \[10y = 15.\overline{15}\]
   \[100y = 15 + y\]
   \[100y - y = 15 + y - y\]
   \[99y = 15\]
   \[99y = 15\]
   \[99 - 99\]
   \[y = \frac{5}{33}\]

   \[3.0\overline{15} = \frac{199}{66}\]
1. a. Let \( x = 0.6\overline{3}1 \). Explain why multiplying both sides of this equation by \( 10^3 \) will help us determine the fractional representation of \( x \).

   When we multiply both sides of the equation by \( 10^3 \), on the right side we will have \( 631.631631 \ldots \). This is helpful because we will be able to subtract the repeating decimal from both sides by subtracting \( x \).

b. After multiplying both sides of the equation by \( 10^3 \), rewrite the resulting equation by making a substitution that will help determine the fractional value of \( x \). Explain how you were able to make the substitution.

\[
x = 0.6\overline{3}1 \\
10^3x = (10^3)0.6\overline{3}1 \\
1,000x = 631.631 \\
1,000x = 631 + 0.631631 \ldots \\
1,000x = 631 + x
\]

Since we let \( x = 0.6\overline{3}1 \), we can substitute the repeating decimal \( 0.6\overline{3}1 \) with \( x \).

c. Solve the equation to determine the value of \( x \).

\[
1,000x - x = 631 + x - x \\
999x = 631 \\
999x = 631 \\
x = \frac{631}{999}
\]

d. Is your answer reasonable? Check your answer using a calculator.

Yes, my answer is reasonable and correct. It is reasonable because the denominator cannot be expressed as a product of 2's and 5's; therefore, I know that the fraction must represent an infinite decimal. Also the number \( 0.6\overline{3}1 \) is closer to \( 0.5 \) than \( 1 \), and the fraction is also closer to \( \frac{1}{2} \) than \( 1 \). It is correct because the division \( \frac{631}{999} \) using the calculator is \( 0.631631 \ldots \).

2. Find the fraction equal to \( 3.4\overline{0}8 \). Check that you are correct using a calculator.

Let \( x = 3.4\overline{0}8 \)

\[
x = 3.4\overline{0}8 \\
10^2x = (10^2)3.4\overline{0}8 \\
100x = 340.8
\]

Let \( y = 0.\overline{8} \)

\[
y = 0.\overline{8} \\
y = 10(0.\overline{8}) \\
y = 8.\overline{8} \\
y = 8 + y \\
10y - y = 8 + y - y \\
9y = 8 \\
y = \frac{8}{9}
\]

\[
x = \frac{3.068}{9} \\
100x = \frac{3.068}{9} \\
x = \frac{3.068}{9} \\
x = \frac{990}{225} = \frac{767}{225}
\]
3. Find the fraction equal to 0.5923. Check that you are correct using a calculator.

\[ x = 0.5923 \]

\[ 10^4x = (10^4)0.5923 \]
\[ 10,000x = 5,923.5923 \]
\[ 10,000x = 5,923 + x \]
\[ 10,000x = 5,923 + x \]
\[ 9,999x = 5,923 \]
\[ 9,999x = 5,923 \]
\[ x = \frac{5,923}{9,999} \]

4. Find the fraction equal to 2.382. Check that you are correct using a calculator.

\[ x = 2.382 \]
\[ y = 0.82 \]
\[ 10x = 23.82 \]
\[ 10^2y = (10^2)0.82 \]
\[ 100y = 82.82 \]
\[ 100y = 82 + y \]
\[ 100y = 82 + y \]
\[ 99y = 82 \]
\[ 99y = 82 \]
\[ y = \frac{82}{99} \]
\[ 2.382 = \frac{2,359}{990} \]

5. Find the fraction equal to 0.714285. Check that you are correct using a calculator.

\[ x = 0.714285 \]
\[ 10^6x = (10^6)0.714285 \]
\[ 1,000,000x = 714,825.714285 \]
\[ 1,000,000x = 714,285 + x \]
\[ 1,000,000x = 714,285 + x \]
\[ 999,999x = 714,285 \]
\[ 999,999x = 714,285 \]
\[ 999,999x = 999,999 \]
\[ x = \frac{714,285}{999,999} \]
\[ x = \frac{5}{7} \]

6. Explain why an infinite decimal that is not a repeating decimal cannot be rational.

Infinite decimals that do repeat can be expressed as a fraction and are therefore rational. The method we learned today to write a repeating decimal as a rational number cannot be applied to infinite decimals that do not repeat. The method requires that we let \( x \) represent the repeating part of the decimal. If the number has a decimal expansion that does not repeat, we cannot express the number as a fraction, i.e., a rational number.
7. In a previous lesson we were convinced that it is acceptable to write $0.\overline{9} = 1$. Use what you learned today to show that it is true.

Let $x = 0.\overline{9}$

\[
\begin{align*}
x &= 0.\overline{9} \\
10x &= (10)0.\overline{9} \\
10x &= 9.\overline{9} \\
10x &= 9 + x \\
10x - x &= 9 + x - x \\
9x &= 9 \\
\frac{9x}{9} &= \frac{9}{9} \\
x &= \frac{9}{9} \\
x &= 1
\end{align*}
\]

8. Examine the following repeating infinite decimals and their fraction equivalents. What do you notice? Why do you think what you observed is true?

\[
\begin{align*}
0.\overline{81} &= \frac{81}{99} \\
0.\overline{4} &= \frac{4}{9} \\
0.\overline{123} &= \frac{123}{999} \\
0.\overline{60} &= \frac{60}{99} \\
0.\overline{9} &= 1.0
\end{align*}
\]

In each case, the fraction that represents the infinite decimal has a numerator that is exactly the repeating part of the decimal and a denominator comprised of $9$’s. Specifically, the denominator has the same number of digits of $9$’s as the number of digits that repeat. For example, $0.\overline{81}$ has two repeating decimal digits, so the denominator has two $9$’s. Since we know that $0.\overline{9} = 1$, we can make the assumption that repeating $9$’s, like $99$ could be expressed as $100$, meaning that the fraction $\frac{81}{99}$ is almost $\frac{81}{100}$, which would then be expressed as $0.81$. 

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Lesson 11: The Decimal Expansion of Some Irrational Numbers

Student Outcomes

- Students use rational approximation to get the approximate decimal expansion of numbers like $\sqrt{3}$ and $\sqrt{28}$.
- Students distinguish between rational and irrational numbers based on decimal expansions.

Lesson Notes

The definition of an irrational number can finally be given and understood completely once students know that the decimal expansion of non-perfect squares like $\sqrt{3}$ and $\sqrt{28}$ are infinite and do not repeat. That is, square roots of non-perfect squares cannot be expressed as rational numbers and are therefore defined as irrational numbers.

Classwork

Opening Exercise (5 minutes)

Lead a discussion where students share their reasoning as to the placement of $\sqrt{28}$ on the number line. Encourage students to critique the reasoning of others while evaluating their own arguments. Consider having students vote on the placement they think is most correct.

Discussion (10 minutes)

- We have studied the properties of rational numbers; today we will finally be able to characterize those numbers that are not rational.
- So far we have been able to estimate the size of a number like $\sqrt{3}$ by stating that it is between the two perfect squares $\sqrt{1}$ and $\sqrt{4}$, meaning that $\sqrt{3}$ is between 1 and 2 but closer to 2. In our work so far we have found the decimal expansion of numbers using long division and by inspecting the denominators for products of 2’s and 5’s. Numbers written with a square root symbol are different and require a different method for determining their decimal expansions. The method we will learn is called rational approximation: using a sequence of rational numbers to get closer and closer to a given number to estimate the value of a number.
Example 1

Recall the Basic Inequality:
Let \( c \) and \( d \) be two positive numbers, and let \( n \) be a fixed positive integer. Then \( c < d \) if and only if \( c^n < d^n \).

Write the decimal expansion of \( \sqrt{3} \).

First approximation:
- We will use the Basic Inequality that we learned in Lesson 3:
  Let \( c \) and \( d \) be two positive numbers, and let \( n \) be a fixed positive integer. Then \( c < d \) if and only if \( c^n < d^n \).
- To write the decimal expansion of \( \sqrt{3} \) we first determine between which two integers the number \( \sqrt{3} \) would lie on the number line. This is our first approximation. What are those integers?
  - The number \( \sqrt{3} \) will be between 1 and 2 on the number line because \( 1^2 = 1 \) and \( 2^2 = 4 \).
- With respect to the Basic Inequality, we can verify that \( \sqrt{3} \) lies between the integers 1 and 2 because \( 1^2 < \sqrt{3} < 2^2 \).
- To be more precise with our estimate of \( \sqrt{3} \), we now look at the tenths between the numbers 1 and 2. This is our second approximation.

Second approximation:
- The question becomes, where exactly would \( \sqrt{3} \) lie on this magnified version of the number line? There are 10 possibilities: \( 1.0 < \sqrt{3} < 1.1, 1.1 < \sqrt{3} < 1.2, 1.2 < \sqrt{3} < 1.3, \ldots, \) or \( 1.9 < \sqrt{3} < 2.0 \). Use of the Basic Inequality can guide us to selecting the correct possibility. Specifically, we need to determine which of the inequalities shown below is correct:
  - \( 1.0^2 < (\sqrt{3})^2 < 1.1^2, \ 1.1^2 < (\sqrt{3})^2 < 1.2^2, \ 1.2^2 < (\sqrt{3})^2 < 1.3^2, \ldots, \) or \( 1.9^2 < (\sqrt{3})^2 < 2.0^2 \).
  - With the help of a calculator we can see that \( 1.7^2 < (\sqrt{3})^2 < 1.8^2 \) because \( 1.7^2 = 2.89 \) and \( 1.8^2 = 3.24 \); therefore, \( 1.7 < \sqrt{3} < 1.8 \).
- What do you think will need to be done to get an even more precise estimate of the number \( \sqrt{3} \)?
  - We will need to look at the interval between 1.7 and 1.8 more closely and repeat the process we did before.
- Looking at the increments between 1.7 and 1.8, we again have 10 possibilities. This is our third approximation.

Third approximation:
The Basic Inequality and the help of a calculator show that $\sqrt{3}$ will be between 1.73 and 1.74 because

$1.73^2 < (\sqrt{3})^2 < 1.74^2$.

Have students verify using a calculator that $1.73^2 = 2.9929$ and $1.74^2 = 3.0276$ and ultimately that $1.73^2 < (\sqrt{3})^2 < 1.74^2$.

What do you think will need to be done to get an even more precise estimate of the number $\sqrt{3}$?

- We will need to look at the interval between 1.73 and 1.74 more closely and repeat the process we did before.

At this point the pattern should be clear. Now to look more carefully at what we are actually doing. We began by looking at the sequence of integers, specifically between two positive integers 1 and 2. Think of this interval as $10^0$ (because it equals 1). Then we looked at the sequence of tenths between 1 and 2; think of this interval as $10^{-1}$ (because it equals $\frac{1}{10}$). Then we looked at the sequence of hundredths between 1.7 and 1.8; think of this interval as $10^{-2}$ (because it equals $\frac{1}{100}$). To determine the location of $\sqrt{3}$, we had to look between points that differ by $10^{-n}$ for any positive integer $n$. The intervals we investigate, i.e., $10^{-n}$, get increasingly smaller as $n$ gets larger.

