Syllabus Objectives: 3.3 – The student will simplify trigonometric expressions and prove trigonometric identities (fundamental identities). 3.4 – The student will solve trigonometric equations with and without technology.

Identity: a statement that is true for all values for which both sides are defined

Example from algebra: $3(x+8)-11=3x+13$

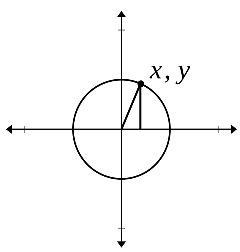
Reciprocal Identities

$$\boxed{\begin{array}{lll} \sin\theta=\dfrac{1}{\csc\theta} & \cos\theta=\dfrac{1}{\sec\theta} & \tan\theta=\dfrac{1}{\cot\theta} \\ \csc\theta=\dfrac{1}{\sin\theta} & \sec\theta=\dfrac{1}{\cos\theta} & \cot\theta=\dfrac{1}{\tan\theta} \end{array}}$$

Quotient Identities

$$\boxed{\tan\theta=\frac{\sin\theta}{\cos\theta} \quad \cot\theta=\frac{\cos\theta}{\sin\theta}}$$

Recall: Unit Circle $r=1,\ x=\cos\theta,\ y=\sin\theta$ or $\cos\theta=\dfrac{x}{1}=x$ $\sin\theta=\dfrac{y}{1}=y$



$$\boxed{\text{Note: } \sin^2\theta=(\sin\theta)^2}$$

Pythagorean Theorem: $x^2+y^2=1 \Rightarrow$ Pythagorean Identity: $\sin^2\theta+\cos^2\theta=1$

To derive the other Pythagorean Identities, divide the entire equation by $\sin^2\theta$ and then by $\cos^2\theta$:

$$\frac{\sin^2\theta}{\sin^2\theta}+\frac{\cos^2\theta}{\sin^2\theta}=\frac{1}{\sin^2\theta} \Rightarrow 1+\cot^2\theta=\csc^2\theta \qquad \frac{\sin^2\theta}{\cos^2\theta}+\frac{\cos^2\theta}{\cos^2\theta}=\frac{1}{\cos^2\theta} \Rightarrow \tan^2\theta+1=\sec^2\theta$$

Pythagorean Identities

$$\boxed{\sin^2\theta+\cos^2\theta=1 \quad 1+\cot^2\theta=\csc^2\theta \quad \tan^2\theta+1=\sec^2\theta}$$

Simplifying Trigonometric Expressions:

- Look for identities
- Change everything to sine and cosine and reduce

Note that the equations in **bold** are the trig identities used when simplifying. All of the other steps are algebra steps.

Ex1: Use basic identities to simplify the expressions.

a) $\cot\theta(1-\cos^2\theta)$ $\qquad \boldsymbol{\cot\theta=\dfrac{\cos\theta}{\sin\theta}}$ $\qquad \boldsymbol{\sin^2\theta+\cos^2\theta=1 \Rightarrow \sin^2\theta=1-\cos^2\theta}$

$$\cot\theta \;\; 1-\cos^2\theta \;=\frac{\cos\theta}{\sin\theta}\;\sin^2\theta \;=\frac{\cos\theta}{\cancel{\sin\theta}}\left(\cancel{\sin\theta}\cdot\sin\theta\right)\boxed{\cos\theta\sin\theta}$$

b) $\tan\theta \;\; \csc\theta$ $\qquad$ $\boldsymbol{\tan\theta=\dfrac{\sin\theta}{\cos\theta}}$ $\qquad$ $\boldsymbol{\csc\theta=\dfrac{1}{\sin\theta}}$

$$\tan\theta \;\; \csc\theta \;=\frac{\cancel{\sin\theta}}{\cos\theta}\left(\frac{1}{\cancel{\sin\theta}}\right)=\frac{1}{\cos\theta}=\boxed{\sec\theta}$$

Ex2: Simplify the expression $\dfrac{\csc x+1 \quad \csc x-1}{\cos^2 x}$.

Use algebra: $\dfrac{\csc x+1 \quad \csc x-1}{\cos^2 x}=\dfrac{\csc^2 x-1}{\cos^2 x}$ $\qquad$ $\boldsymbol{1+\cot^2\theta=\csc^2\theta \Rightarrow \cot^2\theta=\csc^2\theta-1}$

$$\frac{\csc^2 x-1}{\cos^2 x}=\frac{\cot^2 x}{\cos^2 x}=\frac{\;\frac{\cos^2 x}{\sin^2 x}\;}{\cos^2 x}=\frac{\cancel{\cos^2 x}}{\sin^2 x\,\cancel{\cos^2 x}}=\frac{1}{\sin^2 x}=\boxed{\csc^2 x}$$

Solving Trigonometric Equations

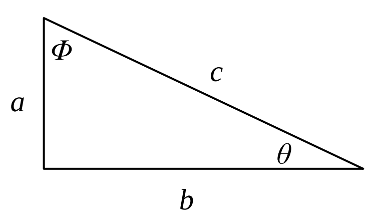
- Isolate the trigonometric function.
- Solve for x using inverse trig functions. Note – There may be more than one solution or no solution.

Ex3: Solve the equation $4\sin^2 x-4=0$ in the interval $0,2\pi$.

$4\sin^2 x-4=0 \Rightarrow \sin^2 x=1 \Rightarrow \sqrt{\sin^2 x}=\sqrt{1} \Rightarrow \sin x=\pm 1$

Find values of x for which $x=\sin^{-1}1$ and $x=\sin^{-1} \;-1$: $\boxed{x=\dfrac{\pi}{2},\dfrac{3\pi}{2}}$

Exploration: Consider a right triangle.



Note that ϕ and θ are complementary. Write the trig functions for each angle. What do you notice?

$\sin\phi=\cos\theta=\dfrac{b}{c}$ $\qquad$ $\cos\phi=\sin\theta=\dfrac{a}{c}$ $\qquad$ $\tan\phi=\cot\theta=\dfrac{b}{a}$

$\csc\phi=\sec\theta=\dfrac{c}{b}$ $\qquad$ $\sec\phi=\csc\theta=\dfrac{c}{a}$ $\qquad$ $\cot\phi=\tan\theta=\dfrac{c}{b}$

**The trig functions of ϕ are equal to the cofunctions of θ, when ϕ and θ are complementary.

<u>Cofunction Identities:</u>

$$\sin(90^\circ-\theta)=\cos\theta \text{ or } \sin\left(\frac{\pi}{2}-\theta\right)=\cos\theta$$

$$\cos(90^\circ-\theta)=\sin\theta \text{ or } \cos\left(\frac{\pi}{2}-\theta\right)=\sin\theta$$

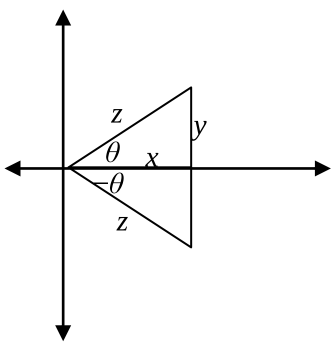
$$\sec(90^\circ-\theta)=\csc\theta \text{ or } \sec\left(\frac{\pi}{2}-\theta\right)=\csc\theta$$

$$\csc(90^\circ-\theta)=\sec\theta \text{ or } \csc\left(\frac{\pi}{2}-\theta\right)=\sec\theta$$

$$\tan(90^\circ-\theta)=\cot\theta \text{ or } \cot\left(\frac{\pi}{2}-\theta\right)=\tan\theta$$

$$\cot(90^\circ-\theta)=\tan\theta \text{ or } \cot\left(\frac{\pi}{2}-\theta\right)=\tan\theta$$

<u>Exploration:</u> Consider an angle, θ, and its opposite, as shown in the coordinate grid. Compare the trig functions of each angle.



