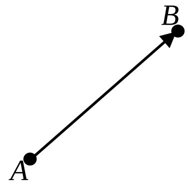
**Syllabus Objectives: 5.1 – The student will explore methods of vector addition and subtraction. 5.2 – The student will develop strategies for computing a vector's direction angle and magnitude given its coordinates. 5.4 – The student will resolve vectors into unit vectors. 5.7 – The student will solve real-world application problems using vectors in two and three dimensions.**

Directed Line Segment: a segment with direction and distance




$\overrightarrow{AB}$ *A*: **Initial Point** (start); *B*: **Terminal Point** (end)

Coordinates of *A*: $A_x, A_y$ Coordinates of *B*: $B_x, B_y$

Magnitude (length) of a Directed Line Segment $\overrightarrow{AB}$: $\left|\overrightarrow{AB}\right| = \sqrt{\ B_x - A_x\ ^2 + \ B_y - A_y\ ^2}$

Note: This is the distance formula!

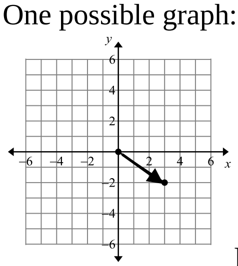
Vector (**v**): the set of all directed line segments that are equivalent to a given directed line segment

Note: **Equivalent** means same magnitude and direction.

Component Form of a Vector: $\langle B_x - A_x, B_y - A_y \rangle$

**Ex:** Graph the vector $\overrightarrow{AB} = \langle 3, -2 \rangle$ and find the magnitude.

One possible graph:




Note: $\overrightarrow{AB} = \langle 3, -2 \rangle$ could be placed anywhere on the coordinate grid. We have placed it in **standard position**, which is with the initial point at the origin.

Magnitude: $\left\|\overrightarrow{AB}\right\| = \sqrt{\ 3-0\ ^2 + \ -2-0\ ^2} = \sqrt{9+4} = \boxed{\sqrt{13}}$

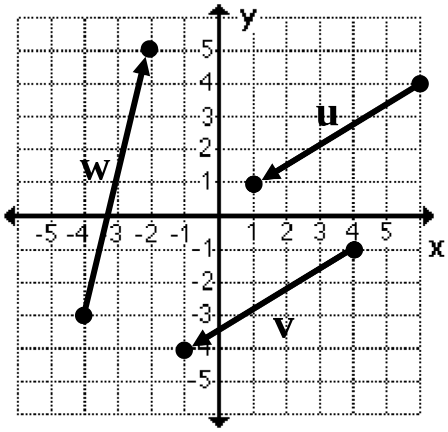
Note: If a vector **u** is written in component form, $\mathbf{u} = \langle u_1, u_2 \rangle$, then the magnitude of **u** is $\|\mathbf{u}\| = \sqrt{\ u_1\ ^2 + \ u_2\ ^2}$. This is because the initial point is the origin, $\ 0,0\ $.

Vector Addition: Let $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$. Then $\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2 \rangle$.

Scalar Multiplication: Let $\mathbf{u} = \langle u_1, u_2 \rangle$ and *k* be any constant. Then $k\mathbf{u} = \langle ku_1, ku_2 \rangle$.

Note: If $k < 0$, then $k\mathbf{u}$ is in the opposite direction.

**Ex:** Use the graph of the vectors to complete each example below.




1. Show that $\mathbf{u} = \mathbf{v}$.

Show that $\|\mathbf{u}\| = \|\mathbf{v}\|$. $\|\mathbf{u}\| = \sqrt{\ 6-1\ ^2 + \ 4-1\ ^2} = \sqrt{25+9} = \sqrt{34}$

$\|\mathbf{v}\| = \sqrt{\ 4- \ -1\ \ ^2 + \ -1- \ -4\ \ ^2} = \sqrt{25+9} = \sqrt{34}$

Show that the direction of **u** is the same as the direction of **v**.

Use slope: Direction of **u** = $\frac{4-1}{6-1} = \frac{3}{5}$; direction of **v** = $\frac{-1- \ -4}{4- \ -1} = \frac{3}{5}$

The direction and magnitude are the same, so $\mathbf{u} = \mathbf{v}$.

2. Find the component form and the magnitude of $\mathbf{u}$ and $\mathbf{w}$.

Component form of **u**: $\mathbf{u} = \langle 1-6, 1-4 \rangle = \langle -5, -3 \rangle$ $\|\mathbf{u}\| = \sqrt{34}$ (see above)

Component form of **w**: $\mathbf{w} = \langle -2- \ -4 \ , 5- \ -3 \ \rangle = \langle 2, 8 \rangle$ $\|\mathbf{w}\| = \sqrt{2^2+8^2} = \sqrt{4+64} = \sqrt{68} = 2\sqrt{17}$

3. Find the component form of $2\mathbf{u} - 3\mathbf{w}$.

$2\mathbf{u} - 3\mathbf{w} = 2\langle -5,-3 \rangle - 3\langle 2,8 \rangle = \langle -10,-6 \rangle + \langle -6,-24 \rangle = \langle -10+-6, -6+-24 \rangle = \boxed{\langle -16,-30 \rangle}$

Unit Vector: a vector with a magnitude of 1

A unit vector in the direction of a vector **v** can be found by dividing **v** by the magnitude of **v**.

Unit Vector in the Direction of **v**: $\frac{\mathbf{v}}{\|\mathbf{v}\|}$

Standard Unit Vectors: unit vectors **i** and **j** in standard position along the positive *x*- and *y*-axes

$\mathbf{i} = \langle 1,0 \rangle$ & $\mathbf{j} = \langle 0,1 \rangle$

Any vector can be written in terms of the standard unit vectors.

**Ex:** Write the vector $\mathbf{v} = \langle -2,5 \rangle$ in terms of the standard unit vectors.

$\mathbf{v} = \langle -2,5 \rangle = -2\langle 1,0 \rangle + 5\langle 0,1 \rangle = \boxed{-2\mathbf{i}+5\mathbf{j}}$

**Ex**:  Find a unit vector in the direction of the given vector.  Verify your answer is a unit vector and give your answer in component form and standard unit vector form.  $2\mathbf{i}-4\mathbf{j}$

Find the magnitude: $\|2\mathbf{i}-4\mathbf{j}\| = \sqrt{2^2 + \ -4\ ^2} = \sqrt{20} = 2\sqrt{5}$

Divide the original vector by its magnitude: $\dfrac{2\mathbf{i}-4\mathbf{j}}{2\sqrt{5}} = \dfrac{2\mathbf{i}}{2\sqrt{5}} - \dfrac{4\mathbf{j}}{2\sqrt{5}} = \dfrac{\mathbf{i}}{\sqrt{5}} - \dfrac{2\mathbf{j}}{\sqrt{5}} = \boxed{\dfrac{\sqrt{5}}{5}\mathbf{i} - \dfrac{2\sqrt{5}}{5}\mathbf{j}}$ (SUV)

Component Form: $\boxed{\left\langle \dfrac{\sqrt{5}}{5}, -\dfrac{2\sqrt{5}}{5} \right\rangle}$

Verify magnitude of unit vector: $\left\| \left\langle \dfrac{\sqrt{5}}{5}, -\dfrac{2\sqrt{5}}{5} \right\rangle \right\| = \sqrt{\left(\dfrac{\sqrt{5}}{5}\right)^2 + \left(\dfrac{2\sqrt{5}}{5}\right)^2} = \sqrt{\dfrac{5}{25}+\dfrac{20}{25}} = \sqrt{\dfrac{25}{25}} = 1$ ☺

**Recall**:  In the unit circle, $x=\cos\theta,\ y=\sin\theta$.  This leads into another way of expressing a vector, in terms of its direction angle, *θ*.