This method of looking at successive intervals is what we call rational approximation. With each new interval we are approximating the value of the number by determining which two rational numbers it lies between.

Discussion (15 minutes)

The following discussion revisits the Opening Exercise. Before you begin, ask students to reevaluate their own reasoning, and if you had them vote, consider asking them to vote again to see if anyone wants to change their mind about the best approximation for $\sqrt{28}$.

Example 2

Write the decimal expansion of $\sqrt{28}$.

First approximation:

- We will use the method of rational approximation to estimate the location of $\sqrt{28}$ on the number line.

- What interval of integers, i.e., an interval equal to $10^0$, do we examine first? Explain.

  - We must examine the interval between 5 and 6 because $5^2 < (\sqrt{28})^2 < 6^2$, i.e., $25 < 28 < 36$.
Now we examine the interval of tenths, i.e., $10^{-1}$, between 5 and 6. Where might $\sqrt{28}$ lie?

**Second approximation:**

- The number $\sqrt{28}$ will lie between 5.0 and 5.1 or 5.1 and 5.2 or...5.9 and 6.0.

How do we determine which interval is correct?

- We must use the Basic Inequality to check each interval. For example, we need to see if the following inequality is true: $5.0^2 < (\sqrt{28})^2 < 5.1^2$

Before we begin checking each interval, let’s think about how we can be more methodical in our approach. We know that $\sqrt{28}$ is between 5 and 6, but which integer is it closer to?

- The number $\sqrt{28}$ will be closer to 5 than 6.

Then we should begin checking the intervals beginning with 5 and work our way up. If the number were closer to 6, then we would begin checking the intervals on the right first and work our way down.

Provide students time to determine which interval the number $\sqrt{28}$ will lie between. Ask students to share their findings, and then continue the discussion.

Now that we know that the number $\sqrt{28}$ lies between 5.2 and 5.3, let’s check intervals of hundredths, i.e., $10^{-2}$.

**Third approximation:**

- Again, we should try to be methodical. Since $5.2^2 = 27.04$ and $5.3^2 = 28.09$, where should we begin checking?

  - We should begin with the interval between 5.29 and 5.30 because 28 is closer to 28.09 compared to 27.04.

Provide students time to determine which interval the number $\sqrt{28}$ will lie between. Ask students to share their findings, and then continue the discussion.

Now we know that the number $\sqrt{28}$ is between 5.29 and 5.30. Let’s go one step further and examine intervals of thousandths, i.e., $10^{-3}$.

**Fourth approximation:**

- Since $5.29^2 = 27.9841$ and $5.30^2 = 28.09$, where should we begin our search?

  - We should begin with the interval between 5.290 and 5.291 because 28 is closer to 27.9841 compared to 28.09.
Provide students time to determine which interval the number \( \sqrt{28} \) will lie between. Ask students to share their findings, and then finish the discussion.

- The number \( \sqrt{28} \) lies between 5.291 and 5.292 because \( 5.291^2 = 27.994681 \) and \( 5.292^2 = 28.005264 \). At this point we have a fair approximation of the value of \( \sqrt{28} \). It is between 5.291 and 5.292 on the number line:

\[
\begin{array}{cccccccccc}
& & & & & & & & & & \\
5.290 & 5.291 & 5.292 & 5.293 & 5.294 & 5.295 & 5.296 & 5.297 & 5.298 & 5.299 & 5.300 \\
\end{array}
\]

- We could continue this process of rational approximation to see that \( \sqrt{28} = 5.291502622 \ldots \) How is this number different from other infinite decimals we have worked with?
  - Other infinite decimals we have worked with have a block of digits that repeat at some point. This infinite decimal does not.

- We know that rational numbers are those that have decimal expansions that eventually repeat. We also know that rational numbers can be expressed as a fraction in the form of a ratio of integers. In the last lesson we learned how to convert a repeating decimal to a fraction. Do you think that same process can be used with a number like \( \sqrt{28} = 5.291502622 \ldots \)?
  - No because the decimal expansion does not repeat.

- Because the number \( \sqrt{28} \) cannot be expressed as a rational number, we say that it is irrational. Any number that cannot be expressed as a rational number is, by definition, an irrational number.

- A common irrational number is pi: \( \pi = 3.14159265 \ldots \) Notice that the decimal expansion of pi is infinite and does not repeat. Those qualities are what make pi an irrational number. Often for computations we give pi a rational approximation of 3.14 or \( \frac{22}{7} \), but those are merely approximations, not the true value of the number pi.

- Another example of an irrational number is \( \sqrt{7} \). What do you expect the decimal expansion of \( \sqrt{7} \) to look like?
  - The decimal expansion of \( \sqrt{7} \) will be infinite without a repeating block.

- The number \( \sqrt{7} = 2.645751311 \ldots \) The decimal expansion is infinite and does not repeat.

- Is the number \( \sqrt{49} \) rational or irrational? Explain.
  - The number \( \sqrt{49} = 7 \). The decimal expansion of \( \sqrt{49} \) can be written as 7.0000 \ldots which is an infinite decimal expansion with a repeat block. Therefore, \( \sqrt{49} \) is a rational number.

- Classify the following numbers as rational or irrational. Be prepared to explain your reasoning.
  \[
  \sqrt{10}, \quad 0.123123123 \ldots, \quad \sqrt{64}, \quad \frac{5}{11}
  \]

Provide students time to classify the numbers. They can do this independently or in pairs. Then select students to share their reasoning. Students should identify \( \sqrt{10} \) as irrational because it has a decimal expansion that can only be approximated by rational numbers. The number 0.123123123 \ldots is a repeating decimal and can be expressed as a fraction and is therefore rational. The number \( \sqrt{64} = 8 \) and is therefore a rational number. The fraction \( \frac{5}{11} \) by definition is a rational number because it is a ratio of integers.

Consider going back to the Opening Exercise to determine whose approximation was the closest.
Exercise 2 (5 minutes)

Students work in pairs to complete Exercise 2.

Exercise 2

Between which interval of hundredths would $\sqrt{14}$ be located? Show your work.

The number $\sqrt{14}$ is between integers 3 and 4 because $3^2 < (\sqrt{14})^2 < 4^2$. Then $\sqrt{14}$ must be checked for the interval of tenths between 3 and 4. Since $\sqrt{14}$ is closer to 4, we will begin with the interval from 3.9 to 4.0. The number $\sqrt{14}$ is between 3.7 and 3.8 because $3.7^2 = 13.69$ and $3.8^2 = 14.44$. Now we must look at the interval of hundredths between 3.7 and 3.8. Since $\sqrt{14}$ is closer to 3.7, we will begin with the interval 3.70 to 3.71. The number $\sqrt{14}$ is between 3.74 and 3.75 because $3.74^2 = 13.9876$ and $3.75^2 = 14.0625$.

Closing (5 minutes)

Summarize, or ask students to summarize, the main points from the lesson:

- We know that any number that cannot be expressed as a rational number is an irrational number.
- We know that to determine the approximate value of an irrational number we must determine between which two rational numbers it would lie.
- We know that the method of rational approximation uses a sequence of rational numbers, in increments of $10^0$, $10^{-1}$, $10^{-2}$, and so on, to get closer and closer to a given number.
- We have a method for determining the approximate decimal expansion of the square root of an imperfect square, which is an irrational number.
Lesson Summary

To get the decimal expansion of a square root of a non-perfect square you must use the method of rational approximation. Rational approximation is a method that uses a sequence of rational numbers to get closer and closer to a given number to estimate the value of the number. The method requires that you investigate the size of the number by examining its value for increasingly smaller powers of 10 (i.e., tenths, hundredths, thousandths, and so on). Since $\sqrt{2}$ is not a perfect square, you would use rational approximation to determine its decimal expansion.

Example:

Begin by determining which two integers the number would lie.

$\sqrt{2}$ is between the integers 4 and 5 because $4^2 < (\sqrt{2})^2 < 5^2$, which is equal to $16 < 2 < 25$.

Next, determine which interval of tenths the number belongs.

$\sqrt{2}$ is between 4.6 and 4.7 because $4.6^2 < (\sqrt{2})^2 < 4.7^2$, which is equal to $21.16 < 2 < 22.09$.

Next, determine which interval of hundredths the number belongs.

$\sqrt{2}$ is between 4.69 and 4.70 because $4.69^2 < (\sqrt{2})^2 < 4.70^2$, which is equal to $21.9961 < 2 < 22.09$.

A good estimate of the value of $\sqrt{2}$ is 4.69 because 22 is closer to 21.9961 than it is to 22.09.

Notice that with each step we are getting closer and closer to the actual value, 22. This process can continue using intervals of thousandths, ten-thousandths, and so on.

Any number that cannot be expressed as a rational number is called an irrational number. Irrational numbers are those numbers with decimal expansions that are infinite and do not have a repeating block of digits.

Exit Ticket (5 minutes)
Lesson 11: The Decimal Expansion of Some Irrational Numbers

Exit Ticket

1. Determine the 3 decimal digit approximation of the number $\sqrt{17}$.

2. Classify the following numbers as rational or irrational, and explain how you know.
   
   $\frac{3}{5}, \ 0.73737373\ldots, \ \sqrt{31}$
Problem Set Sample Solutions

1. Use the method of rational approximation to determine the decimal expansion of \( \sqrt{17} \). Determine which interval of hundredths it would lie in.

   The number \( \sqrt{17} \) is between integers 4 and 5 because \( 4^2 < (\sqrt{17})^2 < 5^2 \). Since \( \sqrt{17} \) is closer to 4, I will start checking the tenths intervals closer to 4. \( \sqrt{17} \) is between 4.1 and 4.2 since 4.1 \( ^2 = 16.81 \) and 4.2 \( ^2 = 17.64 \).
   Checking the hundredths interval, \( \sqrt{17} \) is between 4.12 and 4.13 since 4.12 \( ^2 = 16.9444 \) and 4.13 \( ^2 = 17.0569 \).
   Checking the thousandths interval, \( \sqrt{17} \) is between 4.123 and 4.124 since 4.123 \( ^2 = 16.99129 \) and 4.124 \( ^2 = 17.007376 \). Since 17 is closer to 4.123 \( ^2 \) than 4.124 \( ^2 \), then the three decimal approximation is approximately 4.123.

2. Get a 3 decimal digit approximation of the number \( \sqrt{34} \).

   The number \( \sqrt{34} \) is between 5 and 6 but closer to 6. Looking at the interval of tenths, beginning with 5.9 to 6.0, the number \( \sqrt{34} \) lies between 5.8 and 5.9 because 5.8 \( ^2 = 33.64 \) and 5.9 \( ^2 = 34.81 \) and is closer to 5.8. In the interval of hundredths, the number \( \sqrt{34} \) lies between 5.83 and 5.84 because 5.83 \( ^2 = 33.9889 \) and 5.84 \( ^2 = 34.1056 \) and is closer to 5.83. In the interval of thousandths, the number \( \sqrt{34} \) lies between 5.830 and 5.831 because 5.830 \( ^2 = 33.9889 \) and 5.831 \( ^2 = 34.000561 \) but is closer to 5.831. Since 34 is closer to 5.831 \( ^2 \) than 5.830 \( ^2 \), then the 3 decimal digit approximation of the number is approximately 5.831.