$\sin\theta=\frac{y}{z}$, $\sin(-\theta)=-\frac{y}{z}$ $\qquad$ $\cos\theta=\frac{x}{z}$, $\cos(-\theta)=\frac{x}{z}$ $\qquad$ $\tan\theta=\frac{y}{x}$, $\tan(-\theta)=-\frac{y}{x}$

$\csc\theta=\frac{z}{y}$, $\csc(-\theta)=-\frac{z}{y}$ $\qquad$ $\sec=\frac{z}{x}$, $\sec(-\theta)=\frac{z}{x}$ $\qquad$ $\cot\theta=\frac{x}{y}$, $\cot(-\theta)=-\frac{x}{y}$

**Cosine and secant are the only EVEN $\left(f(-x)=f(x)\right)$ trig functions. All the rest are ODD $\left(f(-x)=-f(x)\right)$.

<u>Odd-Even Identities:</u>

$$\sin(-\theta)=-\sin\theta \text{ and } \csc(-\theta)=-\csc\theta$$

$$\tan(-\theta)=-\tan\theta \text{ and } \cot(-\theta)=-\cot\theta$$

$$\cos(-\theta)=\cos\theta \text{ and } \sec(-\theta)=\sec\theta$$

Ex4: Simplify the expression $\sin(-x)\csc(-x)$.

$$\sin(-x)\csc(-x) = (-\sin x)(-\csc x) = \sin x\left(\frac{1}{\sin x}\right) = \boxed{1} \qquad \mathbf{\sin(-x) = -\sin x} \qquad \mathbf{\csc(-x) = -\csc x}$$

<u>Simplifying Trigonometric Expressions</u>: Simplify using the following strategies. Note that the equations in **bold** are the trig identities used when simplifying. All of the other steps are algebra steps.

Ex5: Simplify the expression by **factoring**. $\cos^3 x + (\cos x)(\sin^2 x)$

$$\cos^3 x + (\cos x)(\sin^2 x) = (\cos x)(\cos^2 x + \sin^2 x) = (\cos x)(1) = \boxed{\cos x} \qquad \mathbf{\sin^2 x + \cos^2 x = 1}$$

Ex6: Simplify the expression by **combining fractions**. $\dfrac{\sin x}{1-\cos x} - \dfrac{\cos x}{\sin x}$

$$\frac{\sin x}{1-\cos x} - \frac{\cos x}{\sin x} = \frac{(\sin x)(\sin x) - (\cos x)(1-\cos x)}{(1-\cos x)(\sin x)} = \frac{\sin^2 x - \cos x + \cos^2 x}{(1-\cos x)(\sin x)} \qquad \mathbf{\sin^2 x + \cos^2 x = 1}$$

$$= \frac{\sin^2 x + \cos^2 x - \cos x}{(1-\cos x)(\sin x)} = \frac{\cancel{1-\cos x}}{\cancel{(1-\cos x)}(\sin x)} = \frac{1}{\sin x} = \boxed{\csc x} \qquad \mathbf{\csc x = \frac{1}{\sin x}}$$

<u>Solving Trigonometric Equations</u>: Solve using the following strategies. Find all solutions for each equation in the interval $[0, 2\pi)$.

Ex7: Solve the equation by **isolating the trig function**. $2\cos x - 1 = 0$

$\cos x = \dfrac{1}{2} \Rightarrow \boxed{x = \dfrac{\pi}{3}, \dfrac{5\pi}{3}}$ These are values of x where the cosine is equal to $\dfrac{1}{2}$.

Ex8: Solve the equation by **extracting square roots**. $4\sin^2 x - 3 = 0$

$\sin^2 x = \dfrac{3}{4} \Rightarrow \sin x = \pm\dfrac{\sqrt{3}}{2} \Rightarrow \boxed{x = \dfrac{\pi}{3}, \dfrac{2\pi}{3}, \dfrac{4\pi}{3}, \dfrac{5\pi}{3}}$ These are values of x where the sine is equal to $\pm\dfrac{\sqrt{3}}{2}$.

Ex9: Solve the equation by **factoring**. $2\cos^2 x + \cos x = 1$

Set equal to zero. $2\cos^2 x + \cos x - 1 = 0$

Factor. $(2\cos x - 1)(\cos x + 1) = 0$

Set each factor equal to zero. $2\cos x - 1 = 0$ $\quad$ $\cos x + 1 = 0$

Solve each equation. $\cos x = \dfrac{1}{2}$, $x = \dfrac{\pi}{3}, \dfrac{5\pi}{3}$ $\quad$ $\cos x = -1$, $x = \pi$ $\quad$ $\boxed{x = \dfrac{\pi}{3}, \pi, \dfrac{5\pi}{3}}$

Note: It may be easier to use *u*-substitution with $u = \cos x$ to help students visualize the equation as a quadratic equation that can be factored.

Ex10: Solve the equation by **factoring**. $2\sec x \sin x - \sec x = 0$

Factor out GCF. $\sec x \;\; 2\sin x - 1 \;\; = 0$

Use zero product property. $\sec x = 0 \qquad 2\sin x - 1 = 0$

Solve each equation. $\dfrac{1}{\cos x} = 0 \qquad \sin x = \dfrac{1}{2}$

$x = \varnothing \qquad \boxed{x = \dfrac{\pi}{6}, \dfrac{5\pi}{6}}$

Note: It is possible for an equation to have no solution.