<u>Direction Angle</u>:  in standard position, the angle the vector makes with the positive *x*-axis (counterclockwise)

<u>Resolving a Vector</u>:  in terms of its direction angle, *θ*, a vector can be written as

$$\|\mathbf{u}\| \cdot \langle \cos\theta, \sin\theta \rangle = \|\mathbf{u}\|\cos\theta\mathbf{i} + \|\mathbf{u}\|\sin\theta\mathbf{j}$$

**Ex:**  Find the magnitude and direction angle of $\mathbf{v} = -2\mathbf{i}+6\mathbf{j}$.

Magnitude: $\|\mathbf{v}\| = \sqrt{\ -2\ ^2 + 6^2} = \sqrt{40} = \boxed{2\sqrt{10}}$

Direction angle: $\|\mathbf{v}\|\cos\theta\mathbf{i} + \|\mathbf{v}\|\sin\theta\mathbf{j} \Rightarrow\ -2 = \|\mathbf{v}\|\cos\theta$

$$-2 = 2\sqrt{10}\cos\theta$$

$$-\frac{1}{\sqrt{10}} = \cos\theta$$

$$\theta = \cos^{-1}\left(-\frac{1}{\sqrt{10}}\right) \approx \boxed{108.43^\circ}$$

OR $6 = \|\mathbf{v}\|\sin\theta$

$6 = 2\sqrt{10}\sin\theta$

$\dfrac{3}{\sqrt{10}} = \sin\theta$ but since we know $\mathbf{v} = -2\mathbf{i}+6\mathbf{j}$ is in Quadrant II, $\theta = 180 - 71.57 = \boxed{108.43^\circ}$

$\theta = \sin^{-1}\left(\dfrac{3}{\sqrt{10}}\right) \approx 71.57^\circ$

**Ex:** Find the component form of **v** given its magnitude and its direction angle. $\|\mathbf{v}\| = 5,\ \theta = 30^\circ$

$$\mathbf{v} = \|\mathbf{v}\|\cos\theta\,\mathbf{i} + \|\mathbf{v}\|\sin\theta\,\mathbf{j} \Rightarrow \mathbf{v} = 5\cos 30^\circ\,\mathbf{i} + 5\sin 30^\circ\,\mathbf{j} \Rightarrow \boxed{\mathbf{v} = \frac{5\sqrt{3}}{2}\mathbf{i} + \frac{5}{2}\mathbf{j}}$$

<u>Application: Resultant Force</u>

**Ex:** Two forces act on an object: $\|\mathbf{u}\| = 3,\ \theta_u = 45^\circ$ and $\|\mathbf{v}\| = 4,\ \theta_u = -30^\circ$. Find the direction and magnitude of the resultant force.

Write each vector in component form:

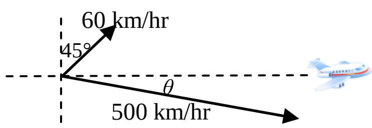
$$\mathbf{u} = \|\mathbf{u}\|\cos\theta_u\mathbf{i} + \|\mathbf{u}\|\sin\theta_u\mathbf{j} \Rightarrow \mathbf{u} = 3\cos 45^\circ\,\mathbf{i} + 3\sin 45^\circ\,\mathbf{j} \Rightarrow \mathbf{u} = \frac{3\sqrt{2}}{2}\mathbf{i} + \frac{3\sqrt{2}}{2}\mathbf{j}$$

$$\mathbf{v} = \|\mathbf{v}\|\cos\theta_v\mathbf{i} + \|\mathbf{v}\|\sin\theta_v\mathbf{j} \Rightarrow \mathbf{v} = 4\cos(-30^\circ)\,\mathbf{i} + 4\sin(-30^\circ)\,\mathbf{j} \Rightarrow \mathbf{v} = 2\sqrt{3}\mathbf{i} - 2\mathbf{j}$$

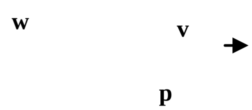
The resultant force is the sum $\mathbf{u}+\mathbf{v}$: $\boxed{\left\langle \frac{3\sqrt{2}}{2} + 2\sqrt{3}, \frac{3\sqrt{2}}{2} - 2 \right\rangle}$

<u>Application: Bearing</u>

**Ex:** A plane flies due east at 500 km/h and there is a 60 km/h with a bearing of 45°. Find the ground speed and the actual bearing of the plane.




Sketch a diagram:




Find the vectors **p** and **w**: $\mathbf{p} = \langle 500\cos\theta, 500\sin\theta \rangle$

$\mathbf{w} = \langle 60\cos 45^\circ, 60\sin 45^\circ \rangle$ Note: The 45° is the direction angle, not the bearing.

Vector **v** is the sum **p** + **w**: $\mathbf{v} = \langle 60\cos 45^\circ + 500\cos\theta, 60\sin 45^\circ + 500\sin\theta \rangle$

The second component of vector **v** must equal zero, because the plane is headed due east.

$$60\sin 45^\circ + 500\sin\theta = 0 \Rightarrow \sin\theta = -\frac{60\sin 45^\circ}{500} \Rightarrow \theta = \sin^{-1}\left(-\frac{60\sin 45^\circ}{500}\right) \Rightarrow \theta \approx -4.868^\circ$$

Bearing of the plane: $90^\circ + |\theta| \approx \boxed{94.868^\circ}$

Ground speed of the plane:

$$|\mathbf{v}| = \sqrt{(60\cos 45^\circ + 500\cos\theta)^2 + 0^2} = |60\cos 45^\circ + 500\cos(-4.868^\circ)| \approx \boxed{540.623\text{ km/hr}}$$

You Try: Find the component form of **v** given its magnitude and the angle it makes with the positive *x*-axis. $\|\mathbf{v}\| = 2$ , direction: $2\mathbf{i} + 3\mathbf{j}$

QOD: In the examples in your notes, we used sine or cosine to find the direction angle of a vector. Explain how you could use tangent to find the direction angle.

**Syllabus Objective: 5.3 – The student will explore methods of vector multiplication. 5.5 – The student will determine if two vectors are parallel or perpendicular (orthogonal). 5.6 – The student will derive an equation of a line or plane by using vector operations. 5.7 – The student will solve real-world application problems using vectors in two and three dimensions.**

<u>Dot Product:</u> Let $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$. The dot product is $\mathbf{u} \bullet \mathbf{v} = u_1 v_1 + u_2 v_2$.

Note: The dot product is a scalar.

