Toothpicks and Transformations

(Quadratic Functions)

Objective

Students will investigate quadratic functions using geometric toothpick designs.

Overview of the Lesson

The lesson begins with a review of transformations of quadratic functions—vertical and horizontal shifts, and stretches and shrinks. First, students match the symbolic form of the function to the appropriate graph, then given the graphs, students analyze the various transformations and determine the equation for the functions. This review is followed by an activity where students explore a mathematical pattern that emerges as they build a geometric design with toothpicks. The pattern is quadratic, and the students determine the mathematical model in several different forms. Students examine the recursive nature of the relationship. An explicit model for the relation is developed, and a third model is developed by examining the scatterplot and determining the equation from the transformations. Finally, the class uses the graphing calculators to develop another model and to verify that all of the models—factored form, vertex form, and general form—are equivalent.

Materials

- graphing calculator overhead unit
- overhead projector

For each group of four:

- small paper cup containing about 100 toothpicks (Note: Flat toothpicks will be easier to work with than round ones.)
- bright-colored construction paper mats (optional)
- graphing calculators

- *Transformations Activities 1 and 2*
- *Toothpicks and Transformations*
- *Toothpicks and Transformation Hints*

Procedure

1. **Introduction:** Explain to the students that they will be reviewing transformations of quadratics and then applying their understanding of quadratics as they explore a mathematical pattern developed by creating a toothpick design.

2. **Review of Quadratics, Parts I and II:** Have the students work in groups of four. Each group should have copies of *Transformation Activities 1 and 2*. Remind the students that they need to think about all of the transformations that they have studied—vertical shifts, horizontal shifts, vertical stretches and shrinks, and reflections—as they complete the activity sheets. Circulate among the groups as the students work on this activity. Encourage students to check their work using their graphing calculators. When most of the groups have completed the activity sheets, have volunteers put the answers on the board. You might have the students examine some of the equations using the overhead calculator. In the video lesson, the TI-83 calculator is used. Suggest to the students that they turn on the grid by going to **FORMAT** by pressing **SECOND** then **ZOOM**. Then they should select **GridOn** and press **ENTER** to select this item. Using the grid is very helpful when the students are trying to determine the stretch or shrink factors.

3. **The Toothpick Activity, Part I, Creating the Table:** Introduce the toothpick activity and have the groups collect their materials and *Toothpick and Transformations* activity sheets. Students work in groups and follow instructions for collecting the data to complete the table of values recording the number of toothpicks on one side of the square and the total number of toothpicks in the square. Students begin by physically creating the designs with the toothpicks and counting the number of toothpicks used. As they become involved in the process, you might notice that some students will begin drawing the design on paper rather than constructing the physical model. This offers several advantages—the students can make the designs smaller and depict many designs on one page, and the drawings will not suffer from bumps or get spilled on the floor!

4. **The Toothpick Activity, Part II, Creating the Model:** Once the students have created the table of values, they analyze this data to determine the pattern. They find a mathematical model that represents the number of toothpicks in a square that has n toothpicks on one side. There are various ways to do this. The students can determine a recursive model or an explicit model. Some students might think to use the graphing calculator to determine a quadratic model. Other students might examine the scatterplot and use some

combination of these models in connection with what they know about transformations to determine the equation of the parabola in vertex form.

How much guidance do the students need? This varies from class to class, and therefore the hint sheets are designed to be a guide for students who need them. You may wish to let students try to determine a model without any guidance and then pass out the various hint sheets as you think particular groups need them. Challenge the groups to determine the model in as many ways as they can.

5. **Putting it all Together:** Give each group a chance to report on the ideas they had and how they arrived at their solutions. Summarize these ideas on the blackboard. As groups are presenting, encourage all of the students in the class to listen intently to make sure they understand the ideas the group is presenting, and to raise questions if there is anything that is not clear. Having a good summary activity is critically important when students do group investigations.

Assessment

Assessment for the purpose of guiding instruction is an ongoing process in the classroom. For this lesson, there are many ways and times that you will observe how well your students understand transformations of quadratic functions and the development of a quadratic model for a particular set of data. As you circulate among the groups, you will hear conversations that give insight into their level of understanding. However, it is impossible for you to monitor every group all of the time. You may have a very good idea of where the groups are, but you don't necessarily have all the details of the thinking that has transpired. For these reasons, it is particularly important to have an entire class discussion of the ideas that each group came up with as they worked on the problem.