Ex11: Solve by rewriting in a **single trig function**. $2\sin^2 x + 3\cos x = 3$

Substitute Pyth. Identity. $2 \;\; 1 - \cos^2 x \;\; + 3\cos x = 3 \qquad \boldsymbol{\sin^2\theta + \cos^2\theta = 1 \Rightarrow \sin^2\theta = 1 - \cos^2\theta}$

Simplify algebraically. $2 - 2\cos^2 x + 3\cos x = 3 \Rightarrow 2\cos^2 x - 3\cos x + 1 = 0$

Factor and solve. $2\cos x - 1 \;\; \cos x - 1 \;\; = 0 \Rightarrow \cos x = \dfrac{1}{2},\ \cos x = 1 \qquad \boxed{x = 0, \dfrac{\pi}{3}, \dfrac{5\pi}{3}}$

Ex12: Solve using **trig substitutions**. $\dfrac{\sin^3 x}{\cos x} = \tan x$

Rewrite $\sin^3 x = \sin x \cdot \sin^2 x$. $\dfrac{\sin^2 x \cdot \sin x}{\cos x} = \tan x \Rightarrow \sin^2 x \cdot \dfrac{\sin x}{\cos x} = \tan x$

Rewrite $\boldsymbol{\dfrac{\sin x}{\cos x} = \tan x}$. $\sin^2 x \cdot \tan x = \tan x$

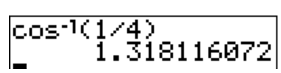
Set equal to zero and factor. $\sin^2 x \cdot \tan x - \tan x = 0 \Rightarrow \tan x \;\; \sin^2 x - 1 \;\; = 0$

Use the zero product property and solve. $\tan x = 0 \Rightarrow x = 0, \pi \qquad \sin x = \pm 1 \Rightarrow x = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$

$\boxed{x = 0, \pi, \dfrac{\pi}{2}, \dfrac{3\pi}{2}}$

Ex13: Find the approximate solution using the **calculator**. $4\cos x = 1$

Isolate the trig function. $\cos x = \dfrac{1}{4}$

To find *x*, we need to find the inverse cosine of ¼. $x = \cos^{-1}\left(\dfrac{1}{4}\right)$ 

$\boxed{x \approx 1.318}$

When solving an equation in the interval $0, 2\pi$, be sure to be in Radian mode.

<u>You Try</u>: Make the suggested trigonometric substitution and then use the Pythagorean Identities to write the resulting function as a multiple of a basic trig function. $\sqrt{4 - x^2}$, $x = 2\cos\theta$

<u>QOD</u>: Explain the relationship between trig functions and their cofunctions.

Syllabus Objective: 3.3 – The student will simplify trigonometric expressions and prove trigonometric identities.

Trigonometric Identity: an equation involving trigonometric functions that is a true equation for all values of *x*

Tips for Proving Trigonometric Identities:

1. Manipulate only one side of the equation. Start with the more complicated side.
2. Look for any identities (use all that you have learned so far).
3. Change everything to sine or cosine.
4. Use algebra (common denominators, factoring, etc) to simplify.
5. Each step should have **one** change only.
6. The final step should have the **same** expression on both sides of the equation.

Note: Your goal when proving a trig identity is to make both sides look identical!

For all of the following examples, prove that the identity is true. The trig identities used in the substitutions are in bold.

Ex1: $\cos^3 x = 1-\sin^2 x \;\; \cos x$

Start with the right side (more complicated).

$$\cos^3 x = 1-\sin^2 x \;\; \cos x$$

$$= \cos^2 x \cos x \qquad\qquad \boldsymbol{\sin^2 x + \cos^2 x = 1 \Rightarrow \cos^2 x = 1-\sin^2 x}$$

$$\cos^3 x = \cos^3 x$$

Ex2: $\dfrac{1}{1-\sin x} + \dfrac{1}{1+\sin x} = 2\sec^2 x$

Start with the left side.

$$\frac{1}{1-\sin x} + \frac{1}{1+\sin x} = 2\sec^2 x$$

Combine fractions. $\dfrac{1+\sin x + 1 - \sin x}{1-\sin x \quad 1+\sin x} =$

Simplify. $\dfrac{2}{1-\sin^2 x} =$

Trig substitution. $\dfrac{2}{\cos^2 x} =$ $\qquad \boldsymbol{\sin^2 x + \cos^2 x = 1 \Rightarrow \cos^2 x = 1-\sin^2 x}$

Identity. $2\sec^2 x = 2\sec^2 x$ $\qquad \boldsymbol{\dfrac{1}{\cos x} = \sec x}$

Ex3: $\tan^2 x + 1 \;\; \cos^2 x - 1 = -\tan^2 x$

Start with the left side. $\tan^2 x + 1 \;\; \cos^2 x - 1 = -\tan^2 x$

Trig substitution. $\sec^2 x \;\; \cos^2 x - 1 =$ $\qquad \mathbf{\tan^2 x + 1 = \sec^2 x}$

Trig substitution. $\sec^2 x \;\; -\sin^2 x =$ $\qquad \mathbf{\sin^2 x + \cos^2 x = 1 \Rightarrow \cos^2 x - 1 = -\sin^2 x}$

Trig substitution. $\dfrac{1}{\cos^2 x} \;\; -\sin^2 x =$ $\qquad \mathbf{\dfrac{1}{\cos x} = \sec x}$

Multiply. $\dfrac{-\sin^2 x}{\cos^2 x} =$

Identitiy. $-\tan^2 x = -\tan^2 x$ $\qquad \mathbf{\dfrac{\sin x}{\cos x} = \tan x}$

Ex4: $\sec x + \tan x = \dfrac{\cos x}{1 - \sin x}$

Start with the left side. $\sec x + \tan x = \dfrac{\cos x}{1 - \sin x}$

Change to sine/cosine. $\dfrac{1}{\cos x} + \dfrac{\sin x}{\cos x} =$ $\qquad \mathbf{\dfrac{1}{\cos x} = \sec x} \qquad \mathbf{\dfrac{\sin x}{\cos x} = \tan x}$

Combine fractions. $\dfrac{1 + \sin x}{\cos x} =$

Multiply num/den by **conjugate**. $\dfrac{1 + \sin x}{\cos x} \cdot \dfrac{1 - \sin x}{1 - \sin x} =$

Multiply. $\dfrac{1 - \sin^2 x}{\cos x \;\; 1 - \sin x} =$

Trig substitution. $\dfrac{\cos^2 x}{\cos x \;\; 1 - \sin x} =$ $\qquad \mathbf{\sin^2 x + \cos^2 x = 1 \Rightarrow \cos^2 x = 1 - \sin^2 x}$

Simplify. $\dfrac{\cos x}{1 - \sin x} = \dfrac{\cos x}{1 - \sin x}$

Ex5: $\dfrac{\sec^2 \theta - 1}{\sec^2 \theta} = \sin^2 \theta$

Start with left side. $\dfrac{\sec^2 \theta - 1}{\sec^2 \theta} = \sin^2 \theta$

Split the fraction. $\dfrac{\sec^2 \theta}{\sec^2 \theta} - \dfrac{1}{\sec^2 \theta} =$

Simplify. $1 - \dfrac{1}{\sec^2 \theta} =$

Trig substitution. $1 - \cos^2 \theta =$ $\qquad \mathbf{\dfrac{1}{\sec \theta} = \cos \theta}$

Identity. $\sin^2 \theta = \sin^2 \theta$ $\qquad \mathbf{\sin^2 \theta + \cos^2 \theta = 1 \Rightarrow \sin^2 \theta = 1 - \cos^2 \theta}$

Challenge: Try to prove the identity above in another way.