**Ex:** Evaluate $\langle 5, -2 \rangle \bullet \langle 3, 4 \rangle$.

$\langle 5, -2 \rangle \bullet \langle 3, 4 \rangle = 5 \;\; 3 \; + \; -2 \;\; 4 \; = 15 - 8 = \boxed{7}$

<u>Properties of the Dot Product:</u>

1. $\mathbf{u} \bullet \mathbf{v} = \mathbf{v} \bullet \mathbf{u}$
2. $\mathbf{u} \bullet \mathbf{u} = \|\mathbf{u}\|^2$
3. $\mathbf{0} \bullet \mathbf{u} = \mathbf{0}$
4. $\mathbf{u} \bullet \;\; \mathbf{v} + \mathbf{w} \;\; = \mathbf{u} \bullet \mathbf{v} + \mathbf{u} \bullet \mathbf{w}$
5. $c\mathbf{u} \;\; \bullet \mathbf{v} = \mathbf{u} \bullet \;\; c\mathbf{v} \;\; = c \;\; \mathbf{u} \bullet \mathbf{v}$

**Ex:** Evaluate the following given $\mathbf{u} = \langle -3, 6 \rangle$; $\mathbf{v} = \langle 1, 0 \rangle$; $\mathbf{w} = \langle 5, -2 \rangle$

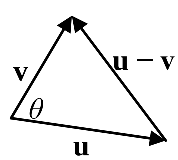
(a) $\mathbf{w} \bullet \mathbf{w}$ $\qquad$ $\mathbf{w} \bullet \mathbf{w} = 5 \;\; 5 \; + \; -2 \;\; -2 \; = \boxed{29}$

(b) $\|\mathbf{w}\|$ $\qquad$ $\|\mathbf{w}\| = \sqrt{\;5\;^2 + \;-2\;^2} = \sqrt{25+4} = \boxed{\sqrt{29}}$

(c) $\;\mathbf{v} + \mathbf{w} \;\; \bullet \mathbf{u}$ $\qquad$ $\;\mathbf{v} + \mathbf{w} \;\; \bullet \mathbf{u} = \langle 1+5, 0+-2 \rangle \bullet \mathbf{u} = \langle 6, -2 \rangle \bullet \langle -3, 6 \rangle = 6 \;\; -3 \; + \; -2 \;\; 6 \; = \boxed{-30}$

(d) $\mathbf{v} \bullet \mathbf{u} + \mathbf{w} \bullet \mathbf{u}$ $\qquad$ $\mathbf{v} \bullet \mathbf{u} + \mathbf{w} \bullet \mathbf{u} = \left[1 \;\; -3 \; + 0 \;\; 6 \;\right] + \left[5 \;\; -3 \; + \; -2 \;\; 6 \;\right] = -3 + \; -27 \; = \boxed{-30}$

<u>Angle Between Two Vectors</u>: $\cos\theta = \dfrac{\mathbf{u} \bullet \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} \Rightarrow \theta = \cos^{-1}\left(\dfrac{\mathbf{u} \bullet \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}\right)$




Proof: Use the triangle.

Law of Cosines: $\qquad \|\mathbf{v} - \mathbf{u}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\|\|\mathbf{v}\|\cos\theta$

Property of Dot Product: $\qquad \;\mathbf{v} - \mathbf{u} \;\; \bullet \;\; \mathbf{v} - \mathbf{u} \;\; = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\|\|\mathbf{v}\|\cos\theta$

Expand: $\qquad \mathbf{v} \bullet \mathbf{v} - \mathbf{v} \bullet \mathbf{u} - \mathbf{u} \bullet \mathbf{v} + \mathbf{u} \bullet \mathbf{u} = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\|\|\mathbf{v}\|\cos\theta$

Property of Dot Product: $\qquad \|\mathbf{v}\|^2 - 2\mathbf{u} \bullet \mathbf{v} + \|\mathbf{u}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\|\|\mathbf{v}\|\cos\theta$

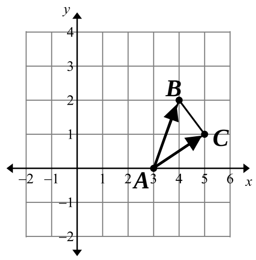
Property of Equality: $\qquad -2\mathbf{u} \bullet \mathbf{v} = -2\|\mathbf{u}\|\|\mathbf{v}\|\cos\theta \Rightarrow \cos\theta = \dfrac{\mathbf{u} \bullet \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}$

**Ex:** Find $\mathbf{u}\bullet\mathbf{v}$, where $\theta$ is the angle between **u** and **v**. $\|\mathbf{u}\| = 6,\ \|\mathbf{v}\| = 8,\ \theta = \frac{5\pi}{6}$

$$\cos\theta = \frac{\mathbf{u}\bullet\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} \Rightarrow \cos\frac{5\pi}{6} = \frac{\mathbf{u}\bullet\mathbf{v}}{6\quad 8} \Rightarrow \mathbf{u}\bullet\mathbf{v} = 48\left(-\frac{\sqrt{3}}{2}\right) \Rightarrow \mathbf{u}\bullet\mathbf{v} = \boxed{-24\sqrt{3}}$$

Application

**Ex:** Find the interior angles of the triangle with vertices .




$$\overrightarrow{AB} = \langle 4-3, 2-0\rangle = \langle 1,2\rangle;\ \overrightarrow{AC} = \langle 5-3, 1-0\rangle = \langle 2,1\rangle \qquad \cos A = \frac{1\ 2\ +2\ 1}{\sqrt{5}\ \sqrt{5}} \Rightarrow A = \cos^{-1}\left(\frac{4}{5}\right) \Rightarrow A \approx 36.87^\circ$$

$$\overrightarrow{BA} = \langle 3-4, 0-2\rangle = \langle -1,-2\rangle;\ \overrightarrow{BC} = \langle 5-4, 1-2\rangle = \langle 1,-1\rangle$$

$$\cos B = \frac{-1\ 1\ +\ -2\ \ -1}{\sqrt{5}\ \sqrt{2}} \Rightarrow B = \cos^{-1}\left(\frac{1}{\sqrt{10}}\right) \Rightarrow B \approx 71.565^\circ$$

$C \approx 180 - \ 36.87 + 71.565\ \approx 71.565^\circ$ Angles: $\boxed{36.87^\circ,\ 71.565^\circ, 71.565^\circ}$

Orthogonal Vectors: two vectors whose dot product is equal to 0

What is the angle between two non-zero orthogonal vectors?

$$\cos\theta = \frac{\mathbf{u}\bullet\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} \Rightarrow \cos\theta = \frac{0}{\|\mathbf{u}\|\|\mathbf{v}\|} \Rightarrow \cos\theta = 0 \Rightarrow \theta = 90^\circ$$

Note: If the angle between the vectors is 90°, we may also say they are perpendicular. The word orthogonal is used instead for vectors because the zero vector is orthogonal to any other vector, but is not perpendicular.

What is the dot product of two vectors that are parallel? The angle “between” them would have to be either 180° or 360°.

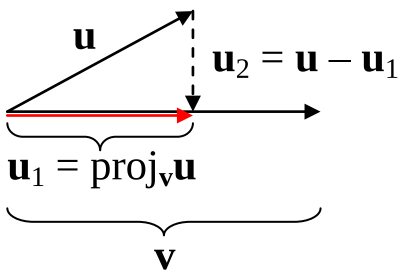
$$\cos\theta = \frac{\mathbf{u}\bullet\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} \Rightarrow \cos 180^\circ = \frac{\mathbf{u}\bullet\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} \Rightarrow -1 = \mathbf{u}\bullet\mathbf{v} \quad \text{or} \quad \cos\theta = \frac{\mathbf{u}\bullet\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} \Rightarrow \cos 360^\circ = \frac{\mathbf{u}\bullet\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} \Rightarrow 1 = \mathbf{u}\bullet\mathbf{v}$$

Parallel Vectors: two vectors whose dot product is equal to $-1$ or 1

**Ex:** Are the vectors orthogonal, parallel, or neither? $\mathbf{v} = 3\mathbf{i} - 2\mathbf{j},\ \mathbf{w} = 3\mathbf{i} + 4\mathbf{j}$

Find $\mathbf{v}\bullet\mathbf{w}$: $\mathbf{v}\bullet\mathbf{w} = 3\ 3\ \ -2\ 4\ \ = 1$ The vectors are **parallel**.

