Math 8 Notes – Unit 4: Ratios, Rates, and Proportions

Syllabus Objective: (4.1) The student will find equivalent ratios.

Ratio: a comparison of two quantities by division

Equivalent ratios: ratios that make the same comparison

Proportion: an equation that states two ratios are equivalent

Writing Ratios

There are three ways to write a ratio. Ratios can be written in words, as a fraction, or separated by a colon.

Examples:
Use the word “to”: $15$ to $5$ tickets

Fraction: \( \frac{15}{5} \)

Using a colon: $15:5$

All three methods give the same information. Remember, a ratio is simply a comparison of two quantities, so changing the display does not change the information.

Finding Equivalent Ratios

You can find equivalent ratios by either multiplying or dividing the numerator and denominator by the same nonzero number. This is essentially multiplying or dividing by one; therefore, you don’t change the value of the original ratio.

Example: Find two ratios that are equivalent to each given ratio.

\[
\frac{9}{27} \quad \text{Given:} \quad \frac{9}{27} \quad \text{or} \quad \frac{9 \div 3}{27 \div 3} = \frac{3}{9}
\]

\[\frac{27}{81} \quad \text{and} \quad \frac{3}{9}\]

are equivalent.
Determining Whether Two Ratios Are in Proportion

There are a couple of ways to determine whether or not two ratios form a proportion. First, have your students simplify the two ratios. They may be surprised at what they find. After students are comfortable with that, you can show them how cross products can be used as a “short-cut” for determining whether two ratios are in proportion.

Method 1: Simplify the ratios

\[
\frac{3}{27} \text{ and } \frac{2}{18}
\]

\[
\frac{3 \div 3}{27 \div 3} = \frac{1}{9}
\]

\[
\frac{2 \div 2}{18 \div 2} = \frac{1}{9}
\]

Since the two original ratios simplify to the same thing, we can conclude that the two original ratios are in proportion.

Method 2: Cross products

\[
\frac{9}{36} \text{ and } \frac{2}{8} \rightarrow \frac{9 \times 8}{36} = 72
\]

Using cross products you get 9 \( \times 8 = 72 \) and 36 \( \times 2 = 72 \).

Example: Tell whether the ratios form a proportion.

\[
\frac{9}{12} \text{ and } \frac{16}{24}
\]

\[
\frac{9}{12} = \frac{3}{4} \text{ and } \frac{16}{24} = \frac{2}{3}
\]

Since \( \frac{3}{4} \neq \frac{2}{3} \) we can conclude that the original ratios do not make a proportion.

Example: Tell whether the ratios form a proportion.

\[
\frac{12}{15} \text{ and } \frac{27}{36}
\]

Using cross products \( \frac{12}{15} \times \frac{36}{27} = \frac{432}{405} \).

Since 432 \( \neq 405 \), you can conclude that the original ratios do not make a proportion.
Example: Tell whether the ratios form a proportion.

\[
\frac{9}{36} \text{ and } \frac{6}{24}
\]

\[
\frac{9}{36} = \frac{9 \div 9}{36 \div 9} = \frac{1}{4}
\]

\[
\frac{6}{24} = \frac{6 \div 6}{24 \div 6} = \frac{1}{4}
\]

Since \(\frac{1}{4} = \frac{1}{4}\), the ratios are in proportion.

Computing Unit Rates and Costs

Syllabus Objective: (4.2) The student will compute unit cost.

Rate: a comparison of two quantities that have different units. For example: miles per hour.

Example: Bob travels 300 miles in 5 hours, find Bob’s rate.

Bob’s rate = \(\frac{300 \text{ miles}}{5 \text{ hours}}\)

Unit rate: a rate in which the second quantity is 1. For example: words per one minute.

Example: Bob can type 50 words per minute.

Unit rate = \(\frac{50 \text{ words}}{1 \text{ minute}}\)

Unit price: unit rate used to compare price per one item.

Finding Unit Rates

Finding unit rates is a two-step process. First the students need to write the rate and then divide.

Example: Geoff can type 120 words in 4 minutes. How many words can he type in one minute?

This is asking for the unit rate because we want to know how many words will be typed compared to a second quantity of one (in this case one minute).
Step 1: Write the rate: 
\[
\frac{120 \text{ words}}{4 \text{ minutes}}
\]

Step 2: Divide to find per minute: 
\[
\frac{120 \div 4}{4 \div 4} = \frac{30}{1}
\]

This means that if Geoff can type 120 words in 4 minutes, then his *unit* rate is 30 words per 1 minute.

**Finding Unit Price**

A unit price is a form of unit rate and is calculated the same way.

**Example:** Find the unit price of ribbon if 3 yards of ribbon cost $1.50.

Step 1: Write the rate: 
\[
\frac{$1.50}{3 \text{ yards}}
\]

Step 2: Divide to find price for 1 yard: 
\[
\frac{$1.50 \div 3}{3 \text{ yards} \div 3} = \frac{$0.50}{1 \text{ yard}}
\]

Finding unit prices can be used to compare costs.