You Try: Prove the identity. $\left(\cos x - \sin x\right)^2 + \left(\cos x - \sin x\right)^2 = 2$

QOD: List at least 5 strategies you can use when proving trigonometric identities.

Syllabus Objective: 3.3 – The student will simplify trigonometric expressions and prove trigonometric identities (sum and difference identities).

Recall: $\sqrt{36+64} = \sqrt{100} = 10$ $\qquad\qquad$ $\sqrt{36+64} \neq \sqrt{36} + \sqrt{64} = 6+8 = 14$

So in general, $\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$

and

$2+3^{\;2} = \;5^{\;2} = 25$ $\qquad\qquad$ $2+3^{\;2} \neq \;2^{\;2} + \;3^{\;2} = 13$

So in general, $a+b^{\;2} \neq a^2 + b^2$

Sum and Difference Identities

$$\sin\;u \pm v\; = \sin u \cos v \pm \cos u \sin v$$

$$\cos\;u \pm v\; = \cos u \cos v \mp \sin u \sin v$$

$$\tan\;u \pm v\; = \frac{\tan u \pm \tan v}{1 \mp \tan u \tan v}$$

Note: Be careful with +/− signs!

Simplifying Expressions with Sum and Differences

1. Rewrite the expression using a sum/difference identity.
2. Simplify the expression and evaluate if necessary.

Ex1: Write the expression as the sine of an angle. Then give the exact value.

$$\sin\frac{\pi}{4}\cos\frac{\pi}{12} - \cos\frac{\pi}{4}\sin\frac{\pi}{12}$$

$$\sin\frac{\pi}{4}\cos\frac{\pi}{12} - \cos\frac{\pi}{4}\sin\frac{\pi}{12} = \sin\left(\frac{\pi}{4} - \frac{\pi}{12}\right)$$ $\qquad$ $\boldsymbol{\sin\;u - v\; = \sin u \cos v - \cos u \sin v}$

$$= \sin\frac{2\pi}{12} = \sin\frac{\pi}{6} = \boxed{\frac{1}{2}}$$

Evaluating Trigonometric Expressions with Non-Special Angles

1. Rewrite the angle as a sum or difference of two special angles. or
2. Rewrite the expression using a sum/difference identity.
3. Evaluate the expression.

Ex2: Find the exact value of $\cos 195^\circ$.

1. $195^\circ = 150^\circ + 45^\circ$ $\qquad$ $\cos 195^\circ = \cos\;150^\circ + 45^\circ$

2. $\boldsymbol{\cos\;u + v\; = \cos u \cos v - \sin u \sin v}$ $\qquad$ $\cos\;150^\circ + 45^\circ\; = \cos 150^\circ \cos 45^\circ - \sin 150^\circ \sin 45^\circ$

3. $\cos 150^\circ \cos 45^\circ - \sin 150^\circ \sin 45^\circ = \left(-\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) - \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) = -\frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} = \boxed{\frac{-\sqrt{6}-\sqrt{2}}{4}}$

Ex3: Write as one trig function and find an exact value. $\frac{\tan 80^\circ + \tan 55^\circ}{1 - \tan 80^\circ \tan 55^\circ}$

1. $\mathbf{\tan(u+v) = \frac{\tan u + \tan v}{1 - \tan u \tan v}}$ $\qquad \frac{\tan 80^\circ + \tan 55^\circ}{1 - \tan 80^\circ \tan 55^\circ} = \tan(80^\circ + 55^\circ)$
2. $\tan(80^\circ + 55^\circ) = \tan(135^\circ)$
3. $\tan(135^\circ) = \boxed{-1}$

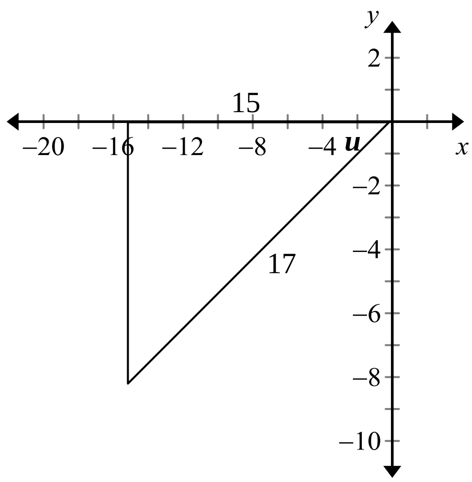
<u>Evaluating Trig Functions Given Other Trig Function(s)</u>

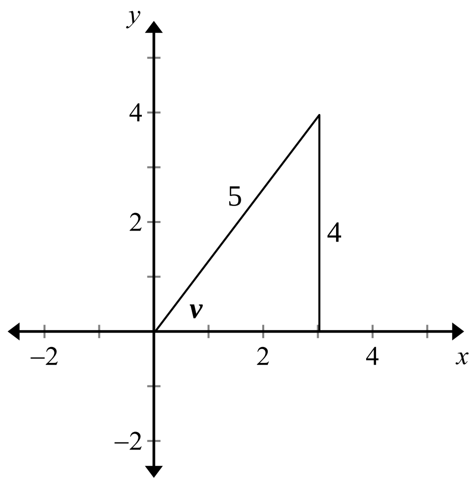
Ex4: Find $\cos(u-v)$ given $\cos u = -\frac{15}{17}$, $\pi < u < \frac{3\pi}{2}$ and $\sin v = \frac{4}{5}$, $0 < v < \frac{\pi}{2}$.

$\mathbf{\cos(u-v) = \cos u \cos v + \sin u \sin v}$ We must find $\cos v$ and $\sin u$.

Draw the appropriate right triangles in the coordinate plane.

$\cos u = -\frac{15}{17}$, $\pi < u < \frac{3\pi}{2}$: $\qquad \sin v = \frac{4}{5}$, $0 < v < \frac{\pi}{2}$:





Use the Pythagorean Theorem to find the missing sides.

$15^2 + U^2 = 17^2$ $\qquad V^2 + 4^2 = 5^2$

$U = 8$ $\qquad V = 3$

In Quadrant III, sine is negative, so $\sin u = -\frac{8}{17}$. In Quadrant I, cosine is positive, so $\cos v = \frac{3}{5}$.