3. Write the decimal expansion of \( \sqrt{47} \) to at least 2 decimal digits.

   The number \( \sqrt{47} \) is between 6 and 7 but closer to 7 because \( 6^2 < (\sqrt{47})^2 < 7^2 \). In the interval of tenths, the number \( \sqrt{47} \) is between 6.8 and 6.9 because 6.8 \( ^2 = 46.24 \) and 6.9 \( ^2 = 47.61 \). In the interval of hundredths, the number \( \sqrt{47} \) is between 6.85 and 6.86 because 6.85 \( ^2 = 46.9225 \) and 6.86 \( ^2 = 47.0596 \). Therefore, to 2 decimal digits, the number \( \sqrt{47} \) is approximately 6.85 but when rounded will be approximately 6.86 because \( \sqrt{47} \) is closer to 6.86 but not quite 6.86.
4. Write the decimal expansion of \( \sqrt{46} \) to at least 2 decimal digits.

   The number \( \sqrt{46} \) is between integers 6 and 7 because \( 6^2 < (\sqrt{46})^2 < 7^2 \). Since \( \sqrt{46} \) is closer to 7, I will start checking the tenths intervals between 6.9 and 7. \( \sqrt{46} \) is between 6.7 and 6.8 since \( 6.7^2 = 44.89 \) and \( 6.8^2 = 46.24 \). Checking the hundredths interval, \( \sqrt{46} \) is between 6.78 and 6.79 since \( 6.78^2 = 45.9684 \) and \( 6.79^2 = 46.1041 \). Since 46 is closer to 6.78 than 6.79, then the two decimal approximation is 6.78.

5. Explain how to improve the accuracy of decimal expansion of an irrational number.

   In order to improve the accuracy of the decimal expansion of an irrational number, you must examine increasingly smaller increments on the number line. Specifically, increments of decreasing powers of 10. The Basic Inequality allows us to determine which interval a number will be between. We begin by determining which two integers the number lies between and then decrease the power of 10 to look at the interval of tenths. Again using the Basic Inequality, we can narrow down the approximation to a specific interval of tenths. Then we look at the interval of hundredths and use the Basic Inequality to determine which interval of hundredths the number would lie between. Then we examine the interval of thousandths. Again the Basic Inequality allows us to narrow down the approximation to thousands. The more intervals that are examined, the more accurate the decimal expansion of an irrational number will be.

6. Is the number \( \sqrt{125} \) rational or irrational? Explain.

   The number \( \sqrt{125} \) is an irrational number because it has a decimal expansion that is infinite and does not repeat. That is, the number \( \sqrt{125} \) cannot be expressed as a rational number; therefore, it is irrational.

7. Is the number 0.646464646 ... rational or irrational? Explain.

   The number 0.646464646 ... = \( \frac{64}{99} \); therefore, it is a rational number. Not only is the number \( \frac{64}{99} \) a quotient of integers, but its decimal expansion is infinite with a repeating block of digits.

8. Is the number 3.741657387 ... rational or irrational? Explain.

   The number 3.741657387 ... is an irrational number because it has a decimal expansion that is infinite and does not repeat. That is, the number 3.741657387 ... cannot be expressed as a rational number; therefore, it is irrational.

9. Is the number \( \sqrt{99} \) rational or irrational? Explain.

   The number \( \sqrt{99} \) is an irrational number because it has a decimal expansion that is infinite and does not repeat. That is, the number \( \sqrt{99} \) cannot be expressed as a rational number; therefore, it is irrational.

10. Challenge: Get a 2 decimal digit approximation of the number \( \sqrt{9} \).

    The number \( \sqrt{9} \) is between integers 2 and 3 because \( 2^3 < (\sqrt{9})^3 < 3^3 \). Since \( \sqrt{9} \) is closer to 2, I will start checking the tenths intervals between 2 and 3. \( \sqrt{9} \) is between 2 and 2.1 since \( 2^3 = 8 \) and \( 2.1^3 = 9.261 \). Checking the hundredths interval, \( \sqrt{9} \) is between 2.08 and 2.09 since \( 2.08^2 = 8.9984 \) and \( 2.09^2 = 9.128129 \). Since 9 is closer to 2.08 than 2.09, the two decimal approximation is 2.08.
Lesson 12: Decimal Expansions of Fractions, Part 2

Student Outcomes
- Students apply the method of rational approximation to determine the decimal expansion of a fraction.
- Students relate the method of rational approximation to the long division algorithm.

Lesson Notes
In this lesson, students use the idea of intervals of tenths, hundredths, thousandths, and so on to determine the decimal expansion of rational numbers. Since there is an explicit value that can be determined, students use what they know about mixed numbers and operations with fractions to pin down specific digits as opposed to the guess and check method used with irrational numbers. The general strategy is for students to compare a fractional value, say \( \frac{2}{11} \), to a known decimal digit, that is \( \frac{2}{11} = 0.1 + \text{“something.”} \) Students find the difference between these two values, then work to find the next decimal digit in the expansion. The process continues until students notice a pattern in their work, leading them to recognize that the decimal expansion must be that of an infinite, repeating decimal block.

This lesson includes a fluency activity that will take approximately 10 minutes to complete. The fluency activity is a personal white board exchange with problems on volume that can be found at the end of the exercises for this lesson.

Classwork
Discussion (20 minutes)

Example 1
Write the decimal expansion of \( \frac{35}{11} \).

- Our goal is to write the decimal expansion of a fraction, in this case \( \frac{35}{11} \). To do so, begin by locating \( \frac{35}{11} \) on the number line. What is its approximate location? Explain.

The number \( \frac{35}{11} \) would lie between 3 and 4 on the number line because \( \frac{35}{11} = \frac{33}{11} + \frac{2}{11} = 3 + \frac{2}{11} \).

- The goal is to use rational approximation to determine the decimal expansion of a number, instead of having to check a series of intervals as we did with the decimal expansions of irrational numbers. To determine the decimal expansion of \( \frac{35}{11} \), focus only on the fraction \( \frac{2}{11} \). Then, methodically determine between which interval of tenths \( \frac{2}{11} \) would lie. Given that we are looking at an interval of tenths, can you think of a way to do this?
Provide time for students to discuss strategies in small groups; then, share their ideas with the class. Encourage students to critique the reasoning of their classmates.

- We know that \(\frac{35}{11}\) has a decimal expansion beginning with 3 in the ones place because \(\frac{35}{11} = 3 + \frac{2}{11}\). Now we want to determine the tenths digit, the hundredths digit, and then the thousandths digit.

3.

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- To figure out the tenths digit, we will use an inequality based on tenths. We are looking for the consecutive integers, \(m\) and \(m + 1\), that \(\frac{2}{11}\) would lie between when \(m\) and \(m + 1\) are intervals of tenths, i.e.:

\[
\frac{m}{10} < \frac{2}{11} < \frac{m + 1}{10}
\]

Give students time to make sense of the above inequality. Since the intervals of tenths are represented by \(\frac{m}{10}\) and \(\frac{m+1}{10}\), consider using concrete numbers, which is clearer than looking at consecutive intervals of tenths on the number line. The chart below may help students make sense of the intervals and the inequality.

Scaffolding:
An alternative way of asking this question is: “In which interval could we place the fraction \(\frac{2}{11}\)?” Show students the number line labeled with tenths.
Multiplying through by 10, we get:

\[
m < 10 \left( \frac{2}{11} \right) < m + 1
\]

\[
10 \left( \frac{2}{11} \right) = \frac{20}{11} = \frac{11 + 9}{11} = 1 + \frac{9}{11}
\]

This implies that \( m = 1 \). Why does the statement that \( 10 \left( \frac{2}{11} \right) = 1 + \frac{9}{11} \) imply that \( m = 1 \)?

- It implies that \( m = 1 \) because \( m \) and \( m + 1 \) are consecutive integers. Since \( 10 \left( \frac{2}{11} \right) = 1 + \frac{9}{11} = 1 + \frac{9}{11} \), the number \( 1 + \frac{9}{11} \) would be between the two consecutive integers 1 and 2, thus implying that \( m = 1 \).

Now we know that the decimal expansion of \( \frac{35}{11} \) has a one in the tenths place:

\[
\begin{array}{c|c|c|c|c}
\text{Ones} & \text{Tenths} & \text{Hundredths} & \text{Thousandths} \\
\hline
& & & \\
\end{array}
\]

Since \( \frac{35}{11} = 3 + \frac{2}{11} \) and the decimal expansion of the number is 3.1 = 3 + \( \frac{1}{10} \), we need to find the difference between these two representations. In other words, we need to find out what is left over after we remove the \( \frac{1}{10} \) from the fraction \( \frac{2}{11} \):

\[
\frac{2}{11} - \frac{1}{10} = \frac{20}{110} - \frac{11}{110} = \frac{9}{110}
\]

The next step is to find out which interval of hundredths will contain the fraction \( \frac{9}{110} \).

Provide time for students to make a prediction and possibly develop a plan for determining the answer.

- The process is the same as looking for the interval of tenths. That is, we are looking for consecutive integers \( m \) and \( m + 1 \) so that

\[
\frac{m}{100} < \frac{9}{110} < \frac{m + 1}{100}
\]

By what number should we multiply each term of the inequality to make our work here easier?

- Multiplying through by 100 will eliminate the fractions at the beginning and at the end of the inequality.

- Multiplying through by 100, we get:

\[
m < \frac{900}{110} < m + 1.
\]
Lesson 12: Decimal Expansions of Fractions, Part 2

Date: 1/31/14

Between which two integers, \( m \) and \( m + 1 \), will we find the fraction \( \frac{900}{110} \)? Explain.

- The fraction \( \frac{900}{110} \) is between 8 and 9. The reason is that \( \frac{900}{110} = \frac{880}{110} + \frac{20}{110} = 8 + \frac{2}{11} \).

Now we know that the decimal expansion of \( \frac{35}{11} \) has an 8 in the hundredths place:

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<td>Ones</td>
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Back to our original goal:

\[
\frac{35}{11} = 3 + \frac{2}{11}.
\]

By substitution we got:

\[
\frac{35}{11} = 3 + \frac{1}{10} + \frac{900}{110} = 3.1 + \frac{8}{11}.
\]

We know that \( \frac{35}{11} = 3 + \frac{2}{11} \) and our work so far has shown the decimal expansion to be \( 3.18 = 3 + \frac{1}{10} + \frac{8}{100} \).