Vector Projection: the projection of **u** onto **v** is denoted by: $\text{proj}_{\mathbf{v}}\mathbf{u} = \left(\dfrac{\mathbf{u}\bullet\mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v}$




Note: The projection of **u** onto **v** is the "shadow" formed by vector **u** onto **v** as light comes straight down.

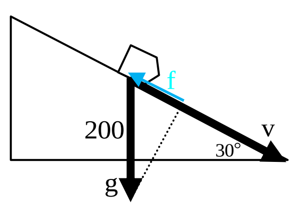
**Ex:** Find the projection of **v** onto **w**. Then write **v** as the sum of two orthogonal vectors, with one the $\text{proj}_{\mathbf{w}}\mathbf{v}$. $\mathbf{v} = \langle 1,3\rangle$; $\mathbf{w} = \langle 1,1\rangle$

$$\text{proj}_{\mathbf{w}}\mathbf{v} = \left(\frac{\mathbf{v}\bullet\mathbf{w}}{\|\mathbf{w}\|^2}\right)\mathbf{w} \Rightarrow \text{proj}_{\mathbf{w}}\mathbf{v} = \frac{1\ 1\ +3\ 1}{\sqrt{1^2+1^2}^{\,2}}\langle 1,1\rangle \Rightarrow \text{proj}_{\mathbf{w}}\mathbf{v} = 2\langle 1,1\rangle \Rightarrow \boxed{\text{proj}_{\mathbf{w}}\mathbf{v} = \langle 2,2\rangle}$$

$$\mathbf{v} - \text{proj}_{\mathbf{w}}\mathbf{v} = \langle 1,3\rangle - \langle 2,2\rangle = \langle -1,1\rangle \qquad \boxed{\mathbf{v} = \langle 2,2\rangle + \langle -1,1\rangle}$$

Application: Force

**Ex:** Find the force required to keep a 200-lb cart from rolling down a 30° incline.




Draw a diagram and label

The force due to gravity: $\mathbf{g} = -200\mathbf{j}$ (gravity acts vertically downward)

Incline vector: $\mathbf{v} = \cos 30^\circ\ \mathbf{i} - \sin 30^\circ\ \mathbf{j} = \dfrac{\sqrt{3}}{2}\mathbf{i} - \dfrac{1}{2}\mathbf{j}$

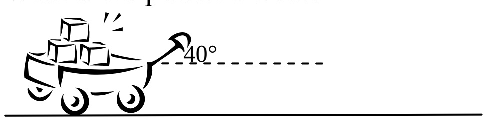
Force vector required to keep the cart from rolling: $\mathbf{f} = \text{proj}_{\mathbf{v}}\mathbf{g} = \left(\dfrac{\mathbf{g}\bullet\mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v}$ $\qquad \|\mathbf{v}\| = \sqrt{\left(\dfrac{\sqrt{3}}{2}\right)^2 + \left(-\dfrac{1}{2}\right)^2} = \sqrt{\dfrac{3}{4}+\dfrac{1}{4}} = 1$

$$\mathbf{f} = \left(\frac{\mathbf{g}\bullet\mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v} = \left(\frac{\langle 0,-100\rangle \bullet \left\langle \frac{\sqrt{3}}{2}, -\frac{1}{2}\right\rangle}{1^2}\right)\left\langle \frac{\sqrt{3}}{2}, -\frac{1}{2}\right\rangle = 100\left\langle \frac{\sqrt{3}}{2}, -\frac{1}{2}\right\rangle = \langle 50\sqrt{3}, -50\rangle$$

Magnitude of Force: $\|\mathbf{f}\| = \left\|\langle 50\sqrt{3}, -50\rangle\right\| = \sqrt{(50\sqrt{3})^2 + (-50)^2} \approx \boxed{100 \text{ pounds}}$

Application: Work $W = \cos\theta$ force distance

**Ex:** A person pulls a wagon with a constant force of 15 lbs at a constant angle of 40° for 500 ft. What is the person's work?




$w = \cos 40^{\circ}$ 15 lbs 500 ft $\approx$ $\boxed{5745.33 \text{ foot lbs}}$

You Try: Find the projection of **v** onto **u**. Then write **v** as the sum of two orthogonal vectors, with one the $\text{proj}_{\mathbf{u}}\mathbf{v}$. $\mathbf{v} = 2\mathbf{i} - 3\mathbf{j}$; $\mathbf{u} = \mathbf{i} - \mathbf{j}$

QOD: If **u** is a unit vector, what is $\mathbf{u} \cdot \mathbf{u}$? Explain why.

**Syllabus Objective: 1.10 – The student will solve problems using parametric equations.**

Parametric Curve: the set of all points $(x, y)$, where $x = f(t)$ and $y = g(t)$ are continuous functions of $t$ on an interval $I$ (called the **parameter interval**)

Parameter: the variable $t$

Parametric Equations: $x = f(t)$ and $y = g(t)$

Orientation: the directions that results from plotting the points as the values of $t$ increase

Graphing Parametric Equations

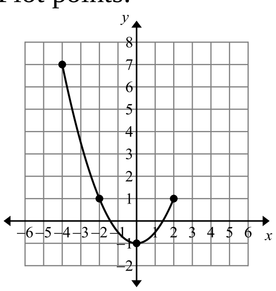
**Ex:** Graph $x = -2t,\ y = 2t^2 - 1,\ -1 \le t \le 2$

Note: Choose appropriate values for $t$ first. Then substitute in and find $x$ and $y$.

Make a table:

| $t$ | $x$ | $y$ |
|---|---|---|
| −1 | 2 | 1 |
| 1 | −2 | 1 |
| 0 | 0 | −1 |
| 2 | −4 | 7 |

Plot points:




Eliminating the Parameter

1. Solve one of the equations for $t$. (Or if a trig function, isolate the trig function.)
2. Substitute for $t$ in the other equation. (Or use an identity if a trig function.)

**Ex:** Write the parametric equation as a function of $y$ in terms of $x$.

a) $x = -2t,\ y = 2t^2 - 1$

Solve for $t$ in the $x$-equation (easier to solve for): $x = -2t \Rightarrow t = \dfrac{x}{-2}$

Substitute into the $y$-equation: $y = 2t^2 - 1 \Rightarrow y = 2\left(\dfrac{x}{-2}\right)^2 - 1 \Rightarrow \boxed{y = \dfrac{1}{2}x^2 - 1}$

b) $x = \dfrac{1}{\sqrt{t+2}},\ y = \dfrac{t}{t+2}$

Solve for $t$ in the $x$-equation (easier to solve for):

$$x = \frac{1}{\sqrt{t+2}} \Rightarrow x^2 = \frac{1}{t+2} \Rightarrow (t+2)x^2 = 1 \Rightarrow t+2 = \frac{1}{x^2} \Rightarrow t = \frac{1}{x^2} - 2$$

Substitute into the *y*-equation: $y=\frac{t}{t+2}\Rightarrow y=\frac{\frac{1}{x^2}-2}{\frac{1}{x^2}-2+2}\Rightarrow y=\frac{\frac{1-2x^2}{x^2}}{\frac{1}{x^2}}\Rightarrow \boxed{y=1-2x^2}$

**Ex:** Write the parametric equation as a function of *y* and graph.

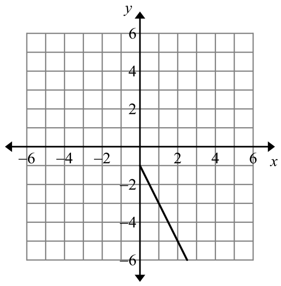
$x=\tan\theta,\ y=-2\tan\theta-1,\ 0\le\theta<\frac{\pi}{2}$

Note: A parametric equation can be written in terms of $\theta$ instead of $t$.