$\mathbf{\cos(u-v) = \cos u \cos v + \sin u \sin v} = \left(-\frac{15}{17}\right)\left(\frac{3}{5}\right) + \left(-\frac{8}{17}\right)\left(\frac{4}{5}\right) = -\frac{45}{85} - \frac{32}{85} = \boxed{-\frac{77}{85}}$

Proving Identities

Ex5: Verify the identity. $\dfrac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta} = \tan\alpha + \tan\beta$

Start with the left side.

$$\frac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta} = \tan\alpha + \tan\beta \qquad \mathbf{\sin(\boldsymbol{u}+\boldsymbol{v}) = \sin \boldsymbol{u}\cos \boldsymbol{v} + \cos \boldsymbol{u}\sin \boldsymbol{v}}$$

Trig substitution: $\dfrac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta} =$

Split the fraction: $\dfrac{\sin\alpha\cancel{\cos\beta}}{\cos\alpha\cancel{\cos\beta}} + \dfrac{\cancel{\cos\alpha}\sin\beta}{\cancel{\cos\alpha}\cos\beta} =$

Simplify: $\dfrac{\sin\alpha}{\cos\alpha} + \dfrac{\sin\beta}{\cos\beta} =$

Trig substitution: $\tan\alpha + \tan\beta = \tan\alpha + \tan\beta$

You Try: Verify the cofunction identity $\sin\left(\dfrac{\pi}{2} - \theta\right) = \cos\theta$ using the angle difference identity.

QOD: Give an example of a function for which $f(a+b) = f(a) + f(b)$ for all real numbers *a* and *b*. Then give an example of a function for which $f(a+b) \neq f(a) + f(b)$ for all real numbers *a* and *b*.

Syllabus Objective: 3.3 – The student will simplify trigonometric expressions and prove trigonometric identities (double angle and power-reducing identities).

Ex1: Derive the double angle identities using the sum identities.

1. $\sin\ 2u\ = \sin\ u+u$

   $\sin\ u+u\ = \sin u \cos u + \sin u \cos u = 2\sin u \cos u$

2. $\cos\ 2u\ = \cos\ u+u$

   $\cos\ u+u\ = \cos u \cos u - \sin u \sin u = \cos^2 u - \sin^2 u$

3. $\tan\ 2u\ = \tan\ u+u$

   $\tan\ u+u\ = \dfrac{\tan u + \tan u}{1-\tan u \tan u} = \dfrac{2\tan u}{1-\tan^2 u}$

<u>Double Angle Identities</u>

$$\boxed{\begin{aligned} &\sin 2\theta = 2\sin\theta\cos\theta \\ &\cos 2\theta = \cos^2\theta - \sin^2\theta \\ &\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} \end{aligned}}$$

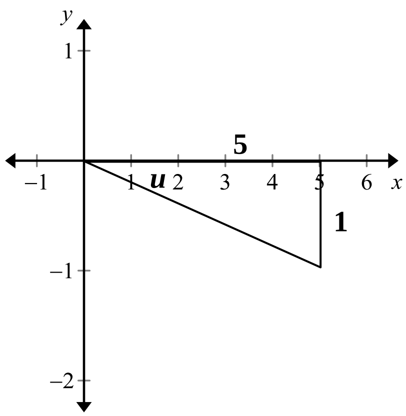
There are two other ways to write the double angle identity for cosine. Use the Pythagorean identity.

$$\begin{aligned} &\sin^2\theta + \cos^2\theta = 1 \\ &\sin^2\theta = 1-\cos^2\theta \\ &\cos^2\theta = 1-\sin^2\theta \end{aligned}$$

$$\boxed{\begin{aligned} &\cos 2\theta = \cos^2\theta - \sin^2\theta \\ &\cos 2\theta = 1-2\sin^2\theta \\ &\cos 2\theta = 2\cos^2\theta - 1 \end{aligned}}$$

Evaluating Double-Angle Trigonometric Functions

Ex: Find the exact value of $\cos 2u$ given $\cot u = -5,\ \dfrac{3\pi}{2} < u < 2\pi$.



u will be in Quadrant IV and forms a right triangle as labeled.

Using the Pythagorean Theorem, we have $c^2 = 5^2 + 1^2 \Rightarrow c = \sqrt{26}$

Double Angle Identity: $\cos 2u = \cos^2 u - \sin^2 u$ $\qquad \cos u = \dfrac{5}{\sqrt{26}},\ \sin u = -\dfrac{1}{\sqrt{26}}$

$$\cos 2u = \left(\frac{5}{\sqrt{26}}\right)^2 - \left(-\frac{1}{\sqrt{26}}\right)^2 = \frac{25}{26} - \frac{1}{26} = \frac{24}{26} = \boxed{\frac{12}{13}}$$

Note: If u is in Quadrant IV, $\dfrac{3\pi}{2} < u < 2\pi$, then for $2u$ we have

$2 \cdot \dfrac{3\pi}{2} < 2u < 2 \cdot 2\pi \Rightarrow 3\pi < 2u < 4\pi$, which is in Quadrant IV. So it makes sense that $\cos 2u$ is positive.

Solving Trigonometric Equations

Ex2: Find the solutions to $4\sin x \cdot \cos x = 1$ in $0, 2\pi$.

Rewrite the equation. $\qquad 2 \cdot 2\sin x \cdot \cos x = 1$

Trig substitution. $\qquad 2\sin\ 2x\ = 1$ $\qquad$ $\mathbf{\sin 2x = 2\sin x \cos x}$

Isolate trig function. $\qquad \sin 2x = \dfrac{1}{2}$

Solve for the argument. $\qquad 2x = \sin^{-1}\left(\dfrac{1}{2}\right)$

Because the argument is $2x$, we must revisit the domain. $0, 2\pi$ is the restriction for x. So $0 \le x < 2\pi$. Therefore, $2 \cdot 0 \le 2x < 2 \cdot 2\pi \Rightarrow 0 \le 2x < 4\pi$.

$$2x = \frac{\pi}{6},\ 2x = \frac{5\pi}{6},\ 2x = \frac{13\pi}{6},\ 2x = \frac{17\pi}{6}$$

Solve for x. $\qquad \boxed{x = \dfrac{\pi}{12}, \dfrac{5\pi}{12}, \dfrac{13\pi}{12}, \dfrac{17\pi}{12}}$

Rewriting a Multiple Angle Trig Function to a Single Angle

Ex3: Express $\sin 3x$ in terms of $\sin x$.

Rewrite argument as a sum $\quad \sin 3x = \sin(2x + x)$

Sum identity $\quad = \sin x \cos 2x + \sin 2x \cos x$

Double angle identities $\quad = \sin x\left(1 - 2\sin^2 x\right) + 2\sin x \cos x \cos x$

$\quad = \sin x - 2\sin^3 x + 2\sin x \cos^2 x$

Pythagorean identity $\quad = \sin x - 2\sin^3 x + 2\sin x\left(1 - \sin^2 x\right)$

$\quad = \sin x - 2\sin^3 x + 2\sin x - 2\sin^3 x$

Simplify $\quad = \boxed{3\sin x - 4\sin^3 x}$

Verifying a Trig Identity

Ex4: Verify $\sin 2\theta = \dfrac{2\tan\theta}{1+\tan^2\theta}$.