Students and groups have opportunities for self-assessment as they report their findings and listen to the ideas of others. For example, what were their ideas; and what did others come up with that they did not have? As groups continue to work and report back, the idea of synergy is one that students should begin to appreciate. Together, as a class, they can learn more and do more than any one person or one group could do alone.

For assessment purposes you could collect the activity sheets that the students complete on transformations; or you could ask students to write up the toothpick investigation, stating the problem, giving the table of values for the data, and explaining the mathematical models that could be used to represent that data. As always, be sure to give students specific information about what you are grading, and how you are grading it. Students need to know the grading criteria in advance.

Extensions & Adaptations

- Students can develop models for other toothpick patterns. You could start with a triangle which takes three toothpicks; the next triangle would have a base of two and take nine toothpicks; and so on. Challenge the students to come up with patterns and mathematical models that represent the number of toothpicks required to make any particular stage of the pattern.

- Parts of this lesson could be used with any class studying quadratic functions. Having students understand the connections among the graphs, the tables, and the symbolic forms is very important for all students.

Mathematically Speaking

The transformations that are dealt with in this lesson are the basic transformations that will be applied to other functions as students continue their study of mathematics. The method used by the teacher in this lesson models a way to determine a quadratic equation in vertex form by using the graph and analyzing the transformations. The students consider the parent function, $f(x) = x^2$ as they examine the graph to determine the values of a (the stretch or shrink factor), h (the horizontal shift), and k (the vertical shift), where $f(x) = a(x - h)^2 + k$. Identifying the vertex is a key task in writing the equation for a graph in vertex form. Once the students have identified the vertex, they know the values for h and k because the vertex has been moved h units horizontally and k units vertically putting the vertex at (h, k). To find the value for a, the students simply move one unit to the right of the vertex and move to find the corresponding value of the function. If they had to move up 3 units, then a is 3; if they had to move down 3 units, then a is -3. The students are actually just evaluating the stretch or shrink factor because they are simply looking for the relative change. If they were looking at $f(x) = x^2$, then they would go over 1 and up 1 to be on the parabola and a would be 1, but when the relative change is different, you have a different value for a. Be sure that the students understand that this stretch or shrink has nothing to do with positive or negative. If the parabola is changing or growing faster than $f(x) = x^2$, then you know that a is greater than 1 and you say that a is the stretch factor. If the parabola is changing or growing slower than $f(x) = x^2$, then you know that a is less than 1 and you say that a is the shrink factor. The sign of a simply tells the students whether the parabola opens up or down.

Tips From Ellen

A Constructivist Approach to Learning

It has been said that we have a written curriculum, a taught curriculum, and a learned curriculum. The distinction between the first two is obvious; the distinction between the last two has to do in part with what we know about how students learn. This lesson demonstrates the importance of a constructivist approach to learning and the importance of learning styles.

Constructivism, based on brain research, holds that people's understanding of any concept depends entirely on their experience in deriving that concept for themselves. Teachers can guide the process, but students must develop their own understandings. People remember an experience based on what their pre-existing knowledge and cognitive structures allow them to absorb—regardless of a teacher's intentions or the quality of an explanation. This lesson illustrates how a teacher can scaffold a learning experience, building from a very concrete activity in which toothpicks are manipulated, to visual drawings of the same, to representations via tables, to abstract models. Students can build their understanding and move backwards from any point that their thinking becomes uncertain to a less abstract form of reasoning. The teacher is available to move the learning forward through interventions and guided discussion at appropriate times. The brain likes to create connections and patterns; it is essential that the desired connections are found!

At the same time, it is very apparent that individual students are responding to the instruction differently. These students are very articulate about their learning preferences and styles, e.g.,

— *"I'm not too good with the visual aspect."*
— *"I tend to work intuitively, so to have to verbalize—it helps me to crystallize my thinking."*
— *"When I'm doing something I don't know, I have to learn it by hand to understand it."*

Learning styles address individuals' preferences or strengths in processing information. There are many processing models in the literature including the familiar visual, auditory, or tactile/kinesthetic; field sensitive versus field dependent; and more recently Howard Gardner's seven intelligences, to name a few.