The *x*-equation is already solved for the trig function: $x=\tan\theta$

Substitute into the *y*-equation: $y=-2x-1$

Graph:




<u>Using a Trig Identity</u>

**Ex:** Eliminate the parameter. $x=6\cos t,\ y=6\sin t,\ 0\le t\le 2\pi$

Solving for a trig function won't help, so we need to use the identity $\sin^2 t+\cos^2 t=1$.

Square both equations: $x^2=36\cos^2 t,\ y^2=36\sin^2 t$

Add the equations: $x^2+y^2=36\cos^2 t+36\sin^2 t\Rightarrow x^2+y^2=36\ \ \cos^2 t+\sin^2 t$

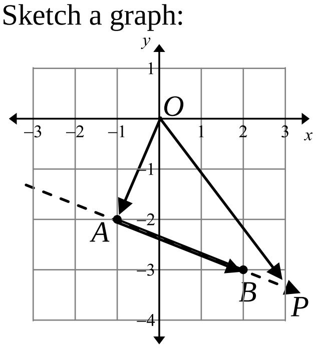
Trig identity: $x^2+y^2=36\ \ \cos^2 t+\sin^2 t\ \Rightarrow \boxed{x^2+y^2=36}$

Note: The graph is a circle. The parameter interval lets us know that it would go around 1 time.

<u>Writing a Parameterization</u>

**Ex:** Find the parameterization of the line segment through the points $A=\ -1,-2$ and $B=\ 2,-3$ .

Sketch a graph:




$\overrightarrow{OP}=\overrightarrow{OA}+\overrightarrow{AP}$; $\overrightarrow{AP}$ is a scalar multiple of $\overrightarrow{AB}$, so $\overrightarrow{OP}=\overrightarrow{OA}+t\cdot\overrightarrow{AB}$

$\overrightarrow{OP} = \overrightarrow{OA} + t \cdot \overrightarrow{AB} \Rightarrow \langle x, y\rangle = \langle -1, -2\rangle + t\langle 2 - \ -1\ , -3 - \ -2\ \rangle \Rightarrow \langle x, y\rangle = \langle -1, -2\rangle + t\langle 3, -1\rangle$

$\langle x, y\rangle = \langle -1+3t, -2-t\rangle$ $\qquad x = -1+3t,\ y = -2-t$ These equations define the LINE.

Find the parameter interval for the line segment: We want $-1 \le x \le 2$.

$x = -1+3t$: $\qquad -1 = -1+3t \Rightarrow t = 0 \qquad 2 = -1+3t \Rightarrow t = 1 \qquad$ So, $0 \le t \le 1$

Solution: $\boxed{x = -1+3t,\ y = -2-t,\ 0 \le t \le 1}$

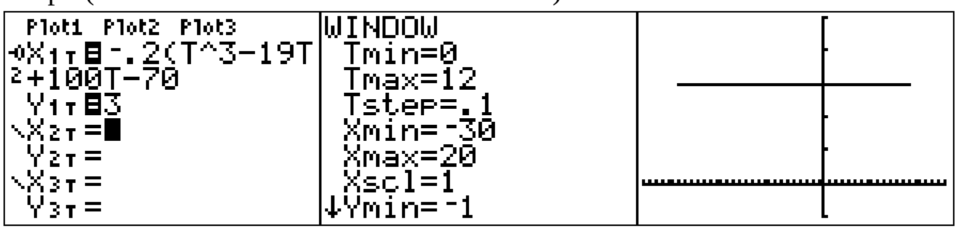
Simulating Horizontal Motion

**Ex:** A dog is running on a horizontal path with the coordinates of his position (in meters) given by $s = -0.2\ t^3 - 19t^2 + 100t - 70$ where $0 \le t \le 15$. Use parametric equations and a graphing calculator to simulate the dog's motion.

Choose any horizontal line to simulate the motion: We will choose $y = 3$.

Parametric Equations: $x = -0.2\ t^2 - 19t^2 + 100t - 70\ ,\ y = 3,\ 0 \le t \le 15$

Graph (Calculator must be in Parametric mode):




Note: To see the motion, change the type of line to a "bubble". If you would like the bubble to move slower, make the Tstep smaller.

Parametric Equations for Projectile Motion

distance: $x = \ v_0 \cos\theta\ \ t$ $\qquad$ height: $y = -\frac{1}{2}gt^2 + \ v_0 \sin\theta\ \ t + h_0$

Note: On Earth, $g = 32$ ft/sec$^2$ or $g = 9.8$ m/sec$^2$. $v_0$ is initial velocity; $h_0$ is initial height.

**Ex:** A golf ball is hit at 150 ft/sec at a 30° angle to the horizontal.

a) When does it reach its maximum height?

Height: $y = -\frac{1}{2}gt^2 + \ v_0 \sin\theta\ \ t + h_0 \Rightarrow y = -16t^2 + 150\sin 30^\circ\ t + 0$

($h_0 = 0$ because a golf ball is hit from the ground)

Simplify: $\qquad y = -16t^2 + 150\sin 30^\circ\ t + 0 \Rightarrow y = -16t^2 + 75t$

Maximum height is at vertex: $\qquad t = -\frac{b}{2a} \Rightarrow t = -\frac{75}{2\ -16\ } = \frac{75}{32} = \boxed{2\frac{11}{32}\text{ sec}}$

b) How far does it go before it hits the ground?

Hits the ground when $y = 0$: $\qquad y = -16t^2 + 75t = 0 \Rightarrow t\ \ -16t + 75\ = 0 \Rightarrow -16t + 75 = 0 \Rightarrow t = 4\frac{11}{16}$ sec

Note: We could have doubled the time it took for the ball to reach its highest point!

Distance: $x = \left(v_0 \cos\theta\right) t = 150\cos 30^\circ \left(4\frac{11}{16}\right) \approx \boxed{608.84 \text{ ft}}$

c) Does the ball hit a 6 ft tall golfer, standing directly in the path of the ball 580 feet away?

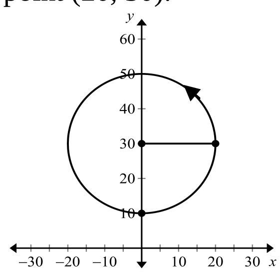
Find the time it takes for the ball to be 580 ft away: $x = \left(v_0 \cos\theta\right) t \Rightarrow 580 = 150\cos 30^\circ t \Rightarrow t \approx 4.46\,\text{sec}$

Find the height of the ball at this time: $y = -16t^2 + 75t = -16\left(4.46\right)^2 + 75\left(4.46\right) \approx 16.23\,\text{ft}$

**No** – the ball misses him by about 10 feet.

Application: Ferris Wheel

**Ex:** Zac is on a Ferris wheel of radius 20 ft that turns counterclockwise at a rate of one revolution every 24 sec. The lowest point of the Ferris wheel (6 o'clock) is 10 ft above ground level at the point (0, 10) on a rectangular coordinate system. Find the parametric equations for the position of Zac as a function of time *t* (in seconds) if the Ferris wheel starts (*t* = 0) with Zac at the point (20, 30).




Remember: $x = r\cos\theta,\ y = r\sin\theta$

Time to complete one revolution = 24 sec: $\frac{2\pi}{24} = \frac{\pi}{12}$

So, in one second, the ferris wheel travels through an angle of $\frac{\pi}{12}$.