Start with left side. $\quad \sin 2\theta = \dfrac{2\tan\theta}{1+\tan^2\theta}$

Pythagorean identity $\quad = \dfrac{2\tan\theta}{\sec^2\theta}$

Rewrite in sines/cosines $\quad = \dfrac{2\dfrac{\sin\theta}{\cos\theta}}{\dfrac{1}{\cos^2\theta}}$

Simplify $\quad = 2\dfrac{\sin\theta}{\cos\theta}\cdot\dfrac{\cos^2\theta}{1}$

Double angle identity $\quad \sin 2\theta = 2\sin\theta\cos\theta$

Recall:
$$\boxed{\begin{aligned}&\mathbf{\cos 2\theta = \cos^2\theta - \sin^2\theta}\\ &\mathbf{\cos 2\theta = 1 - 2\sin^2\theta}\\ &\mathbf{\cos 2\theta = 2\cos^2\theta - 1}\end{aligned}}$$

Solving for $\sin^2\theta$ and $\cos^2\theta$, we can derive the power reducing identities.

$$\cos 2\theta = 1 - 2\sin^2\theta \qquad \cos 2\theta = 2\cos^2\theta - 1$$

$$\sin^2\theta = \frac{1-\cos 2\theta}{2} \qquad \cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$\tan^2\theta = \frac{\sin^2\theta}{\cos^2\theta} = \frac{\dfrac{1-\cos 2\theta}{2}}{\dfrac{1+\cos 2\theta}{2}} = \frac{1-\cos 2\theta}{1+\cos 2\theta}$$

Power Reducing Identities

$$\boxed{\begin{aligned}\sin^2\theta &= \frac{1-\cos 2\theta}{2}\\ \cos^2\theta &= \frac{1+\cos 2\theta}{2}\\ \tan^2\theta &= \frac{1-\cos 2\theta}{1+\cos 2\theta}\end{aligned}}$$

Ex5: Express $\cos^5 x$ in terms of trig functions with no power greater than 1.

Rewrite as a product $\quad \cos^5 x = \cos^2 x \cos^2 x \cos x$

Power reducing identity $\quad = \frac{1+\cos 2x}{2}\cdot\frac{1+\cos 2x}{2}\cdot\cos x$

Multiply $\quad = \frac{1}{4}\ 1+2\cos 2x+\cos^2 2x\ \ \cos x$

Power reducing identity $\quad = \boxed{\frac{1}{4}\left(1+2\cos 2x+\frac{1+\cos 4x}{2}\right)\cos x}$

You Try:

1. Find the solutions to $2\cos x+\sin 2x=0$ in $\ 0,2\pi\ $.
2. Verify $\frac{2\cos 2\alpha}{\sin 2\alpha}=\cot\alpha-\tan\alpha$.

QOD: How do you convert from a cosine function to a sine function? Explain.

Syllabus Objective: 3.3 – The student will simplify trigonometric expressions and prove trigonometric identities (half angle identities).

Recall: $\mathbf{\sin^2\theta = \frac{1-\cos 2\theta}{2}}$ Let $\theta = \frac{u}{2}$. We have $\sin^2\left(\frac{u}{2}\right) = \frac{1-\cos u}{2}$

Solving for $\sin\left(\frac{u}{2}\right)$, we have $\sin\left(\frac{u}{2}\right) = \pm\sqrt{\frac{1-\cos u}{2}}$. All of the other half-angle identities can be derived in a similar manner.

Half-Angle Identities

$$\boxed{\begin{aligned}\sin\frac{u}{2} &= \pm\sqrt{\frac{1-\cos u}{2}}\\ \cos\frac{u}{2} &= \pm\sqrt{\frac{1+\cos u}{2}}\\ \tan\frac{u}{2} &= \pm\sqrt{\frac{1-\cos u}{1+\cos u}}\end{aligned}}$$

Note: There are 2 others for tangent.

$$\boxed{\begin{aligned}\tan\frac{u}{2} &= \frac{1-\cos u}{\sin u}\\ \tan\frac{u}{2} &= \frac{\sin u}{1+\cos u}\end{aligned}}$$

Note: The $\pm$ will be decided based upon which quadrant $\frac{u}{2}$ lies in.

Evaluating Trig Functions

Ex1: Find the exact value of $\cos\frac{\pi}{12}$.

Rewrite as a half angle $\quad \cos\frac{\pi}{12} = \cos\left(\frac{1}{2}\cdot\frac{\pi}{6}\right)$

Half angle identity $\quad = \pm\sqrt{\frac{1+\cos\frac{\pi}{6}}{2}}$

Evaluate $\quad = \pm\sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}}$

$\quad = \pm\sqrt{\frac{2+\sqrt{3}}{4}}$

Choose sign $\quad = \boxed{\frac{\sqrt{2+\sqrt{3}}}{2}}$ $\quad \frac{\pi}{12}$ is in Quadrant I, where cosine is positive.

Note: This can also be found using a difference identity. $\cos\frac{\pi}{12} = \cos\left(\frac{\pi}{3}-\frac{\pi}{4}\right)$

Solving a Trig Equation

Ex2: Solve the equation $\sin x = \sin\frac{x}{2}$ in $\ 0, 2\pi\ $.

Half-angle identity $\quad \sin x = \pm\sqrt{\frac{1-\cos x}{2}}$

Square both sides $\quad \sin^2 x = \frac{1-\cos x}{2}$

Pythagorean identity $\quad 1-\cos^2 x = \frac{1-\cos x}{2}$

Set equal to zero $\quad 2-2\cos^2 x = 1-\cos x$

$2\cos^2 x - \cos x - 1 = 0$

Factor $\quad 2\cos x + 1 \quad \cos x - 1 \ = 0$

Zero product property

$2\cos x + 1 = 0$ $\qquad$ $\cos x - 1 = 0$

$\cos x = -\frac{1}{2}$ $\qquad$ $\cos x = 1$

$x = \frac{2\pi}{3}, \frac{4\pi}{3}$ $\qquad$ $x = 0$

$$\boxed{x = 0, \frac{2\pi}{3}, \frac{4\pi}{3}}$$

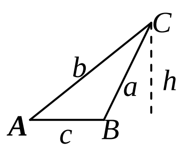
You Try:

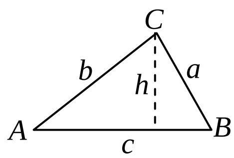
1. Find the exact value of $\tan 22.5^\circ$.
2. Solve the equation $\sin^2\frac{x}{2} + \cos x = 0$ in $\ 0, 2\pi\ $.

QOD: Explain why two of the half-angle identities do not have +/− signs.

Syllabus Objective: 3.5 – The student will solve application problems involving triangles (Law of Sines).

<u>Deriving the Law of Sines:</u> Consider the two triangles.





In the acute triangle, $\sin A = \dfrac{h}{b}$ and $\sin B = \dfrac{h}{a}$. In the obtuse triangle, $\sin(\pi - B) = \sin B = \dfrac{h}{a}$.