Some implications and ideas:

* Organize instruction around a few powerful ideas. A constructivist approach is more time-consuming, but the learning is deeper and persists longer.
* Move from concrete to abstract reasoning.
* Remember that "hands on" is not always "minds on." Students can follow directions as mindlessly when using objects as they can completing a activity sheets. Design activities that require complex reasoning.

* Remember that there is a role for other kinds of learning, such as rote memorization.
* Develop an awareness of students' learning preferences and styles for yourself and among your students. Learning style inventories are fun and informative.
* Teach to a variety of styles within each lesson. This may take the form of tapping into the various strengths of students at different times (e.g. using manipulatives, writing a description, drawing a picture). It can also take the form of offering students choices in how to complete a task.
* Encourage students to make their own adaptations to suit their styles.
* Encourage students to formulate their learning in a variety of ways.

Resources

National Council of Teachers of Mathematics. *Curriculum and Evaluation Standards for School Mathematics Addenda Series, Grades 9 - 12 : Connecting Mathematics*. Reston, Virginia: National Council of Teachers of Mathematics, 1991.

Mamchur, Carolyn, *A Teacher's Guide to Cognitive Type Theory and Learning Style*. Alexandria, Virginia: Association for Supervision and Curriculum Development, 1996.

Keirsey, D. and M. Bates. *Please Understand Me*. Del Mar: Prometheus Nemesis Books, 1978.

Lawrence, G. *People Types and Tiger Stripes*. Gainsville: Center for Applications of Psychological Type, 1979.

Myers, I. *The Myers-Briggs Type Indicator*. Princeton: Educational Testing Service, 1962.

Internet location:
http://viihills,nw-georgia.resak12.ga.us/InternetResources/Conselor.html This address takes you to The Counselor's Closet which has topics such as colleges and universities, financial aid, academic testing, financial aid and careers in addition to personality testing. Direct your students to the Keirsey Temperament Sorter.

Internet location:
http://www.dc.peachnet.edu/~shale/humanities/composition/assignments/Keirsey.html This location describes the Keirsey Temperament Sorter, allows you to take the test, and gives you an analysis of the results.

Internet location: **http//www.esu10.k12.ne.us/~loupcity/academic/cassette.html** The lesson uses a quadratic function to model sales of a popular cassette. Collecting data, analyzing graphs, and examining local maxima and minimum values are also covered.

Ideas for Online Discussion

(*Some ideas may apply to more than one standard of the* ***NCTM Professional Standards for Teaching Mathematics.)***

Standard 1: Worthwhile Mathematical Tasks

1. This particular lesson allowed students to focus on mathematical patterns, discuss those patterns, and represent those patterns in graphs, tables, and symbols; but it did not include a real-world application of a quadratic model, such as business profits, or projectile motion. What sort of balance is needed between math for the sake of math, and math connected to the real world as teachers seek to select tasks that meet the criteria set forth in Standard 1?

2. How can a teacher who is using a traditional textbook accomplish this standard in a realistic fashion? Isn't it almost like asking the actor to also write the script? How can teachers identify worthwhile tasks and weave them through a traditional course?

Standard 2: The Teacher's Role in Discourse

3. Did the teacher in this lesson let students, or a student, struggle? Is this easy for teachers to do? Why is letting the students struggle important? How do you determine when students have struggled enough?

Standard 3: Students' Role in Discourse

4. Sometimes classes have exceptional students who always know the answer and are eager to tell and explain to others how to solve a problem, often before other individuals or the group has had time to process the problem completely. How do you support and encourage such students, while keeping them from simply giving answers to the other students?

Standard 4: Tools for Enhancing Discourse

5. Could this lesson have been taught without the graphing calculator? Should students be expected to purchase graphing calculators? How is this handled in your school?

Standard 5: Learning Environment

6. How do you make mathematics really meaningful for students in your classroom?

Transformations
Activity Sheet 1

In your group, determine which function rule matches each of the following graphs. Be sure to remember to consider stretches and shrinks, horizontal shifts, vertical shifts, and reflections over the x-axis. Note that each tick mark on the axes represents one unit.