**Example:** Pens can be purchased in a 5-pack for $1.95 or a 15-pack for $6.15. Which is the better buy?

In this case students need to calculate the unit rate for both and then decide which is less expensive.

Step 1: Write both rates: 
\[
\frac{$1.95}{5 \text{ pens}} \text{ and } \frac{$6.15}{15 \text{ pens}}
\]

Step 2: Calculate unit price for each: 
\[
\frac{$1.95 \div 5}{5 \text{ pens} \div 5} = \frac{$0.39}{1 \text{ pen}} \text{ and } \frac{$6.15 \div 15}{15 \text{ pens} \div 15} = \frac{$0.41}{1 \text{ pen}}
\]

Step 3: State your conclusion: The 5-pack of pens is a better buy because you are paying less per pen.
Solving Proportions

*Syllabus Objective: (4.3) The student will solve proportions.*

To solve problems, most people use either equivalent fractions or cross products to solve proportions. Generally, you use equivalent fractions when either the numerator or denominator of a fraction is a multiple of the numerator or denominator of the other fraction. If that is not immediately obvious, then use cross products.

**Example:** Find the value of $x$.

\[
\frac{6}{10} = \frac{36}{x}
\]

In this example, one numerator is a multiple of the other so it is appropriate to find equivalent fractions.

\[
\frac{6 \cdot 6}{10 \cdot 6} = \frac{36}{60}
\]

Therefore you can conclude that $x = 60$.

**Example:** Solve the proportion.

\[
\frac{d}{12} = \frac{96}{8}
\]

Since multiples are not obvious, use cross products on this problem.

\[
8d = (12)(96)
\]

\[
d = 1152
\]

\[
d = 144
\]

**Example:** J&A Department Store is selling 3 pairs of children’s socks for $5. Mrs. Wagner wants to buy a dozen pairs of socks. How much will this cost?

**Step 1:** Write the proportion.

\[
\frac{3 \text{ pairs}}{5} = \frac{12 \text{ pairs}}{x}
\]

**Step 2:** Solve.

\[
3x = 60
\]

\[
x = 20
\]
Applications of Proportions:  
Congruent and Similar Polygons

Syllabus Objective:  (4.5) The student will determine similar figures.  (4.6) The student will apply the properties of proportionality to congruent or similar shapes.

*Congruent polygons* have the *exact* same shape and size.  
Here are several examples:

**Triangles** $ABC$ and $XYZ$ shown below are congruent.  

![Triangles](image)

We can write that as $\triangle ABC \cong \triangle XYZ$.  The symbol “$\cong$” means “is congruent to”.  If we could lift $\triangle ABC$ and place it on top of $\triangle XYZ$, $A$ would fall on $X$, $B$ on $Y$, and $C$ on $Z$.

These matching vertices are called *corresponding vertices*.  Angles at corresponding vertices are *corresponding angles*, and the sides joining corresponding vertices are *corresponding sides*.

Polygons are congruent if:
1. Corresponding angles of congruent figures are congruent.
2. Corresponding sides of congruent figures are congruent.

When we name two congruent figures we list corresponding vertices in the same order.  Thus when we see $\triangle ABC \cong \triangle XYZ$ or $\triangle CAB \cong \triangle ZXY$, we know that:

$\angle A \cong \angle X$,  \hspace{10pt}  \angle B \cong \angle Y$,  \hspace{10pt}  \angle C \cong \angle Z$  

$\overline{AB} \cong \overline{XY}$,  \hspace{10pt}  $\overline{BC} \cong \overline{YZ}$,  \hspace{10pt}  $\overline{CA} \cong \overline{ZX}$

Knowing corresponding parts can help us to determine missing measurements in congruent figures.
Example: Find the values of $x$ and $y$, given: $ABCD \cong FGHJ$

![Diagram of congruent figures]

Because the figures are congruent, the corresponding angles are congruent and the corresponding sides are congruent. So,

$\angle B \cong \angle G$, so $m\angle B = m\angle G = 70^\circ$; $x = 70^\circ$

$BC \cong GH$, so $BC = GH = 5\ cm$; $y = 5\ cm$

Similar polygons have the same shape, but not necessarily the same size. Some examples are shown below.

![Diagram of similar polygons]

Corresponding parts of polygons are in the same relative position. For instance, in the similar figures below,

$\overline{AB}$ corresponds to $\overline{EF}$

$\angle A$ corresponds to $\angle E$

$\overline{BC}$ corresponds to $\overline{FG}$

$\angle B$ corresponds to $\angle F$

$\overline{CD}$ corresponds to $\overline{GH}$

$\angle C$ corresponds to $\angle G$

$\overline{AD}$ corresponds to $\overline{EH}$

$\angle D$ corresponds to $\angle H$
Two polygons are similar if:

a) the measure of their corresponding angles are equal and
b) the ratio of the lengths of their corresponding sides are proportional

The notation “~” means “is similar to”. For instance, in the above example we can write $ABCD \sim EFGH$ which we would read “rectangles $ABCD$ is similar to rectangle $EFGH$.”