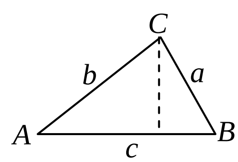
Solve for *h*. $h = b\sin A$ and $h = a\sin B$

Substitute. $a\sin B = b\sin A$ can be rewritten as $\dfrac{\sin B}{b} = \dfrac{\sin A}{a}$

The same type of argument can be used to show that $\dfrac{\sin C}{c} = \dfrac{\sin B}{b} = \dfrac{\sin A}{a}$.

<u>Law of Sines</u>: The ratio of the sine of an angle to the length of its opposite side is the same for all three angles of any triangle.

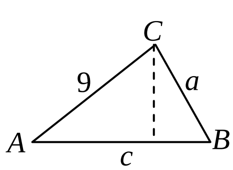
$$\boxed{\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c} \quad \textbf{or} \quad \dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}}$$



<u>Solving a Triangle</u>: finding all of the missing sides and angles

Note: The Law of Sines can be used to solve triangles given **AAS** and **ASA**.

Ex1: Solve the triangle ΔABC given that $\angle B = 15^\circ$, $\angle C = 52^\circ$, and $b = 9$.



We are given AAS, so we will use the Law of Sines.

$\dfrac{\sin B}{b} = \dfrac{\sin C}{c} \Rightarrow \dfrac{\sin 15^\circ}{9} = \dfrac{\sin 52^\circ}{c}$ Solve for *c*: $c\sin 15^\circ = 9\sin 52^\circ \Rightarrow c = \dfrac{9\sin 52^\circ}{\sin 15^\circ} \approx 27.4$

Find $m\angle A$ using the triangle sum. $m\angle A = 180 - (15 + 52) = 113^\circ$

$\dfrac{\sin B}{b} = \dfrac{\sin A}{a} \Rightarrow \dfrac{\sin 15^\circ}{9} = \dfrac{\sin 113^\circ}{a}$ Solve for *a*: $a\sin 15^\circ = 9\sin 113^\circ \Rightarrow c = \dfrac{9\sin 113^\circ}{\sin 15^\circ} \approx 32$

$\boxed{\Delta ABC : m\angle A = 113^\circ,\ m\angle B = 15^\circ,\ m\angle C = 52^\circ,\ a = 32,\ b = 9,\ c = 27.4}$

<u>Ambiguous Case</u>: When given **SSA**, there could be 2 triangles, 1 triangle, or no triangles that can be created with the given information.

Ex1: Solve the triangle ΔABC (if possible) when $m\angle C = 54^\circ$, $a = 10$, $c = 7$.

Given SSA, use Law of Sines. $\dfrac{\sin C}{c} = \dfrac{\sin A}{a} \Rightarrow \dfrac{\sin 54^\circ}{7} = \dfrac{\sin A}{10}$

Solve for *A*.

$$10\sin 54^\circ = 7\sin A \Rightarrow \sin A = \frac{10\sin 54^\circ}{7} \Rightarrow A = \sin^{-1}\left(\frac{10\sin 54^\circ}{7}\right) = \sin^{-1}\ 1.1557 = \varnothing$$

Solution: There is **no possible triangle** with the given information because there is no angle with a sine greater than 1.

Ex2: Solve the triangle ΔABC (if possible) when $m\angle C = 31^\circ$, $b = 46$, $c = 29$.

Given SSA, use Law of Sines. $\dfrac{\sin C}{c} = \dfrac{\sin B}{b} \Rightarrow \dfrac{\sin 31^\circ}{29} = \dfrac{\sin B}{46}$

Solve for ***B***.

$$46\sin 31^\circ = 29\sin B \Rightarrow \sin B = \frac{46\sin 31^\circ}{29} \Rightarrow B = \sin^{-1}\left(\frac{46\sin 31^\circ}{29}\right) \approx 54.8^\circ$$

Note that the calculator only gives the **acute** angle measure for *B*. There does exist an **obtuse** angle *B* with the same sine. $m\angle B = 180 - 54.8 = 125.2^\circ$ This is also an appropriate measure of an angle in a triangle, so there are **2 triangles** that can be formed with the given information.

Triangle 1

$$m\angle A = 180 - (54.8 + 31) = 94.2^\circ$$

$$\frac{\sin C}{c} = \frac{\sin A}{a} \Rightarrow \frac{\sin 31^\circ}{29} = \frac{\sin 94.2^\circ}{a}$$

$$a\sin 31^\circ = 29\sin 94.2^\circ \Rightarrow a \approx 56.2$$

Triangle 2

$$m\angle A = 180 - (125.2 + 31) = 23.8^\circ$$

$$\frac{\sin C}{c} = \frac{\sin A}{a} \Rightarrow \frac{\sin 31^\circ}{29} = \frac{\sin 23.8^\circ}{a}$$

$$a\sin 31^\circ = 29\sin 23.8^\circ \Rightarrow a \approx 22.7$$

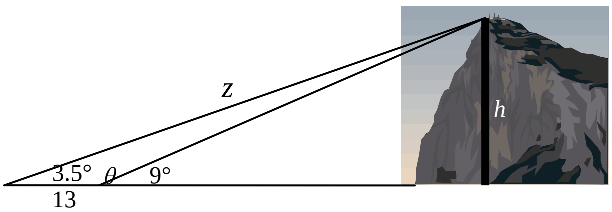
Triangle 1: $\boxed{m\angle A = 94.2^\circ,\ m\angle B = 54.8^\circ,\ m\angle C = 31^\circ,\ a = 56.2,\ b = 46,\ c = 29}$

Triangle 2: $\boxed{m\angle A = 23.8^\circ,\ m\angle B = 125.2^\circ,\ m\angle C = 31^\circ,\ a = 22.7,\ b = 46,\ c = 29}$

<u>Application Problems</u>

1. Draw a picture!
2. Use the Law of Sines to solve for what is asked in the problem.

Ex3: The angle of elevation to a mountain is 3.5°. After driving 13 miles, the angle of elevation is 9°. Approximate the height of the mountain.



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First, find θ: $\theta = 180 - 9 = 171^\circ$. Therefore, the third angle in the small triangle = 5.5°.

Using the law of sines, we know that $\dfrac{13}{\sin 5.5} = \dfrac{z}{\sin 171} \Rightarrow \sin 5.5 z = 13 \sin 171 \Rightarrow z \approx 21.22$.

Now we can use right triangle trig to find h: $\sin 3.5 = \dfrac{h}{21.22} \Rightarrow \boxed{h \approx 1.3 \text{miles}}$

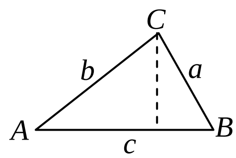
<u>You Try</u>: Solve the triangle ΔABC (if possible) when $m\angle B = 98^\circ$, $b = 10$, $c = 3$.

<u>QOD</u>: Explain why SSA is the ambiguous case when solving triangles.