When $t = 0,\ x = 20\ \&\ y = 30$: $\boxed{x = 20\cos\left(\frac{\pi}{12}t\right),\ y = 30 + 20\sin\left(\frac{\pi}{12}t\right),\ t \ge 0}$

You Try: A baseball is hit at 3 ft above the ground with an initial speed of 160 ft/sec at an angle of 17° with the horizontal. Will the ball clear a 20-ft wall that is 400 ft away?

QOD: How would you write a parametrization for a semicircle?

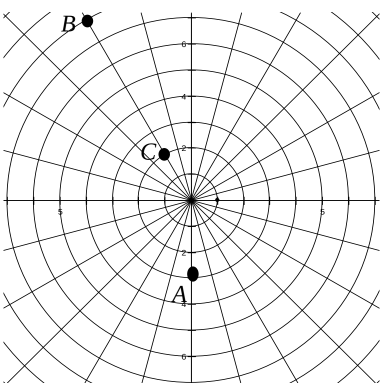
**Syllabus Objectives: 3.3 – The student will differentiate between polar and Cartesian (rectangular) coordinates. 6.2 – The student will transform functions between Cartesian and polar form. 6.4 – The student will solve real-world application problems using polar coordinates.**

Polar Coordinate: $r,\theta$ ; r: the directed distance from the **pole** (origin); $\theta$: the directed angle from the **polar axis** ($x$-axis)

Plotting Points on a Polar Graph

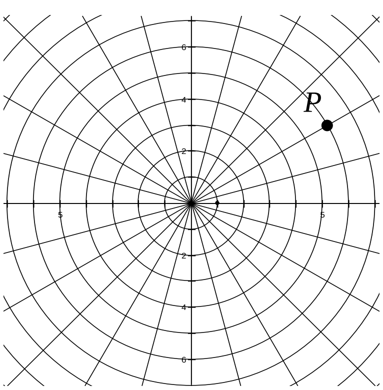
**Ex:** Plot the points $A\left(3,\frac{3\pi}{2}\right)$, $B\ 8,-240^\circ$ , & $C\left(-2,\frac{5\pi}{6}\right)$.

Point $A$: Start at the polar axis and go counter-clockwise $\frac{3\pi}{2}$ (270°). Place the point 3 units from the pole (origin). Point $B$: Start at the polar axis and go clockwise 240°. Place the point 8 units from the pole. (Note: Each radius drawn in the grid is 15°.) Point $C$: Start by going counter-clockwise $\frac{5\pi}{3}$ (300°) from the polar axis. Place a point 2 units from the pole. Because $r=-2$, you must place the point on the opposite side of the pole.




Writing the Polar Coordinates of a Point

**Ex:** Find four different polar coordinates of $P$.




Starting at the polar axis and going counter-clockwise: $6,30^\circ$

Starting at the polar axis and going clockwise: $6, -330^\circ$

Going counter-clockwise more than one revolution: $6, 390^\circ$

Using $r < 0$ and rotating counter-clockwise: $-6, 210^\circ$

Note: There are infinitely many correct answers!

Polar Conversions

Polar to Rectangular:
$$r\cos\theta = x$$
$$r\sin\theta = y$$

Rectangular to Polar:
$$\tan^{-1}\left(\frac{y}{x}\right) = \theta$$
$$x^2 + y^2 = r^2 \Rightarrow r = \sqrt{x^2 + y^2}$$

Converting from Polar to Rectangular Coordinates

**Ex:** Convert to rectangular coordinates.

a) $5\sqrt{2}, 45^\circ$

$r\cos\theta = x$

$r\sin\theta = y$

$$x = 5\sqrt{2}\cos 45^\circ = 5\sqrt{2}\left(\frac{1}{\sqrt{2}}\right) = 5$$
$$y = 5\sqrt{2}\sin 45^\circ = 5\sqrt{2}\left(\frac{1}{\sqrt{2}}\right) = 5$$

$\boxed{5, 5}$

b) $\left(-3, \frac{\pi}{3}\right)$

$r\cos\theta = x$

$r\sin\theta = y$

$$x = -3\cos\frac{\pi}{3} = -3\frac{1}{2} = -\frac{3}{2}$$
$$y = -3\sin\frac{\pi}{3} = -3\frac{\sqrt{3}}{2} = -\frac{3\sqrt{3}}{2}$$

$\boxed{\left(-\frac{3}{2}, -\frac{3\sqrt{3}}{2}\right)}$

Converting from Rectangular to Polar Coordinates (Note: Be careful with the quadrant!)

**Ex:** Convert to polar coordinates.

a) $-1, \sqrt{3}$

$\tan^{-1}\left(\frac{y}{x}\right) = \theta$

$x^2 + y^2 = r^2 \Rightarrow r = \sqrt{x^2 + y^2}$

$\theta = \tan^{-1} -\sqrt{3} = 60^\circ$

$r = \sqrt{1+3} = 2$

$-1, \sqrt{3}$ is in Quadrant II, so the polar coordinates are $\boxed{2, 120^\circ}$.

b) $0, -4$

This point is on the negative $y$-axis, so we know $\theta = 270^\circ$. $\boxed{4, 270^\circ}$

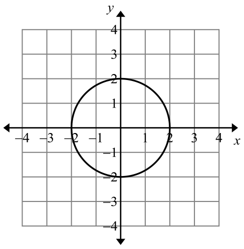
Note: There are other possible answers to these!

Converting from Polar to Rectangular Equations

**Ex:** Convert the equations and sketch the graph.

a) $r = 2$

$r = 2 \Rightarrow r^2 = 4 \Rightarrow \boxed{x^2 + y^2 = 4}$ Graph is a circle with center at origin & radius 2.

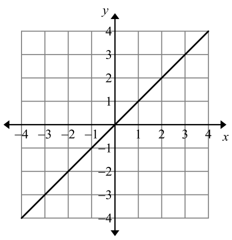



b) $\theta = \dfrac{\pi}{4}$

$x = r\cos\theta \Rightarrow x = r\cos\dfrac{\pi}{4} \Rightarrow x = r\dfrac{\sqrt{2}}{2}$

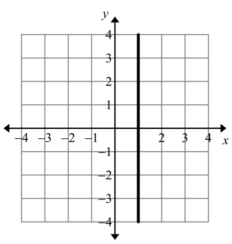
$y = r\sin\theta \Rightarrow y = r\sin\dfrac{\pi}{4} \Rightarrow y = r\dfrac{\sqrt{2}}{2}$

$\boxed{y = x}$




c) $r = \sec\theta$

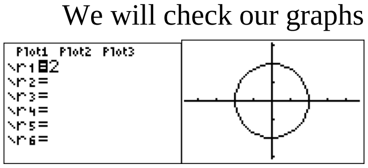
$r = \sec\theta \Rightarrow r = \dfrac{1}{\cos\theta} \Rightarrow r\cos\theta = 1 \Rightarrow \boxed{x = 1}$

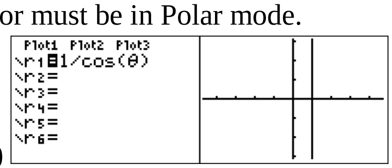




Graphing in Polar Coordinates on the Calculator

We will check our graphs above. Calculator must be in Polar mode.

a) 


c) 


Note: We cannot check the graph of b) on the calculator, but the line $y = x$ represents the angle $\dfrac{\pi}{4} = 45^\circ$ for all values of *r*.