As before, we need to find the difference between \( \frac{2}{11} \) and \( \left( \frac{1}{10} + \frac{8}{100} \right) \):

\[
\frac{2}{11} - \left( \frac{1}{10} + \frac{8}{100} \right) = \frac{2}{11} - \frac{18}{100} = \frac{2}{110} - \frac{18}{1100} = \frac{2}{1100}.
\]

Then, again, by substitution:

\[
\frac{35}{11} = 3 + \frac{1}{10} + \frac{900}{110} = 3 + \frac{8}{100} + \frac{2}{1100} = 3.18 + \frac{2}{1100}.
\]

Now, look at the interval of thousandths. Where do you expect \( \frac{2}{1100} \) to lie on the number line? Write and explain a plan for determining the interval of thousandths in which the number belongs.
Provide students time to make a prediction and develop a plan for determining the answer. Students should recognize that \( \frac{2}{1100} = \frac{2}{11} \times \frac{1}{100} \) and that we’ve placed the fraction \( \frac{2}{11} \) first, but for a different place value.

- Note that \( \frac{2}{1100} = \frac{2}{11} \times \frac{1}{100} \). The reappearance of the fraction \( \frac{2}{11} \) is meaningful in that we can expect a decimal digit to repeat, but in a different place value since we are now looking for the thousandths digit. We are looking for consecutive integers \( m \) and \( m+1 \) so that

\[
\frac{m}{1000} < \frac{2}{1100} < \frac{m+1}{1000}.
\]

What should we multiply each term by?

- Multiplying through by 1,000 will eliminate the fractions at the beginning and at the end of the inequality.

- Multiplying through by 1,000, we get:

\[
m < \frac{20}{11} < m + 1.
\]

However, we already know that:

\[
\frac{20}{11} = \frac{11}{11} + \frac{9}{11} = 1 + \frac{9}{11}
\]

- Therefore, the next digit in the decimal expansion of \( \frac{35}{11} \) will be 1:

\[
3.1818...
\]

- As before, we have the reappearance of the fraction \( \frac{9}{11} \). So, we can expect the next decimal digit to be 8, followed by the reappearance of \( \frac{2}{11} \) and so on. Therefore, the decimal expansion of \( \frac{35}{11} = 3.1818... \).

- Perform the long division algorithm on the fraction \( \frac{35}{11} \), and be prepared to share your observations.

Provide time for students to work. Ask students: How is this method of rational approximation similar to the long division algorithm? Students should notice that the algorithm became repetitive with the appearance of the numbers 2 and 9, alternating with each step. Conclude the discussion by pointing out that the method of rational approximation is similar to the long division algorithm.
Exercises 1–3 (5 minutes)

Students work independently or in pairs to complete Exercises 1–3.

Exercises 1–3

1. Use rational approximation to determine the decimal expansion of $\frac{5}{3}$.

\[
\frac{5}{3} = \frac{3}{3} + \frac{2}{3} = 1 + \frac{2}{3}
\]

In the sequence of tenths, we are looking for integers $m$ and $m+1$ so that

\[
\frac{m}{10} < \frac{2}{3} < \frac{m+1}{10},
\]

which is the same as

\[
m < 10 \left( \frac{2}{3} \right) < m + 1
\]

\[
\frac{20}{3} = \frac{18}{3} + \frac{2}{3} = 6 + \frac{2}{3}
\]

The tenths digit is 6. The difference between $\frac{2}{3}$ and $\frac{6}{10}$ is

\[
\frac{2}{3} - \frac{6}{10} = \frac{2}{30}
\]

In the interval of hundredths, we are looking for integers $m$ and $m+1$ so that

\[
\frac{m}{100} < \frac{2}{30} < \frac{m+1}{100},
\]

which is the same as

\[
m < \frac{20}{3} < m + 1.
\]

But we already know that $\frac{20}{3} = 6 + \frac{2}{3}$; therefore, the hundredths digit is 6. Because we keep getting $\frac{2}{3}$, we can assume the digit of 6 will continue to repeat. Therefore, the decimal expansion of $\frac{5}{3} = 1.666 \ldots$
2. Use rational approximation to determine the decimal expansion of \( \frac{5}{11} \).

In the sequence of tenths, we are looking for integers \( m \) and \( m + 1 \) so that
\[
\frac{m}{10} < \frac{5}{11} < \frac{m + 1}{10}
\]

which is the same as
\[
\frac{m}{11} < \frac{50}{11} < \frac{m + 1}{10}
\]

The tenths digit is 4. The difference between \( \frac{5}{11} \) and \( \frac{4}{10} \) is
\[
\frac{5}{11} = \frac{44}{110} + \frac{6}{110} = 4 + \frac{6}{11}
\]

In the sequence of hundredths, we are looking for integers \( m \) and \( m + 1 \) so that
\[
\frac{m}{100} < \frac{6}{110} < \frac{m + 1}{100}
\]

which is the same as
\[
\frac{m}{11} < \frac{60}{110} < \frac{m + 1}{100}
\]

So the hundredths digit is 5. Again, we see the fraction \( \frac{5}{11} \), which means the next decimal digit will be 4, as it was in the tenths place. This means we will again see the fraction \( \frac{6}{11} \), meaning we will have another digit of 5. Therefore, the decimal expansion of \( \frac{5}{11} \) is 0.4545 \ldots.

3. a. Determine the decimal expansion of the number \( \frac{23}{99} \) using rational approximation and long division.

In the interval of tenths, we are looking for integers \( m \) and \( m + 1 \) so that
\[
\frac{m}{10} < \frac{23}{99} < \frac{m + 1}{10}
\]

which is the same as
\[
\frac{m}{99} < \frac{230}{99} < \frac{m + 1}{10}
\]

The tenths digit is 2. The difference between \( \frac{23}{99} \) and \( \frac{2}{10} \) is
\[
\frac{23}{99} = \frac{198}{99} + \frac{32}{99} = 2 + \frac{32}{99}
\]

The tenths digit is 2. The difference between \( \frac{23}{99} \) and \( \frac{2}{10} \) is
\[
\frac{23}{99} = \frac{23}{99} + \frac{2}{10} = \frac{990}{990}
\]
In the interval of hundredths, we are looking for integers \( m \) and \( m + 1 \) so that

\[
\frac{m}{100} < \frac{32}{99} \frac{1}{10} < \frac{m + 1}{100}
\]

which is the same as

\[
m < \frac{320}{99} < m + 1
\]

\[
\frac{320}{99} = \frac{297}{99} + \frac{23}{99} = 3 + \frac{23}{99}
\]

So, the hundredths digit is 3. Again, we see the fraction \( \frac{23}{99} \) which means the next decimal digit will be 2, as it was in the tenths place. This means we will again see the fraction \( \frac{32}{99} \) meaning we will have another digit of 3. Therefore, the decimal expansion of \( \frac{23}{99} \) is \( 0.2323 \ldots \).

Long division gives us the same decimal expansion of \( \frac{23}{99} = 0.2323 \ldots \).

**b.** When comparing rational approximation to long division, what do you notice?

The first thing I notice is that the method of rational approximation gives the same decimal expansion as the long division algorithm. This makes sense because when doing long division, I put zeros past the 23, dividing into tenths, hundredths, thousandths, and so on. When I use the method of rational approximation, I do the same thing.

**Fluency Exercise (10 minutes)**

Please see the White Board Exchange exercise at the end of this lesson. Display the problems one at a time on a whiteboard, document camera, or PowerPoint. Give students about 1 minute to solve each problem, and go over them as a class.

**Closing (5 minutes)**

Summarize, or ask students to summarize, the main points from the lesson:

- Using rational approximation to write the decimal expansion of a fraction is similar to using the long division algorithm.
- Use the method of rational approximation to write the decimal expansion of a fraction instead of guessing and checking the intervals of tenths, hundredths, thousandths, etc. Determine the interval that a decimal digit is in using computation.
Lesson Summary

The method of rational approximation, used earlier to write the decimal expansion of irrational numbers, can also be used to write the decimal expansion of fractions (rational numbers).

When used with rational numbers, there is no need to guess and check to determine the interval of tenths, hundredths, thousandths, etc. in which a number will lie. Rather, computation can be used to determine between which two consecutive integers, \( m \) and \( m + 1 \), a number would lie for a given place value. For example, to determine where the fraction \( \frac{1}{8} \) lies in the interval of tenths, compute using the following inequality:

\[
\frac{m}{10} < \frac{1}{8} < \frac{m + 1}{10}
\]

Use the denominator of 10 because of our need to find the tenths digit of \( \frac{1}{8} \).

\[
m < \frac{10}{8} < m + 1
\]

Multiply through by 10

\[
m < \frac{1}{4} < m + 1
\]

Simplify the fraction \( \frac{10}{8} \).

The last inequality implies that \( m = 1 \) and \( m + 1 = 2 \), because \( 1 < \frac{1}{4} < 2 \). Then the tenths digit of the decimal expansion of \( \frac{1}{8} \) is 1.

Next, find the difference between the number \( \frac{1}{8} \) and the known tenths digit value, \( \frac{1}{10} \), i.e., \( \frac{1}{8} - \frac{1}{10} = \frac{2}{80} = \frac{1}{40} \).

Use the inequality again, this time with \( \frac{1}{40} \), to determine the hundredths digit of the decimal expansion of \( \frac{1}{8} \):

\[
\frac{m}{100} < \frac{1}{40} < \frac{m + 1}{100}
\]

Use the denominator of 100 because of our need to find the hundredths digit of \( \frac{1}{8} \).

\[
m < \frac{100}{40} < m + 1
\]

Multiply through by 100

\[
m < \frac{1}{2} < m + 1
\]

Simplify the fraction \( \frac{100}{40} \).

The last inequality implies that \( m = 2 \) and \( m + 1 = 3 \), because \( 2 < \frac{1}{2} < 3 \). Then the hundredths digit of the decimal expansion of \( \frac{1}{8} \) is 2.

Exit Ticket (5 minutes)
Lesson 12: Decimal Expansions of Fractions, Part 2

Exit Ticket

Use rational approximation to determine the decimal expansion of \( \frac{41}{6} \).
Exit Ticket Sample Solutions

Use rational approximation to determine the decimal expansion of $\frac{41}{6}$.

$\frac{41}{6} = \frac{36}{6} + \frac{5}{6}$

$= 6 + \frac{5}{6}$

The ones digit is 6. In the interval of tenths, we are looking for integers $m$ and $m+1$ so that

$$\frac{m}{10} < \frac{5}{6} < \frac{m+1}{10}$$

which is the same as

$$m < \frac{50}{6} < m + 1$$

The tenths digit is 8. The difference between $\frac{5}{6}$ and $\frac{8}{10}$ is

$$\frac{5}{6} - \frac{8}{10} = \frac{1}{30}$$

In the interval of hundredths, we are looking for integers $m$ and $m+1$ so that

$$\frac{m}{100} < \frac{1}{30} < \frac{m+1}{100}$$

which is the same as

$$m < \frac{10}{3} < m + 1$$

$$\frac{10}{3} = \frac{9}{3} + \frac{1}{3}$$

$$= 3 + \frac{1}{3}.$$

The hundredths digit is 3. Again, we see the fraction $\frac{1}{3}$, which means the next decimal digit will be 3, as it was in the hundredths place. This means we will again see the fraction $\frac{1}{3}$, meaning we will have another digit of 3. Therefore, the decimal expansion of $\frac{41}{6}$ is 6.8333....
Lesson 12

Decimal Expansions of Fractions, Part 2

Problem Set Sample Solutions

1. Explain why the tenths digit of $\frac{3}{11}$ is 2, using rational approximation.

   In the interval of tenths, we are looking for integers $m$ and $m + 1$ so that
   \[
   m \frac{3}{11} < \frac{m + 1}{10}
   \]
   which is the same as
   \[
   m < \frac{30}{11} < m + 1
   \]
   \[
   30 = \frac{22}{11} + \frac{8}{11} = 2 + \frac{8}{11}
   \]

   In looking at the interval of tenths, we see that the number $\frac{3}{11}$ must be between $\frac{2}{10}$ and $\frac{3}{10}$ because $\frac{2}{10} < \frac{3}{11} < \frac{3}{10}$.