Syllabus Objective: 3.5 – The student will solve application problems involving triangles (Law of Cosines).

Law of Cosines: For any triangle, *ABC*

$$\boxed{\begin{array}{l} \boldsymbol{c^2 = a^2 + b^2 - 2ab\cos C} \\ \boldsymbol{b^2 = a^2 + c^2 - 2ac\cos B} \\ \boldsymbol{a^2 = b^2 + c^2 - 2bc\cos A} \end{array}}$$



Note: In a right triangle, $c^2 = a^2 + b^2 - 2ab\cos 90^\circ \Rightarrow c^2 = a^2 + b^2 - 2ab\ \ 0 \ \Rightarrow c^2 = a^2 + b^2$ (Pythagorean Theorem)

The Law of Cosines can be used to solve triangles when given SAS or SSS.

Ex1: Solve the triangle *ABC* when $m\angle A = 49^\circ$, $b = 42$, & $c = 15$.

Note: The given information is SAS. Use $\boldsymbol{a^2 = b^2 + c^2 - 2bc\cos A}$.

$$a^2 = 42^2 + 15^2 - 2\ \ 42\ \ \ 15\ \ \cos 49^\circ \Rightarrow a^2 = 1162.36 \Rightarrow a \approx 34.09$$

Now that we have a matching pair of a side and angle, we can use the Law of Sines.

$$\frac{a}{\sin A} = \frac{b}{\sin B} \Rightarrow \frac{34.09}{\sin 49^\circ} = \frac{42}{\sin B} \Rightarrow \sin B = \frac{42\sin 49^\circ}{34.09} \Rightarrow B \approx 68.4^\circ \text{ or } B = 180 - 68.4 = 111.6^\circ$$

Now find the two possibilities for $m\angle C$ using the triangle sum:

$$C = 180 - \ \ 49 + 68.4 \ \ = 62.6 \text{ or } C = 180 - \ \ 49 + 111.6 \ \ = 19.4$$

Since *c* is the shortest side, it must be opposite the smallest angle. So $m\angle C = 19.4^\circ$.

$$\boxed{m\angle A = 49^\circ,\ m\angle B = 111.6^\circ,\ m\angle C = 19.4^\circ,\ a = 34.09,\ b = 42,\ c = 15}$$

Ex2: Solve the triangle *ABC* when $a = 31$, $b = 52$, & $c = 28$.

Note: The given information is SSS. Use $\boldsymbol{a^2 = b^2 + c^2 - 2bc\cos A}$. (Or any of them!)

$$31^2 = 52^2 + 28^2 - 2\ \ 52\ \ \ 28\ \ \cos A \Rightarrow \cos A = \frac{2527}{2912} \Rightarrow A \approx 29.8^\circ$$

Now that we have a matching pair of a side and angle, we can use the Law of Sines.

$$\frac{a}{\sin A} = \frac{b}{\sin B} \Rightarrow \frac{31}{\sin 29.8^\circ} = \frac{52}{\sin B} \Rightarrow \sin B = \frac{52\sin 29.8^\circ}{31} \Rightarrow B \approx 56.5^\circ \text{ or } B = 180 - 56.5 = 123.5^\circ$$

Now find the two possibilities for $m\angle C$ using the triangle sum:

$$C = 180 - \ \ 29.8 + 56.5 \ \ = 93.7 \text{ or } C = 180 - \ \ 29.8 + 123.5 \ \ = 26.7$$

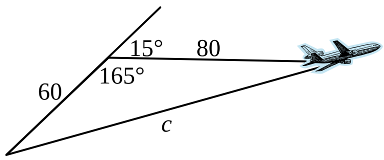
Since *c* is the shortest side, it must be opposite the smallest angle. So $m\angle C = 26.7^\circ$.

$$\boxed{m\angle A = 29.8^\circ,\ m\angle B = 123.5^\circ,\ m\angle C = 26.7^\circ,\ a = 31,\ b = 52,\ c = 28}$$

Application Problems

1. Draw a picture!
2. Use the Law of Cosines to solve for what is asked in the problem.

Ex3: A plane takes off and travels 60 miles, then turns 15° and travels for 80 miles. How far is the plane from the airport?



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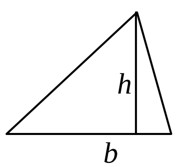
Using the picture, we can find the angle in the triangle to give us SAS.

Use the Law of Cosines: $\boldsymbol{c^2 = a^2 + b^2 - 2ab\cos C}$

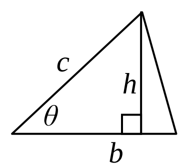
$c^2 = 60^2 + 80^2 - 2(60)(80)\cos 165 \Rightarrow c^2 = 19272.89 \Rightarrow \boxed{c \approx 138.8\text{ miles}}$

Area of a Triangle

Recall: $A = \frac{1}{2}bh$

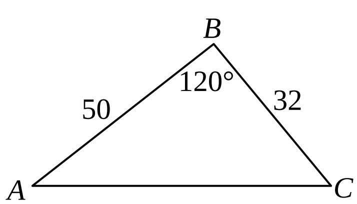


$\sin\theta = \frac{h}{c} \Rightarrow h = c\sin\theta$; $\therefore A = \frac{1}{2}bc\sin\theta$



Formula for the Area of a Triangle Given SAS: $\boxed{\boldsymbol{A = \frac{1}{2}ab\sin C}}$

Ex: Find the area of triangle *ABC* shown.



We will use $\boxed{\boldsymbol{A = \frac{1}{2}ac\sin B}}$.

$A = \frac{1}{2}(32)(50)\sin 120^\circ \Rightarrow A = (16)(50)\left(\frac{\sqrt{3}}{2}\right) \Rightarrow A = \boxed{400\sqrt{3}\text{ sq units}}$

Heron's Area Formula

Semi-Perimeter: $s = \frac{a+b+c}{2}$

Area of a Triangle Given SSS: $\boxed{\boldsymbol{A = \sqrt{s(s-a)(s-b)(s-c)}}}$

Ex4: Find the area of the triangle with side lengths 5 m, 6 m, and 9 m.

Semiperimeter: $s=\dfrac{a+b+c}{2} \Rightarrow s=\dfrac{5+6+9}{2} \Rightarrow s=10$

$\boldsymbol{A=\sqrt{s\left(s-a\right)\left(s-b\right)\left(s-c\right)}}$

$$A=\sqrt{10\left(10-5\right)\left(10-6\right)\left(10-9\right)} \Rightarrow A=\sqrt{10\left(5\right)\left(4\right)\left(1\right)} \Rightarrow A=\sqrt{200} \Rightarrow A=\boxed{10\sqrt{2}\text{ m}^2}$$

<u>You Try</u>: Two ships leave port with a 19° angle between their planned routes. If they are traveling at 23 mph and 31 mph, how far apart are they in 3 hours?

<u>QOD</u>: Can there be an ambiguous case when using the Law of Cosines? Explain why or why not.