Remember, the ratios of the lengths of corresponding sides of similar figures are equal. This property can be used to help us determine missing lengths in similar figures.

**Example:** Find the values of $x$, given these two rectangles are similar.

\[
\begin{array}{cc}
\text{4 in} & x \\
10 \text{ in} & 5 \text{ in}
\end{array}
\]

Since the two rectangles are similar, their sides must be in proportion. That is, the left side is to the bottom in the large rectangle as the left side is to the bottom in the small rectangle. Another way of saying that is “the width is to the length in the large rectangle as the width is to the length in the small rectangle.”

\[
\frac{4}{10} = \frac{x}{5}
\]

\[
10x = 4 \cdot 5
\]

\[
10x = \frac{20}{10}
\]

\[
x = 2 \text{ inches}
\]

A scale factor is the ratio formed by the corresponding sides. What is the scale factor for the previous problem?

\[
\frac{4}{2} = \frac{2}{1} \text{ or } 2:1
\]

Again, in order for figures to be similar the corresponding angles must be congruent and the corresponding sides must be proportional.
**Example**: Which rectangles are similar?

<table>
<thead>
<tr>
<th>Diagram</th>
<th>Corresponding Parts</th>
</tr>
</thead>
<tbody>
<tr>
<td>△ABC</td>
<td>△EFG</td>
</tr>
<tr>
<td></td>
<td>(\angle A = 53^\circ)</td>
</tr>
<tr>
<td></td>
<td>(\angle B = 90^\circ)</td>
</tr>
<tr>
<td></td>
<td>(\angle C = 37^\circ)</td>
</tr>
<tr>
<td></td>
<td>(\angle E = 53^\circ)</td>
</tr>
<tr>
<td></td>
<td>(\angle F = 90^\circ)</td>
</tr>
<tr>
<td></td>
<td>(\angle G = 37^\circ)</td>
</tr>
<tr>
<td></td>
<td>(\Delta ABC \sim \Delta EFG)</td>
</tr>
</tbody>
</table>

Since the three figures are rectangles, we know that the corresponding angles are congruent (since angles of a rectangle measure 90 degrees). Therefore, we just need to test the corresponding sides to see if they are proportional.

How does \(\frac{4}{10}\) compare to \(\frac{2}{5}\) and \(\frac{5}{12}\)?

Answer: \(\frac{4}{10} = \frac{2}{5}\), but \(\frac{5}{12} \neq \) either ratio.

Therefore you can conclude that rectangle J is similar to rectangle K because the corresponding angles are congruent and the corresponding sides are proportional.
You can use scale factors to find missing dimensions.

**Example:** A picture is 10 inches tall and 14 inches wide is to be scaled to 1.5 inches tall to be displayed on a Web page. How wide should the picture be on the Web page for the two pictures to be similar?

Draw a picture.

```
10 in

14 in
```

```
1.5 in
```

\[\frac{10}{1.5} = \frac{14}{x}\]

\[10x = (14)(1.5)\]

\[10x = 21\]

\[x = 2.1\]

The missing side is 2.1 in long.

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**Dilatations**

**Syllabus Objective: (4.4) The student will create dilations of plane figures.**

*Dilation* is a transformation that changes the size, but not the shape of a figure. A dilation can enlarge or reduce a figure.

The *center of dilation* is the point of intersection of lines through each pair of corresponding vertices in the dilation.

**Things to remember:** a scale factor describes how much a figure is enlarged or reduced. It can be expressed as a decimal, fraction, or percent. A scale factor between 0 and 1 *reduces* a figure, whereas a scale factor greater than 1 *enlarges* a figure.

Let’s use the origin as the center of dilation.
**Example:** Dilate the triangle with vertices $A(3, 2)$, $B(1, 1)$, $C(3, 1)$ by a scale factor of 3.

Step 1: Multiply each vertex by the scale factor.

Step 2: Graph the dilation.

$$
\begin{align*}
A (3, 2) &\rightarrow A'(3 \cdot 3, 2 \cdot 3) \rightarrow A'(9, 6) \\
B (1, 1) &\rightarrow B'(1 \cdot 3, 1 \cdot 3) \rightarrow B'(3, 3) \\
C (3, 1) &\rightarrow C'(3 \cdot 3, 1 \cdot 3) \rightarrow C'(9, 3)
\end{align*}
$$

Students can now graph $A'$, $B'$, and $C'$ on the same coordinate plane as the original figure.

In the following example, students are asked to find the scale factor based on the picture.

$$
\begin{align*}
A(−2,−2) &\text{ becomes } A'(−4,−4) \\
B(−1, 2) &\text{ becomes } B'(−2, 4) \\
C(2, 1) &\text{ becomes } C'(4, 2)
\end{align*}
$$

Now students need to figure out what number the original ordered pairs was multiplied by to generate the new ordered pair.

In this case if you multiply $A$ by a scale factor of 2, you will get $A'$. The same is true for $B$ and $C$.

Therefore, the scale factor is 2.

Note: There are many opportunities during this unit for students to work in groups on hands-on activities—provide those opportunities for your students. Look online for interactive applications of these concepts to share with your students.