   For this reason, the tenths digit of the decimal expansion of $\frac{3}{11}$ must be 2.

2. Use rational approximation to determine the decimal expansion of $\frac{25}{9}$

   $\frac{25}{9} = \frac{18}{9} + \frac{7}{9} = 2 + \frac{7}{9}$

   The ones digit is 2. In the interval of tenths, we are looking for integers $m$ and $m + 1$ so that
   \[
   m \frac{7}{9} < \frac{m + 1}{10}
   \]
   which is the same as
   \[
   m < \frac{70}{9} < m + 1
   \]
   \[
   70 = \frac{63}{9} + \frac{7}{9} = 7 + \frac{7}{9}
   \]

   The tenths digit is 7. The difference between $\frac{7}{9}$ and $\frac{7}{10}$ is
   \[
   \frac{7}{9} - \frac{7}{10} = \frac{7}{90}
   \]

   In the interval of hundredths, we are looking for integers $m$ and $m + 1$ so that
   \[
   m \frac{7}{90} < \frac{m + 1}{100}
   \]
   which is the same as
   \[
   m < \frac{70}{9} < m + 1.
   \]

   But we already know that $\frac{70}{9} = 7 + \frac{7}{9}$; therefore, the hundredths digit is 7. Because we keep getting $\frac{7}{9}$, we can assume the digit of 7 will continue to repeat. Therefore, the decimal expansion of $\frac{25}{9} = 2.777 \ldots$
3. Use rational approximation to determine the decimal expansion of \(\frac{11}{41}\) to at least 5 digits.

   In the interval of tenths, we are looking for integers \(m\) and \(m + 1\) so that
   \[
   \frac{m}{10} < \frac{11}{41} < \frac{m + 1}{10},
   \]

   which is the same as
   \[
   m < \frac{110}{41} < m + 1.
   \]

   The tenths digit is 2. The difference between \(\frac{11}{41}\) and \(\frac{2}{10}\) is
   \[
   \frac{11}{41} - \frac{2}{10} = \frac{410}{4100} - \frac{200}{4100} = \frac{210}{4100}.
   \]

   In the interval of hundredths, we are looking for integers \(m\) and \(m + 1\) so that
   \[
   \frac{m}{100} < \frac{28}{410} < \frac{m + 1}{100},
   \]

   which is the same as
   \[
   m < \frac{280}{41} < m + 1.
   \]

   The hundredths digit is 6. The difference between \(\frac{11}{41}\) and \(\left(\frac{2}{10} + \frac{6}{100}\right)\) is
   \[
   \frac{11}{41} - \left(\frac{20}{100} + \frac{6}{100}\right) = \frac{11}{41} - \frac{26}{100} = \frac{34}{4100}.
   \]

   In the interval of thousandths, we are looking for integers \(m\) and \(m + 1\) so that
   \[
   \frac{m}{1000} < \frac{34}{4100} < \frac{m + 1}{1000},
   \]

   which is the same as
   \[
   m < \frac{340}{41} < m + 1.
   \]

   The thousandths digit is 8. The difference between \(\frac{11}{41}\) and \(\left(\frac{2}{10} + \frac{6}{100} + \frac{8}{1000}\right)\) is
   \[
   \frac{11}{41} - \left(\frac{20}{100} + \frac{6}{100} + \frac{8}{1000}\right) = \frac{11}{41} - \frac{268}{4100} = \frac{12}{41000}.
   \]

   In the interval of ten-thousandths, we are looking for integers \(m\) and \(m + 1\) so that
   \[
   \frac{m}{10000} < \frac{12}{41000} < \frac{m + 1}{10000},
   \]

   which is the same as
   \[
   m < \frac{120}{41} < m + 1.
   \]
The ten-thousandths digit is 2. The difference between \( \frac{11}{41} \) and \( \left( \frac{2}{10} + \frac{6}{100} + \frac{8}{1000} + \frac{2}{10000} \right) \) is

\[
\frac{11}{41} - \left( \frac{2}{10} + \frac{6}{100} + \frac{8}{1000} + \frac{2}{10000} \right) = \frac{11}{41} - \frac{2682}{10000} = \frac{38}{410000}
\]

In the interval of hundred-thousandths, we are looking for integers \( m \) and \( m + 1 \) so that

\[
\frac{m}{100000} < \frac{38}{410000} < \frac{m + 1}{100000}
\]

which is the same as

\[
m < \frac{380}{41} < m + 1\]

\[
\frac{380}{41} = \frac{369}{41} + \frac{11}{41} = 9 + \frac{11}{41}
\]

The hundred-thousandths digit is 9. We see again the fraction \( \frac{11}{41} \), so we can expect the decimal digits to repeat at this point. Therefore, the decimal approximation of \( \frac{11}{41} \) is 0.2682926829 ...

4. Use rational approximation to determine which number is larger, \( \sqrt{10} \) or \( \frac{28}{9} \).

The number \( \sqrt{10} \) is between 3 and 4. In the sequence of tenths, \( \sqrt{10} \) is between 3.1 and 3.2 because

\[
3.1^2 < (\sqrt{10})^2 < 3.2^2
\]

In the sequence of hundredths, \( \sqrt{10} \) is between 3.16 and 3.17 because

\[
3.16^2 < (\sqrt{10})^2 < 3.17^2
\]

The decimal expansion of \( \sqrt{10} \) is approximately 3.162 ...

\[
\frac{28}{9} = \frac{27 + 1}{9} = 3 + \frac{1}{9}
\]

In the interval of tenths, we are looking for the integers \( m \) and \( m + 1 \) so that

\[
m \frac{1}{10} < \frac{m + 1}{10}
\]

which is the same as

\[
m < \frac{10}{9} < m + 1
\]

\[
\frac{10}{9} = \frac{9 + 1}{9} = 1 + \frac{1}{9}
\]

The tenths digit is 1. Since the fraction \( \frac{1}{9} \) has reappeared, then we can assume that the next digit is also 1, and the work will continue to repeat. Therefore, the decimal expansion of \( \frac{28}{9} \) is 3.1111 ...

Therefore, \( \frac{28}{9} < \sqrt{10} \).
5. Sam says that \( \frac{7}{11} \approx 0.63 \), and Jaylen says that \( \frac{7}{11} \approx 0.636 \). Who is correct? Why?

In the interval of tenths, we are looking for integers \( m \) and \( m + 1 \) so that

\[
\frac{m}{10} < \frac{7}{11} < \frac{m + 1}{10}
\]

which is the same as

\[
\frac{m}{11} < \frac{7}{11} < \frac{m + 1}{10}
\]

The tenths digit is 6. The difference between \( \frac{7}{11} \) and \( \frac{6}{10} \) is

\[
\frac{7}{11} - \frac{6}{10} = \frac{4}{110}
\]

In the interval of hundredths, we are looking for integers \( m \) and \( m + 1 \) so that

\[
\frac{m}{100} < \frac{4}{110} < \frac{m + 1}{100}
\]

which is the same as

\[
\frac{m}{110} < \frac{4}{110} < \frac{m + 1}{110}
\]

The hundredths digit is 3. Again, we see the fraction \( \frac{7}{11} \), which means the next decimal digit will be 6, as it was in the tenths place. This means we will again see the fraction \( \frac{4}{11} \), meaning we will have another digit of 3. Therefore, the decimal expansion of \( \frac{7}{11} \) is 0.636363….

Then, technically, both Sam and Jaylen are incorrect because the fraction \( \frac{7}{11} \) is an infinite decimal. However, Sam is correct to the first two decimal digits of the number, and Jaylen is correct to the first three decimal digits of the number.
Fluency Exercise: White Board Exchange [Key]

1. Find the area of the square shown below.
   \[ A = 4^2 = 16 \text{ m}^2 \]

2. Find the volume of the cube shown below.
   \[ V = 4^3 = 64 \text{ m}^3 \]

3. Find the area of the rectangle shown below.
   \[ A = 8(4) = 32 \text{ cm}^2 \]

4. Find the volume of the rectangular prism shown below.
   \[ V = 32(6) = 192 \text{ cm}^3 \]

5. Find the area of the circle shown below.
   \[ A = 7^2\pi = 49\pi \text{ m}^2 \]
6. Find the volume of the cylinder shown below.

\[ V = 49\pi(12) \]
\[ = 588\pi \text{ m}^3 \]

7. Find the area of the circle shown below.

\[ A = 6^2\pi \]
\[ = 36\pi \text{ in.}^2 \]

8. Find the volume of the cone shown below.

\[ V = \frac{1}{3}36\pi(10) \]
\[ = 120\pi \text{ in.}^3 \]

9. Find the area of the circle shown below.

\[ A = 8^2\pi \]
\[ = 64\pi \text{ mm}^2 \]

10. Find the volume of the sphere shown below.

\[ V = \frac{4}{3}\pi(64)(8) \]
\[ = \frac{2048}{3}\pi \text{ mm}^3 \]
Lesson 13: Comparison of Irrational Numbers

Student Outcomes

- Students use rational approximations of irrational numbers to compare the size of irrational numbers.
- Students place irrational numbers in their approximate locations on a number line.

Classwork

Exploratory Challenge Exercises 1–11 (30 minutes)

Students work in pairs to complete Exercises 1–11. The first exercise may be used to highlight the process of answering and explaining the solution to each question. An emphasis should be placed on students’ ability to explain their reasoning. Consider allowing students to use a calculator to check their work, but all decimal expansions should be done by hand. At the end of the Exploratory Challenge, consider asking students to state or write a description of their approach to solving each exercise.

Exercises 1–11

1. Rodney thinks that $\sqrt[3]{64}$ is greater than $\frac{17}{4}$. Sam thinks that $\frac{17}{4}$ is greater. Who is right and why?

   $\sqrt[3]{64} = \sqrt[3]{4^3} = 4$

   $\frac{17}{4} = \frac{16}{4} + \frac{1}{4} = 4 + \frac{1}{4} = 4\frac{1}{4}$

   Because $4 < 4\frac{1}{4}$, then $\sqrt[3]{64} < \frac{17}{4}$. So, $\sqrt[3]{64}$ is smaller. The number $\frac{17}{4}$ is equivalent to the mixed number $4\frac{1}{4}$. The cube root of 64 is the whole number 4. Because $4\frac{1}{4}$ is to the right of 4 on the number line, then $4\frac{1}{4}$ is greater than 4, which means that $\frac{17}{4} > \sqrt[3]{64}$; therefore, Sam is correct.
2. Which number is smaller, $\sqrt{27}$ or $2.89$? Explain.

$$\sqrt{27} = \sqrt{3^3} = 3$$

Because $2.89 < 3$, then $2.89 < \sqrt{27}$; so, $2.89$ is smaller. On a number line, $3$ is to the right of $2.89$, meaning that $3$ is greater than $2.89$. Therefore, $2.89 < \sqrt{27}$.

