Illustrative Mathematics

F-LE Exponential growth versus linear growth I

Alignments to Content Standards: F-LE.A.3

Task

Mr. Wiggins gives his daughter Celia two choices of payment for raking leaves:

i. Two dollars for *each* bag of leaves filled,

ii. She will be paid for the number of bags of leaves she rakes as follows: two cents for filling one bag, four cents for filling two bags, eight cents for filling three bags, and so on, with the amount doubling for each additional bag filled.

a. If Celia rakes enough to five bags of leaves, should she opt for payment method 1 or 2? What if she fills ten bags of leaves?

b. How many bags of leaves would Celia have to fill before method 2 pays more than method 1?

IM Commentary

The purpose of this task it to have students discover how (and how quickly) an exponentially increasing quantity eventually surpasses a linearly increasing quantity. Students' intuitions will probably have them favoring Option A for much longer than is actually the case, especially if they are new to the phenomenon of exponential growth. Teachers might use this surprise as leverage to segue into a more involved task comparing linear and exponential growth.

Solutions

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Solution: Table

a. A table of values giving the number of bags of leaves and the amount paid using methods 1 and 2 shows that method 1 pays more up to and including eleven bags.

b. The table also shows that method 2 pays more as soon as Celia rakes at least twelve bags of leaves. We know that method 2 will always pay more, beyond the twelfth bag, because doubling an amount x gives a larger increase than adding 2 as soon as x is greater than 2:

$$2x > x + 2 \quad \text{whenever} \quad x > 2.$$

Number of Bags	Payment Method 1 (dollars)	Payment Method 2 (dollars)
1	2	0.02
2	4	0.04
3	6	0.08
4	8	0.16
5	10	0.32
6	12	0.64
7	14	1.28
8	16	2.56
9	18	5.12
10	20	10.24
11	22	20.48

12	24	40.96

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Solution: 2. arithmetic and geometric sequences

The numbers in the second column of the table in the first solution form part of the arithmetic sequence which starts with 2 and increases each time by 2: the n^{th} term in this arithmetic sequence is $2n$ and this is the number in the n^{th} row of the second column of the table. The third column of the table is a geometric sequence which starts at 0.02 and increases by multiples of 2 each time. The n^{th} term in this sequence, found in the n^{th} row of the third column, is

$$\frac{2^n}{100}.$$

The numerator 2^n shows the geometric sequence, while the denominator 100 reflects the fact that the sequence began at $\frac{2}{100}$.

Geometric sequences grow exponentially. Since the multiplier two is larger than one, the geometric sequence grows faster than, and eventually surpasses, the linear arithmetic sequence. To see this more clearly, note that each additional bag of leaves makes Celia two dollars with method 1 while with method 2 it doubles her payment. Hence as soon as payment method 2 is worth more than two dollars (that is after 8 bags of leaves) method 2 pays more than method 1 for every additional bag and so the deficit is quickly made up.