Converting from Rectangular to Polar Equations

**Ex:** Convert the equations.

a) $x = 4$

$x = r\cos\theta$: $r\cos\theta = 4 \Rightarrow \boxed{r = 4\sec\theta}$

b) $3x - 6y + 2 = 0$

$$3x - 6y + 2 = 0 \Rightarrow 3r\cos\theta - 6r\sin\theta + 2 = 0 \Rightarrow r\left(3\cos\theta - 6\sin\theta\right) = -2 \Rightarrow r = \frac{-2}{3\cos\theta - 6\sin\theta} \text{ or } \boxed{r = \frac{2}{6\sin\theta - 3\cos\theta}}$$

c) $(x+2)^2 + (y-1)^2 = 5$

Expand: $x^2 + 4x + 4 + y^2 - 2y + 1 = 5 \Rightarrow x^2 + y^2 + 4x - 2y = 0$

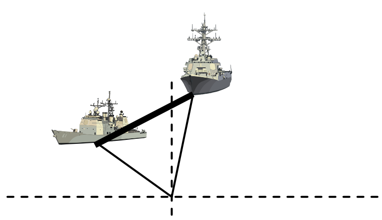
Substitute: $r^2 + 4r\cos\theta - 2r\sin\theta = 0 \Rightarrow r\left(r + 4\cos\theta - 2\sin\theta\right) = 0$

So $r = 0$ or $r + 4\cos\theta - 2\sin\theta = 0$. But $r = 0$ is a single point. So $\boxed{r = -4\cos\theta + 2\sin\theta}$

Application: Finding Distance

**Ex:** The location of two ships from the shore patrol station, given in polar coordinates, are $(2\text{ mi}, 150^\circ)$ & $(3\text{ mi}, 80^\circ)$. Find the distance between the ships.

Sketch a diagram:



Note: The angle between the ships (from the patrol station) is $150^\circ - 80^\circ = 70^\circ$.

Using the Law of Cosines: $d^2 = 2^2 + 3^2 - 2(2)(3)\cos 70^\circ \Rightarrow d = \sqrt{13 - 12\cos 70^\circ} \approx \boxed{2.983\text{ mi}}$

You Try: Convert the coordinates. Polar: $(2, \pi)$; Rectangular: $(-7, 7)$

QOD: How could you write an expression for all of the possible polar coordinates of a point?

**Syllabus Objectives: 6.3 – The student will sketch the graph of a polar function and analyze it.**

Tests for Symmetry of Polar Curves

1. Symmetry about $x$-axis: $r,\theta$ is equivalent to $r,-\theta$
2. Symmetry about $y$-axis: $r,\theta$ is equivalent to $-r,-\theta$
3. Symmetry about origin: $r,\theta$ is equivalent to $-r,\theta$

Graphing Polar Curves

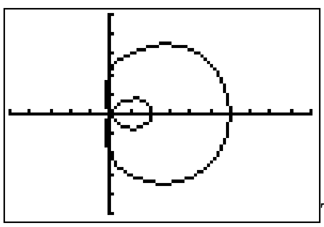
**Ex:** Graph and find the domain, range, symmetry, and maximum $r$-value.

a) $r = 2 + 4\cos\theta$

| $\boldsymbol{\theta}$ | 0 | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ | $\frac{2\pi}{3}$ | $\pi$ |
|---|---|---|---|---|---|
| $\boldsymbol{r}$ | 6 | 4 | 2 | 0 | −2 |

Domain: All real numbers Range: $-2 \le r \le 6$ Max $r$-value: $r = 6$

Symmetry: Substitute $-\theta$. $r = 2 + 4\cos\ -\theta\ = 2 + 4\cos\theta$ Symmetric about $x$-axis



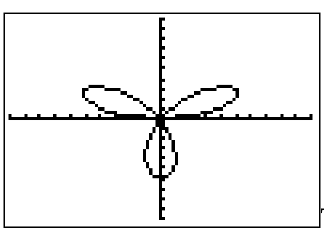
This curve is called a **limaçon**.

b) $r = 6\sin\ 3\theta$

| $\boldsymbol{\theta}$ | 0 | $\frac{\pi}{18}$ | $\frac{\pi}{6}$ | $\frac{5\pi}{18}$ | $\frac{\pi}{3}$ | $\frac{7\pi}{18}$ | $\frac{\pi}{2}$ |
|---|---|---|---|---|---|---|---|
| $\boldsymbol{r}$ | 0 | 3 | 6 | 3 | 0 | −3 | −6 |

Domain: All real numbers Range: $-6 \le r \le 6$ Max $r$-value: $r = 6$

Symmetry: Substitute $-r$ & $-\theta$. $-r = 6\sin\ -3\theta\ \Rightarrow -r = -6\sin\ 3\theta\ \Rightarrow r = 6\sin\ 3\theta$ Symmetric about $y$-axis



This curve is called a **rose**.

c) $r^2 = 4\cos\ 2\theta$

| $\boldsymbol{\theta}$ | 0 | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ |
|---|---|---|---|
| $\boldsymbol{r}$ | $\pm 2$ | $\pm\sqrt{2}$ | 0 |

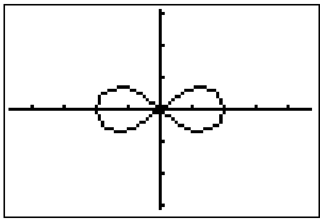
Domain: All real numbers Range: $-2 \le r \le 2$ Max $r$-value: $r = 2$

Symmetry:

Substitute $-\theta$. $r^2 = 4\cos\ -2\theta\ \Rightarrow r^2 = 4\cos\ 2\theta$ Symmetric about $x$-axis

Substitute $-r$ . $\quad (-r)^2 = 4\cos(2\theta) \Rightarrow r^2 = 4\cos(2\theta)$ Symmetric about origin

Substitute $-r$ & $-\theta$. $\quad (-r)^2 = 4\cos(-2\theta) \Rightarrow r^2 = 4\cos(2\theta)$ Symmetric about $y$-axis


This curve is called a **lemniscate**.

Classifications of Polar Curves

- Limaçon Curves: $r = a \pm b\sin\theta$ and $r = a \pm b\cos\theta$
- Rose Curves: $r = a\cos(n\theta)$ and $r = a\sin(n\theta)$

  Petals: odd = $n$ and even = $2n$
- Lemniscate Curves: $r^2 = a^2\cos(2\theta)$ and $r^2 = a^2\sin(2\theta)$

<u>You Try:</u> Use your graphing calculator to explore variations of $r = a \cdot \sin(n\theta)$ . Describe the effects of changing the window, the $\theta$-step, $a$, $n$, and changing $\sin\theta$ to $\cos\theta$.

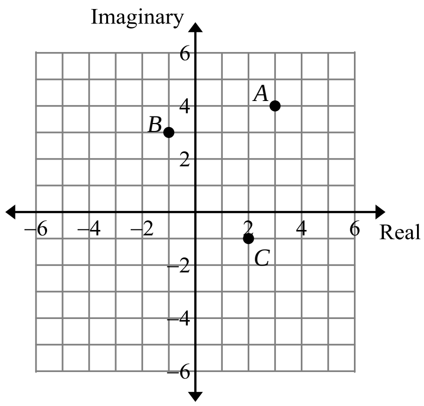
<u>QOD:</u> Are all polar curves bounded? Explain.