3. Which number is smaller, $\sqrt{121}$ or $\sqrt{125}$? Explain.

$$\sqrt{121} = \sqrt{11^2} = 11$$

$$\sqrt{125} = \sqrt{5^3} = 5$$

Because $5 < 11$, then $\sqrt{125} < \sqrt{121}$. So, $\sqrt{125}$ is smaller. On a number line, the number $5$ is to the left of $11$, meaning that $5$ is less than $11$. Therefore, $\sqrt{125} < \sqrt{121}$.

4. Which number is smaller, $\sqrt{49}$ or $\sqrt{216}$? Explain.

$$\sqrt{49} = \sqrt{7^2} = 7$$

$$\sqrt{216} = \sqrt{6^3} = 6$$

Because $6 < 7$, then $\sqrt{216} < \sqrt{49}$. So, $\sqrt{216}$ is smaller. On the number line, $7$ is to the right of $6$, meaning that $7$ is greater than $6$. Therefore, $\sqrt{216} < \sqrt{49}$.

5. Which number is greater, $\sqrt{50}$ or $\frac{319}{45}$? Explain.

Note that students may have used long division or the method of rational approximation to determine the decimal expansion of the fraction.

The number $\frac{319}{45}$ is equal to $7.0\bar{8}$.

The number $\sqrt{50}$ is between $7$ and $8$ because $7^2 < (\sqrt{50})^2 < 8^2$. The number $\sqrt{50}$ is between $7$ and $7.1$ because $7^2 < (\sqrt{50})^2 < 7.1^2$. The number $\sqrt{50}$ is between $7.07$ and $7.08$ because $7.07^2 < (\sqrt{50})^2 < 7.08^2$. The approximate decimal value of $\sqrt{50}$ is $7.07$. Since $7.07 < 7.08$, then $\sqrt{50} < \frac{319}{45}$; therefore, the fraction $\frac{319}{45}$ is greater.

6. Which number is greater, $\frac{5}{11}$ or $0.4\bar{5}$? Explain.

Note that students may have used long division or the method of rational approximation to determine the decimal expansion of the fraction.

The number $\frac{5}{11}$ is equal to $0.45$. Since $0.44444... < 0.45454545...$, then $0.4 < \frac{5}{11}$; therefore, the fraction $\frac{5}{11}$ is greater.
7. Which number is greater, $\sqrt{38}$ or $\frac{154}{25}$? Explain.

Note that students may have used long division or equivalent fractions to determine the decimal expansion of the fraction.

$$\frac{154}{25} = \frac{154 \times 4}{25 \times 4} = \frac{616}{100} = 6.16$$

The number $\frac{154}{25}$ is between 6 and 7 because $6^2 < (\frac{154}{25})^2 < 7^2$. The number $\sqrt{38}$ is between 6.1 and 6.2 because $6.1^2 < (\sqrt{38})^2 < 6.2^2$. The number $\frac{154}{25}$ is between 6.16 and 6.17 because $6.16^2 < (\sqrt{38})^2 < 6.17^2$. Since $\frac{154}{25}$ is greater than 6.16, then $\sqrt{38}$ is greater than $\frac{154}{25}$.

8. Which number is greater, $\sqrt{2}$ or $\frac{15}{9}$? Explain.

Note that students may have used long division or the method of rational approximation to determine the decimal expansion of the fraction.

The number $\frac{15}{9}$ is equal to $1.6$.

The number $\sqrt{2}$ is between 1 and 2 because $1^2 < (\sqrt{2})^2 < 2^2$. The number $\sqrt{2}$ is between 1.4 and 1.5 because $1.4^2 < (\sqrt{2})^2 < 1.5^2$. Therefore, $\sqrt{2} < \frac{15}{9}$; the fraction $\frac{15}{9}$ is greater.

9. Place the following numbers at their approximate location on the number line: $\sqrt{25}$, $\sqrt{28}$, $\sqrt{30}$, $\sqrt{32}$, $\sqrt{35}$, $\sqrt{36}$.

Solutions shown in red.

The number $\sqrt{25} = \sqrt{5^2} = 5$.

The number $\sqrt{28}$ is between 5 and 6. The number $\sqrt{28}$ is between 5.2 and 5.3 because $5.2^2 < (\sqrt{28})^2 < 5.3^2$.

The number $\sqrt{30}$ is between 5 and 6. The number $\sqrt{30}$ is between 5.4 and 5.5 because $5.4^2 < (\sqrt{30})^2 < 5.5^2$.

The number $\sqrt{32}$ is between 5 and 6. The number $\sqrt{32}$ is between 5.6 and 5.7 because $5.6^2 < (\sqrt{32})^2 < 5.7^2$.

The number $\sqrt{35}$ is between 5 and 6. The number $\sqrt{35}$ is between 5.9 and 6.0 because $5.9^2 < (\sqrt{35})^2 < 6^2$.

The number $\sqrt{36} = \sqrt{6^2} = 6$.

10. Challenge: Which number is larger $\sqrt{5}$ or $\sqrt[3]{11}$?

The number $\sqrt{5}$ is between 2 and 3 because $2^2 < (\sqrt{5})^2 < 3^2$. The number $\sqrt{5}$ is between 2.2 and 2.3 because $2.2^2 < (\sqrt{5})^2 < 2.3^2$. The number $\sqrt{5}$ is between 2.23 and 2.24 because $2.23^2 < (\sqrt{5})^2 < 2.24^2$. The number $\sqrt{5}$ is between 2.236 and 2.237 because $2.236^2 < (\sqrt{5})^2 < 2.237^2$. The decimal expansion of $\sqrt{5}$ is approximately 2.236 ... 

The number $\sqrt[3]{11}$ is between 2 and 3 because $2^3 < (\sqrt[3]{11})^3 < 3^3$. The number $\sqrt[3]{11}$ is between 2.2 and 2.3 because $2.2^3 < (\sqrt[3]{11})^3 < 2.3^3$. The number $\sqrt[3]{11}$ is between 2.22 and 2.23 because $2.22^3 < (\sqrt[3]{11})^3 < 2.23^3$. The decimal expansion of $\sqrt[3]{11}$ is approximately 2.22 ... Since 2.22 ... < 2.236 ..., then $\sqrt[3]{11} < \sqrt{5}$; therefore, $\sqrt{5}$ is larger.
11. A certain chessboard is being designed so that each square has an area of $3 \text{ in}^2$. What is the length, rounded to the tenths place, of one edge of the board? (A chessboard is composed of 64 squares as shown.)

The area of one square is $3 \text{ in}^2$. So, if $x$ is the length of one side of one square,

$x^2 = 3$

$\sqrt{x^2} = \sqrt{3}$

$x = \sqrt{3}$

There are 8 squares along one edge of the board, so the length of one edge is $8 \times \sqrt{3}$. The number $\sqrt{3}$ is between 1 and 2 because $1^2 < (\sqrt{3})^2 < 2^2$. The number $\sqrt{3}$ is between 1.7 and 1.8 because $1.7^2 < (\sqrt{3})^2 < 1.8^2$. The number $\sqrt{3}$ is between 1.73 and 1.74 because $1.73^2 < (\sqrt{3})^2 < 1.74^2$. The number $\sqrt{3}$ is approximately 1.73. So, the length of one edge of the chessboard is $8 \times 1.73 = 13.84 \approx 13.8$ in.

Note: Some students may determine the total area of the board, $64 \times 3 = 192$, then determine the approximate value of $\sqrt{192} \approx 13.8$, to answer the question.

Discussion (5 minutes)

- How do we know if a number is rational or irrational?
  - Numbers that can be expressed as a fraction, i.e., a ratio of integers, are by definition rational numbers. Any number that is not rational is irrational.
- Is the number 1.6 rational or irrational? Explain.
  - The number 1.6 is rational because it is equal to $\frac{15}{9}$.
- Is the number $\sqrt{2}$ rational or irrational? Explain.
  - Since $\sqrt{2}$ is not a perfect square, then $\sqrt{2}$ is an irrational number. This means that the decimal expansion can only be approximated by rational numbers.
- Which strategy do you use to write the decimal expansion of a fraction? What strategy do you use to write the decimal expansion of square and cube roots?
  - Student responses will vary. Students will likely state that they use long division or equivalent fractions to write the decimal expansion of fractions. Students will say that they have to use the definition of square and cube roots or rational approximation to write the decimal expansion of the square and cube roots.

Closing (5 minutes)

Summarize, or ask students to summarize, the main points from the lesson:

- The decimal expansion of rational numbers that are expressed as fractions can be found by using long division, by using what we know about equivalent fractions for finite decimals, or by using rational approximation.
- The approximate decimal expansions of irrational numbers (square roots of imperfect squares and imperfect cubes) can be found using rational approximation.
- Numbers, of any form (e.g., fraction, decimal, square root), can be ordered and placed in their approximate location on a number line.
Lesson Summary

The decimal expansion of rational numbers can be found by using long division, equivalent fractions, or the method of rational approximation.

The decimal expansion of irrational numbers can be found using the method of rational approximation.

Exit Ticket (5 minutes)
Lesson 13: Comparison of Irrational Numbers

Exit Ticket

Place the following numbers at their approximate location on the number line: $\sqrt{12}$, $\sqrt{16}$, $\frac{20}{6}$, 3.53, $\sqrt{27}$. 

![Number line with approximate locations for $\sqrt{12}$, $\sqrt{16}$, $\frac{20}{6}$, 3.53, $\sqrt{27}$]
Exit Ticket Sample Solutions

Place the following numbers at their approximate location on the number line: $\sqrt{12}$, $\sqrt{16}$, $\frac{20}{6}$, $3.53$, $\sqrt{27}$.

Note that students may have used long division or the method of rational approximation to determine the decimal expansion of the fraction.

The number $\sqrt{12}$ is between $3.4$ and $3.5$, since $3.4^2 < (\sqrt{12})^2 < 3.5^2$.

The number $\sqrt{16} = 4$.

The number $\frac{20}{6}$ is equal to $3.3$.

The number $\sqrt{27} = \sqrt{3^3} = 3$.

Solutions in red:

$\sqrt{27}$ $\frac{20}{6}$ $\sqrt{12}$ $3.53$ $\sqrt{16}$

3 3.1 3.2 3.3 3.4 3.5 3.6 3.7 3.8 3.9 4

Problem Set Sample Solutions

1. Which number is smaller, $\sqrt{343}$ or $\sqrt{48}$? Explain.

$\sqrt{343} = \sqrt{7^3} = 7$

The number $\sqrt{48}$ is between 6 and 7, but definitely less than 7. Therefore, $\sqrt{48} < \sqrt{343}$ and $\sqrt{48}$ is smaller.

2. Which number is smaller, $\sqrt{100}$ or $\sqrt{1,000}$? Explain.

$\sqrt{100} = \sqrt{10^2} = 10$

$\sqrt{1,000} = \sqrt{10^3} = 10$

The numbers $\sqrt{100}$ and $\sqrt{1,000}$ are equal because both are equal to 10.

3. Which number is larger, $\sqrt{87}$ or $\frac{929}{99}$? Explain.

Note that students may have used long division or the method of rational approximation to determine the decimal expansion of the fraction.

The number $\frac{929}{99}$ is equal to 9.38.

The number $\sqrt{87}$ is between 9 and 10 because $9^2 < (\sqrt{87})^2 < 10^2$. The number $\sqrt{87}$ is between 9.3 and 9.4 because $9.3^2 < (\sqrt{87})^2 < 9.4^2$. The number $\sqrt{87}$ is between 9.32 and 9.33 because $9.32^2 < (\sqrt{87})^2 < 9.33^2$. The approximate decimal value of $\sqrt{87}$ is 9.32 .... Since 9.32 < 9.38, then $\sqrt{87} < \frac{929}{99}$; therefore, the fraction $\frac{929}{99}$ is larger.
4. Which number is larger, \( \frac{9}{13} \) or \( 0.692 \)? Explain.

Note that students may have used long division or the method of rational approximation to determine the decimal expansion of the fraction.

The number \( \frac{9}{13} \) is equal to \( 0.692307 \). Since \( 0.692307 < 0.692692 \), then \( \frac{9}{13} < 0.692 \); therefore, the decimal 0.692 is larger.

5. Which number is larger, 9.1 or \( \sqrt{82} \)? Explain.

The number \( \sqrt{82} \) is between 9 and 10 because \( 9^2 < (\sqrt{82})^2 < 10^2 \). The number \( \sqrt{82} \) is between 9.0 and 9.1 because \( 9.0^2 < (\sqrt{82})^2 < 9.1^2 \). Since \( \sqrt{82} < 9.1 \), then the number 9.1 is larger than the number \( \sqrt{82} \).

6. Place the following numbers at their approximate location on the number line: \( \sqrt{144}, \sqrt{1000}, \sqrt{130}, \sqrt{110}, \sqrt{120}, \sqrt{115}, \sqrt{133} \). Explain how you knew where to place the numbers.

Solutions shown in red

The number \( \sqrt{144} = \sqrt{12^2} = 12 \).

The number \( \sqrt{1000} = \sqrt{10^3} = 10 \).

The numbers \( \sqrt{110}, \sqrt{115}, \) and \( \sqrt{120} \) are all between 10 and 11 because when squared, their value falls between \( 10^2 \) and \( 11^2 \). The number \( \sqrt{110} \) is between 10.4 and 10.5 because \( 10.4^2 < (\sqrt{110})^2 < 10.5^2 \). The number \( \sqrt{115} \) is between 10.7 and 10.8 because \( 10.7^2 < (\sqrt{115})^2 < 10.8^2 \). The number \( \sqrt{120} \) is between 10.9 and 11 because \( 10.9^2 < (\sqrt{120})^2 < 11^2 \).

The numbers \( \sqrt{130} \) and \( \sqrt{133} \) are between 11 and 12 because when squared, their value falls between \( 11^2 \) and \( 12^2 \). The number \( \sqrt{130} \) is between 11.4 and 11.5 because \( 11.4^2 < (\sqrt{130})^2 < 11.5^2 \). The number \( \sqrt{133} \) is between 11.5 and 11.6 because \( 11.5^2 < (\sqrt{133})^2 < 11.6^2 \).
7. Which of the two right triangles shown below, measured in units, has the longer hypotenuse? Approximately how much longer is it?

Let \( x \) represent the hypotenuse of the triangle on the left.

\[
7^2 + 2^2 = x^2 \\
49 + 4 = x^2 \\
53 = x^2 \\
\sqrt{53} = \sqrt{x^2} \\
\sqrt{53} = x
\]

The number \( \sqrt{53} \) is between 7 and 8 because \( 7^2 < (\sqrt{53})^2 < 8^2 \). The number \( \sqrt{50} \) is between 7. 2 and 7. 3 because \( 7.2^2 < (\sqrt{53})^2 < 7.3^2 \). The number \( \sqrt{53} \) is between 7. 28 and 7. 29 because \( 7.28^2 < (\sqrt{53})^2 < 7.29^2 \). The approximate decimal value of \( \sqrt{53} \) is 7. 28 ...

Let \( y \) represent the hypotenuse of the triangle on the right.

\[
5^2 + 5^2 = y^2 \\
25 + 25 = y^2 \\
50 = y^2 \\
\sqrt{50} = \sqrt{y^2} \\
\sqrt{50} = y
\]

The number \( \sqrt{50} \) is between 7 and 8 because \( 7^2 < (\sqrt{50})^2 < 8^2 \). The number \( \sqrt{50} \) is between 7. 0 and 7. 1 because \( 7.0^2 < (\sqrt{50})^2 < 7.1^2 \). The number \( \sqrt{50} \) is between 7. 07 and 7. 08 because \( 7.07^2 < (\sqrt{50})^2 < 7.08^2 \). The approximate decimal value of \( \sqrt{50} \) is 7. 07 ...

The triangle on the left has the longer hypotenuse. It is approximately 0. 21 units longer than the hypotenuse of the triangle on the right.

Note: Based on their experience, some students may reason that \( \sqrt{50} < \sqrt{53} \). To answer completely, students must determine the decimal expansion to approximate how much longer one hypotenuse is than the other.
Lesson 14: The Decimal Expansion of $\pi$

Student Outcomes

- Students calculate the decimal expansion of $\pi$ using basic properties of area.
- Students estimate the value of expressions such as $\pi^2$.

Lesson Notes

For this lesson, students will need grid paper and a compass. Lead students through the activity that produces the decimal expansion of $\pi$. Quarter circles on grids of 10 by 10 and 20 by 20 are included at the end of the lesson if you would prefer to hand out the grids as opposed to students making their own with grid paper and a compass.

Classwork

Opening Exercises 1–3 (5 minutes)

The purpose of the Opening Exercises is to activate students’ prior knowledge of $\pi$ and of what that number means.

Opening Exercises

1. Write an equation for the area, $A$, of the circle shown.

$$A = \pi 6.3^2 = 39.69\pi$$

The area of the circle is $39.69\pi$ cm$^2$.

2. Write an equation for the circumference, $C$, of the circle shown.

$$C = \pi 2(9.7) = 19.4\pi$$

The circumference of the circle is $19.4\pi$ mm.
3. Each of the squares in the grid below has an area of 1 unit².

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a. Estimate the area of the circle shown by counting squares.

Estimates will vary. The approximate area of the circle is 78 units².

b. Calculate the area of the circle using a radius of 5 units and 3.14 for π.

\[
A = 3.14(5^2) = 78.5
\]

The area of the circle is 78.5 units².

Discussion (25 minutes)

- The number π, π, is defined as the ratio of the circumference to the diameter of a circle. The number π is also the area of a unit circle. A unit circle is a circle with a radius of one unit. Our goal in this lesson is to determine the decimal expansion of π. What do you think that is?
  - Students will likely state that the decimal expansion of π is 3.14 because that is the number they have used in the past to approximate π.

- The number 3.14 is often used to approximate π, but it is not its decimal expansion. How might we determine its real decimal expansion?

Provide time for students to try to develop a plan for determining the decimal expansion of π. Have students share their ideas with the class.

- To determine the decimal expansion of π, we will use the fact that the number π is the area of a unit circle together with the counting strategy used in Exercise 3(a). Since the area of the unit circle is equal to π, and we will be counting squares, we can decrease our work by focusing on the area of just \( \frac{1}{4} \) of the circle. What is the area of \( \frac{1}{4} \) of a unit circle?
Since the unit circle has an area of $\pi$, then $\frac{1}{4}\pi$ will be the area of $\frac{1}{4}$ of the unit circle.

- On a piece of graph paper, mark a center $O$, near the center of the paper. Use your ruler and draw two lines through $O$, one horizontal and one vertical. Our unit will be 10 of the grid squares on the graph paper. Use your compass to measure 10 of the grid squares, and then make an arc to represent the outer edge of the quarter circle. Make sure your arc intersects the horizontal and vertical lines you drew.

Verify that all students have a quarter circles on their graph paper.

- What we have now are inner squares, those that are inside the quarter circle, and outer squares, those that are outside of the quarter circle. What we want to do is mark a border just inside the circle and just outside the circle, but as close to the arc of the circle as possible. Mark a border inside the circle that captures all of the whole squares; that is, you should not include any partial squares in this border (shown in red below). Mark a border just outside the arc that contains all of the whole squares within the quarter circle and parts of the squares that are just outside the circle (shown in black below).
The squares of the grid paper are congruent; that is, they are all equal in size and, thus, area. We will let $r_2$ denote the totality of all of the inner squares and $s_2$ the totality of all of the outer squares. Then, clearly,

$$r_2 < \frac{\pi}{4} < s_2.$$ 

Count how many squares are contained within $r_2$ and $s_2$.

- There are 69 inner squares and 86 outer squares.

If we consider the area of the square with side length equal to 10 squares of the grid paper, then $r_2 = \frac{69}{100}$.

What does $s_2$ equal?

- The area of $s_2 = \frac{86}{100}$.

By substitution we see that

$$r_2 < \frac{\pi}{4} < s_2$$

$$\frac{69}{100} < \frac{\pi}{4} < \frac{86}{100}$$

$$0.69 < \frac{\pi}{4} < 0.86$$

Multiplying by 4 throughout gives

$$2.76 < \pi < 3.44.$$ 

Is this inequality showing a value for $\pi$ that we know to be accurate? Explain.

- Yes, because we frequently use 3.14 to represent $\pi$, and 2.76 < 3.14 < 3.44.

Of course we can improve our estimate of $\pi$ by taking another look at those grid squares. Columns have been labeled at the top, A-H.
Look at the top row, columns A through B. There are some significant portions of squares that were not included in the area of the quarter circle. If we wanted to represent that portion of the circle with a whole number of grid squares, about how many do you think it would be?

Accept any reasonable answers that students give for this and the next few questions about columns A-G. Included in the text below is a possible scenario; however, your students may make better estimations and decide on different numbers of squares to include in the area.

- It looks like there would be at least 2 whole squares, but likely less than 3.
- Now look at columns C and D. Using similar reasoning, about how many grid squares do you think we should add to the area of the quarter circle using just columns C and D?
  - It looks like we should add 1 more to the area of the quarter circle.
- Now look at columns E and F. What should we add to the area of the quarter circle?
  - We should add 1 more to the area.
- What about column G?
  - We should add at least 1 more square to the area.
- Finally, look at column H. What should we add to the area to represent the portion of the quarter circle not accounted for yet?
  - It looks like column H is just like columns A to B, so we should add 2 more to the area.