**Syllabus Objectives: 7.1 – The student will graph a complex number on the complex/Argand plane. 7.2 – The student will represent a complex number in trigonometric (polar) form. 7.3 – The student will simplify expressions involving complex numbers in trigonometric (polar) form. 7.4 – The student will compute the powers of complex numbers using DeMoivre's Theorem and find the nth roots of a complex number.**

Complex Number Plane (Argand Plane): horizontal axis: real axis; vertical axis: imaginary axis

Plotting Points in the Complex Plane

**Ex:** Plot the points $A(3+4i)$, $B(-1+3i)$, & $(2-i)$ in the complex plane.




Absolute Value (Modulus) of a Complex Number: the distance a complex number is from the origin on the complex plane $\quad |\boldsymbol{a}+\boldsymbol{bi}| = \sqrt{\boldsymbol{a}^2+\boldsymbol{b}^2}$ (This can be shown using the Pythagorean Theorem.)

**Ex:** Evaluate $|3-i|$. $\quad |3-i| = \sqrt{3^2+(-1)^2} = \boxed{\sqrt{10}}$

Recall: Trigonometric form of a vector: $\|\mathbf{u}\|\langle\cos\theta, \sin\theta\rangle$

Trigonometric Form of a Complex Number $z$ = a + $bi$: $z = r(\cos\theta + i\sin\theta)$

Note: This can also be written as $z = r\text{cis}\theta$.

$a = r\cos\theta,\ b = r\sin\theta,\ r = \sqrt{a^2+b^2} \qquad \tan\theta = \frac{b}{a}$

$r$ = **modulus**; $\theta$ = **argument**

Writing a Complex Number in Trig Form

**Ex:** Find the trigonometric form of $\sqrt{3}-i$.

Find $r$: $r = \sqrt{a^2+b^2} = \sqrt{(\sqrt{3})^2+(-1)^2} = \sqrt{4} = 2$

Find $\theta$: $\tan\theta = \frac{b}{a} \Rightarrow \tan\theta = \frac{-1}{\sqrt{3}} \Rightarrow \theta = -\frac{\pi}{6} = \frac{11\pi}{6} \qquad \sqrt{3}-i = \boxed{2\left(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}\right)}$ or $\boxed{2\text{cis}\frac{11\pi}{6}}$

Writing a Complex Number in Standard Form ($a + bi$)

**Ex:** Write $9\text{cis}\,\pi$ in standard form.

Expand: $9\text{cis}\,\pi = 9\left(\cos\pi + i\sin\pi\right) = 9\left(-1 + i\left(0\right)\right) = -9 + 0i = \boxed{-9}$

Multiplying and Dividing Complex Numbers

Let $z_1 = r_1\left(\cos\theta_1 + i\sin\theta_1\right)$ and $z_2 = r_2\left(\cos\theta_2 + i\sin\theta_2\right)$.

Multiplication: $\boldsymbol{z_1 \cdot z_2 = r_1 \cdot r_2\left[\cos\left(\theta_1 + \theta_2\right) + i\sin\left(\theta_1 + \theta_2\right)\right]}$

Division: $\boldsymbol{\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}\left[\cos\left(\theta_1 - \theta_2\right) + i\sin\left(\theta_1 - \theta_2\right)\right]}$

**Ex:** Express the product of $z_1$ and $z_2$ in standard form.

$z_1 = 4\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right),\ z_2 = \sqrt{2}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

$$z_1 \cdot z_2 = 4\sqrt{2}\left(\cos\left(\frac{\pi}{4} + \frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{4} + \frac{\pi}{6}\right)\right) = 4\sqrt{2}\left(\cos\left(\frac{5\pi}{12}\right) + i\sin\left(\frac{5\pi}{12}\right)\right) \approx \boxed{1.464 + 5.464i}$$

Powers of a Complex Number: **De Moivre's Theorem** $\boldsymbol{z^n = \left[r\left(\cos\theta + i\sin\theta\right)\right]^n = r^n\left(\cos n\theta + i\sin n\theta\right)}$

**Ex:** Evaluate $\left(-\sqrt{2} + i\sqrt{2}\right)^5$.

Rewrite in trig form: $\tan^{-1}\left(-\frac{\sqrt{2}}{\sqrt{2}}\right) = -45^\circ = 135^\circ$ $\quad r = \sqrt{\left(-\sqrt{2}\right)^2 + \left(\sqrt{2}\right)^2} = \sqrt{4} = 2$

$\left(2\text{cis}135^\circ\right)^5 = 2^5\text{cis}\left(5 \cdot 135^\circ\right) = 32\text{cis}\left(675^\circ\right) = \boxed{32\text{cis}\left(315^\circ\right)}$

$n^{\text{th}}$ Roots of a Complex Number:

$$\boldsymbol{\sqrt[n]{z} = \sqrt[n]{r}\left[\cos\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right) + i\sin\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right)\right] = \sqrt[n]{r}\text{cis}\left(\frac{\theta + 2\pi k}{n}\right),\ k = 0, 1, 2, \ldots\ n - 1}$$

Note: Every complex number has a total of "$n$" $n^{\text{th}}$ roots.

**Ex:** Find the cube roots of $8i$.

Write in trig form: $r = 8,\ \theta = 90^\circ$ $\quad 8\text{cis}90^\circ$

Evaluate the roots: $\sqrt[3]{8\text{cis}90^\circ} = \sqrt[3]{8}\text{cis}\left(\frac{90^\circ + 360^\circ k}{3}\right) = 2\text{cis}\left(30^\circ + 120^\circ k\right)$

$k = 0: 2\text{cis}\left(30^\circ + 0^\circ\right) = 2\text{cis}\left(30^\circ\right) = 2\left(\cos 30^\circ + i\sin 30^\circ\right) = \boxed{\sqrt{3} + i}$

$k = 1: 2\text{cis}\left(30^\circ + 120^\circ\right) = 2\text{cis}\left(150^\circ\right) = 2\left(\cos 150^\circ + i\sin 150^\circ\right) = \boxed{-\sqrt{3} + i}$

$k = 2: 2\text{cis}\left(30^\circ + 240^\circ\right) = 2\text{cis}\left(270^\circ\right) = 2\left(\cos 270^\circ + i\sin 270^\circ\right) = \boxed{-2i}$

Roots of Unity: the $n^{\text{th}}$ roots of 1

**Ex:** Express the fifth roots of unity in standard form and graph them in the complex plane.

5th Roots of Unity: $\sqrt[5]{1+0i}$   $r=1,\ \theta=0$   $1\text{cis}0$

$$\sqrt[5]{1\text{cis}0}=\sqrt[5]{1}\text{cis}\left(\frac{0+2\pi k}{5}\right)=\text{cis}\left(\frac{2\pi k}{5}\right)$$

$k=0:\ \text{cis}0=\boxed{1}$

$k=1:\ \text{cis}\left(\frac{2\pi}{5}\right)\approx\boxed{0.31+0.95i}$

$k=2:\ \text{cis}\left(\frac{4\pi}{5}\right)\approx\boxed{-0.81+0.59i}$

$k=3:\ \text{cis}\left(\frac{6\pi}{5}\right)\approx\boxed{-0.81-0.59i}$

$k=4:\ \text{cis}\left(\frac{8\pi}{5}\right)\approx\boxed{0.31-0.95i}$

<u>You Try</u>:

1. Write each complex number in trigonometric form. Then find the product and the quotient.
   $-1+\sqrt{3}i,\ -2-2\sqrt{3}i$
2. Solve the equation $x^4+1=0$. (You should have 4 solutions!)

<u>QOD</u>: Is the trigonometric form of a complex number unique? Explain.