We began by counting only the number of whole squares within the border of the quarter circle, which totals 69. By estimating partial amounts of squares in columns A through H, we have decided to improve our estimate by adding another 2, 1, 1, 1, 2 squares, making our total number of grid squares represented by the quarter circle 76. Therefore, we have refined \( r_2 \) to 76, which means that

\[
\frac{76}{100} < \frac{\pi}{4} < \frac{86}{100}
\]

which is equal to

\[
\frac{304}{100} < \pi < \frac{344}{100}
\]

\[
3.04 < \pi < 3.44
\]

Does this inequality still represent a value we expect \( \pi \) to be?
- Yes, because 3.04 < 3.14 < 3.44.

We can reason the same way as before to refine the estimate of \( s_2 \). Provide students time to refine their estimate of \( s_2 \). It is likely that students will come up with different numbers, but they should all be very close. Expect students to say that they have refined their estimate of \( s_2 \) to 80, instead of the original 86.

- Thus, we have

\[
\frac{76}{100} < \frac{\pi}{4} < \frac{80}{100}
\]

\[
\frac{304}{100} < \pi < \frac{320}{100}
\]

\[
3.04 < \pi < 3.20
\]

Does this inequality still represent a value we expect \( \pi \) to be?
- Yes, because 3.04 < 3.14 < 3.20.
These are certainly respectable approximations of \( \pi \). What would make our approximation better?

- We could decrease the size of the squares we are using to develop the area of the quarter circle. We could go back and make better estimations of the squares to include in \( r^2 \) and the squares not to include in \( s^2 \).

- As you have stated, one way to improve our approximation is by using smaller squares. Suppose we divide each square horizontally and vertically so that instead of having 100 squares, we have 400 squares.

If time permits, allow students to repeat the process that we just went through when we had only 100 squares in the unit square. If time does not permit, then provide them with the information below.

- Then, the inner region, \( r^2 \), is comprised of 294 squares, and the outer region, \( s^2 \), is comprised of 333 squares. This means that

\[
\frac{294}{400} < \frac{\pi}{4} < \frac{333}{400}
\]

Multiplying by 4 throughout, we have

\[
\frac{294}{100} < \pi < \frac{333}{100}
\]

\[
2.94 < \pi < 3.33
\]

- By looking at partial squares that can be combined, the refined estimate of \( r^2 \) is 310 and \( s^2 \) is 321. Then, the inequality is

\[
\frac{310}{400} < \frac{\pi}{4} < \frac{321}{400}
\]

\[
\frac{310}{100} < \pi < \frac{321}{100}
\]

\[
3.10 < \pi < 3.21
\]
Lesson 14

The Decimal Expansion of $\pi$

How does this inequality compare to what we know $\pi$ to be.

- This inequality is quite accurate as $3.10 < 3.14 < 3.21$; there is only a difference of $\frac{4}{100}$ for the lower region and $\frac{7}{100}$ for the upper region.

We could continue the process of refining our estimate several more times to see that $3.14159 < \pi < 3.14160$ and then continue on to get an even more precise estimate of $\pi$. But at this point, it should be clear that we have a fairly good one already.

We finish by making one more observation about $\pi$ and irrational numbers in general. When we take the square of an irrational number such as $\pi$, we are doing it formally without exactly knowing the value of $\pi^2$. Since we can use a calculator to show that $3.14159 < \pi < 3.14160$, then, we also know that

$$3.14159^2 < \pi^2 < 3.14160^2$$

Notice that the first 4 digits, 9.869, appear in the inequality. Therefore, we can say that $\pi^2 = 9.869$ is correct up to 3 decimal digits.

Exercises 4–7 (5 minutes)

Students work on Exercises 4–7 independently or in pairs. If necessary, model for students how to use the given decimal digits of the irrational number to “trap” the number in the inequality for Exercises 5–7. An online calculator was used to determine the decimal values of the squared numbers in Exercises 5–7. If handheld calculators are used, then the decimal values will be truncated to 8 places. However, this will not affect the estimate of the irrational numbers.

4. Gerald and Sarah are building a wheel with a radius of 6.5 cm and are trying to determine the circumference. Gerald says, "Because $6.5 \times 2 \times 3.14 = 40.82$, the circumference is 40.82 cm." Sarah says, "Because $6.5 \times 2 \times 3.10 = 40.3$ and $6.5 \times 2 \times 3.21 = 41.73$, the circumference is somewhere between 40.3 and 41.73." Explain the thinking of each student.

Gerald is using a common approximation for the number $\pi$ to determine the circumference. That is why he used 3.14 in his calculation. Sarah is using an interval between which the value of $\pi$ falls, based on the work we did in class. We know that $3.10 < \pi < 3.21$; therefore, her calculations of the circumference uses numbers we know $\pi$ to be between.

5. Estimate the value of the irrational number $(6.12486 ...)^2$.

$$6.12486^2 < (6.12486 ...)^2 < 6.12487^2$$
$$37.5139100196 < (6.12486 ...)^2 < 37.5140325169$$

$(6.12486 ...)^2 = 37.51$ is correct up to 2 decimal digits.

6. Estimate the value of the irrational number $(9.204107 ...)^2$.

$$9.204107^2 < (9.204107 ...)^2 < 9.204108^2$$
$$84.715585667449 < (9.204107 ...)^2 < 84.715604075664$$

$(9.204107 ...)^2 = 84.715$ is correct up to 3 decimal digits.
Closing (5 minutes)

Summarize, or ask students to summarize, the main points from the lesson:

- The area of a unit circle is $\pi$.
- We learned a method to estimate the value of $\pi$ using graph paper, a unit circle, and areas.
- When we square the decimal expansion of an irrational number, we are doing it formally. This is similar to using approximation for computations. For that reason, we may only be accurate to a few decimal digits.

Exit Ticket (5 minutes)

Lesson Summary

Irrational numbers, such as $\pi$, are frequently approximated in order to compute with them. Common approximations for $\pi$ are $3.14$ and $\frac{22}{7}$. It should be understood that using an approximate value of an irrational number for computations produces an answer that is accurate to only the first few decimal digits.
Lesson 14: The Decimal Expansion of $\pi$

Exit Ticket

Describe how we found a decimal approximation for $\pi$. 
Exit Ticket Sample Solutions

Describe how we found a decimal approximation for \( \pi \).

To make our work easier, we looked at the number of unit squares in a quarter circle that comprised its area. We started by counting just the whole number of unit squares. Then, we continued to revise our estimate of the area by looking at parts of squares, specifically to see if parts could be combined to make a whole unit square. We looked at the inside and outside boundaries and said that the value of \( \pi \) would be between these two numbers. The inside boundary is a conservative estimate of the value of \( \pi \), and the outside boundary is an overestimate of the value of \( \pi \). We could continue this process with smaller squares in order to refine our estimate for the decimal approximation of \( \pi \).

Problem Set Sample Solutions

Students estimate the values of irrational numbers squared.

1. Caitlin estimated \( \pi \) to be \( 3.10 < \pi < 3.21 \). If she uses this approximation of \( \pi \) to determine the area of a circle with a radius of 5 cm, what could the area be?

   \[ \text{The area of the circle with radius } 5 \text{ cm will be between } 77.5 \text{ cm}^2 \text{ and } 80.25 \text{ cm}^2. \]

2. Myka estimated the circumference of a circle with a radius of 4.5 in. to be 28.44 in. What approximate value of \( \pi \) did she use? Is it an acceptable approximation of \( \pi \)? Explain.

   \[ C = 2\pi r \]
   \[ 28.44 = 2\pi (4.5) \]
   \[ 28.44 = 9\pi \]
   \[ \frac{28.44}{9} = \pi \]

   Myka used 3.16 to approximate \( \pi \). This is an acceptable approximation for \( \pi \) because it is in the interval, \( 3.10 < \pi < 3.21 \), that we approximated \( \pi \) to be in the lesson.

3. A length of ribbon is being cut to decorate a cylindrical cookie jar. The ribbon must be cut to a length that stretches the length of the circumference of the jar. There is only enough ribbon to make one cut. When approximating \( \pi \) to calculate the circumference of the jar, which number in the interval \( 3.10 < \pi < 3.21 \) should be used? Explain.

   In order to make sure the ribbon is long enough, we should use an estimate of \( \pi \) that is closer to 3.21. We know that 3.10 is a fair estimate of \( \pi \), but less than the actual value of \( \pi \). Similarly, we know that 3.21 is a fair estimate of \( \pi \), but greater than the actual value of \( \pi \). Since we can only make one cut, we should cut the ribbon so that there will be a little more, not less than, what we need. For that reason, an approximation of \( \pi \) closer to 3.21 should be used.

4. Estimate the value of the irrational number \((1.86211 \ldots)^2\).

   \[ 1.86211^2 < (1.86211 \ldots)^2 < 1.86212^2 \]
   \[ 3.4674536521 < (1.86211 \ldots)^2 < 3.4674908944 \]

   \((1.86211 \ldots)^2 = 3.4674 \) is correct up to 4 decimal digits.

5. Estimate the value of the irrational number \((5.9035687 \ldots)^2\).

   \[ 5.9035687^2 < (5.9035687 \ldots)^2 < 5.9035688^2 \]
   \[ 34.85212339561969 < (5.9035687 \ldots)^2 < 34.85212457633344 \]

   \((5.9035687 \ldots)^2 = 34.85212 \) is correct up to 5 decimal digits.
6. Estimate the value of the irrational number $(12.30791 \ldots)^2$.

$$12.30791^2 < (12.30791 \ldots)^2 < 12.30792^2$$

$$151.4864485681 < (12.30791 \ldots)^2 < 151.4848974264$$

$$(12.30791 \ldots)^2 = 151.484 \text{ is correct up to 3 decimal digits.}$$

7. Estimate the value of the irrational number $(0.6289731 \ldots)^2$.

$$0.6289731^2 < (0.6289731 \ldots)^2 < 0.6289732^2$$

$$0.39560716052361 < (0.6289731 \ldots)^2 < 0.39560728631824$$

$$(0.6289731 \ldots)^2 = 0.395607 \text{ is correct up to 6 decimal digits.}$$

8. Estimate the value of the irrational number $(1.11223333 \ldots)^2$.

$$1.11223333^2 < (1.11223333 \ldots)^2 < 1.11223334^2$$

$$1.2370407424696289 < (1.11223333 \ldots)^2 < 1.2370407446940756$$

$$(1.11223333 \ldots)^2 = 1.23704074 \text{ is correct up to 8 decimal digits.}$$

9. Which number is a better estimate for $\pi$, $\frac{22}{7}$ or 3.14? Explain.

Allow for both answers to be correct as long as the student provides a reasonable explanation.

Sample answer might be as follows.

*I think that 3.14 is a better estimate because when I find the decimal expansion, $\frac{22}{7} \approx 3.142857 \ldots$; the number 3.14 is closer to the actual value of $\pi$."

10. To how many decimal digits can you correctly estimate the value of the irrational number $(4.56789012 \ldots)^2$?

$$4.56789012^2 < (4.56789012 \ldots)^2 < 4.56789013^2$$

$$20.8656201483936144 < (4.56789012 \ldots)^2 < 20.8656202397514169$$

$$(4.56789012 \ldots)^2 = 20.865620 \text{ is correct up to 6 decimal digits.}$$
10 by 10 Grid
20 by 20 Grid