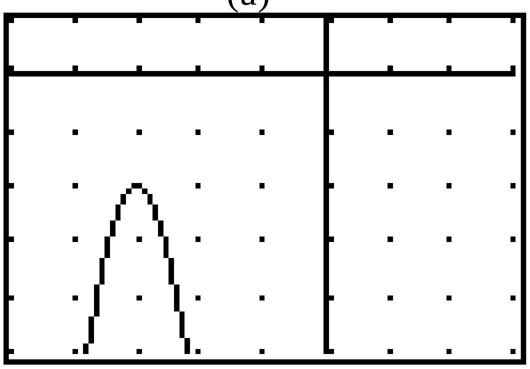
1. $y = \frac{1}{5}(x-3)^2 - 2$

2. $y = -5(x+3)^2 - 2$

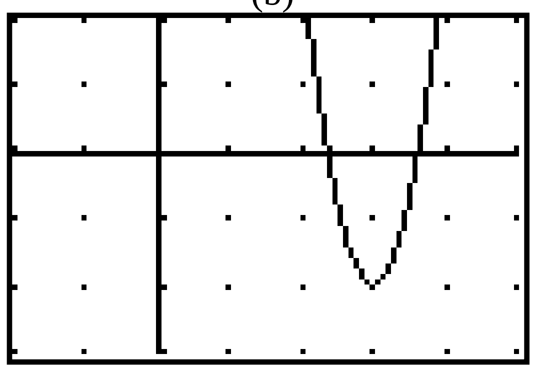
3. $y = 5(x-3)^2 - 2$

4. $y = -(x-2)^2 + 3$

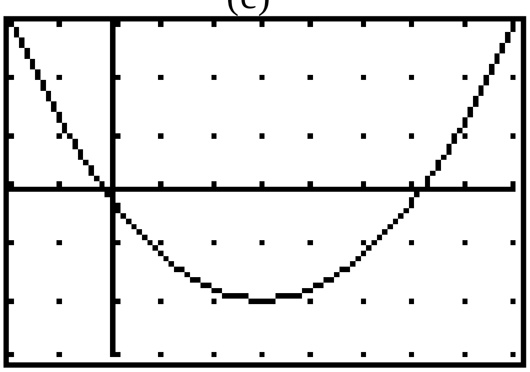
(a)



(b)



(c)



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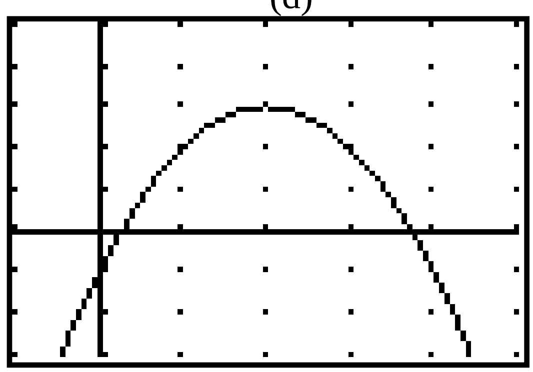
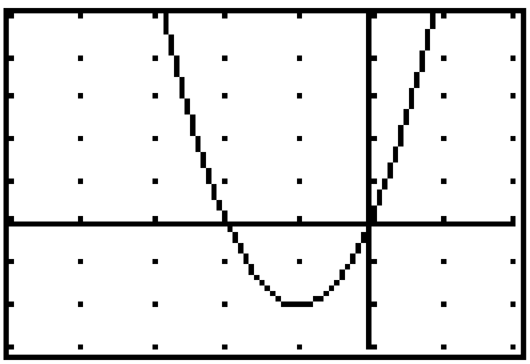


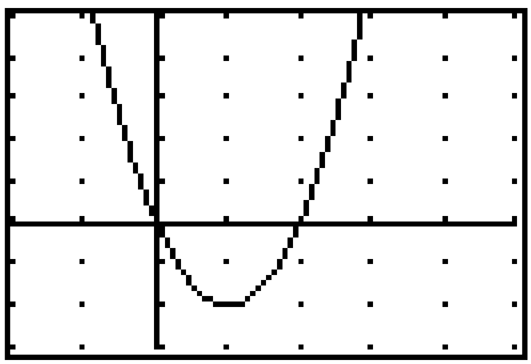
Transformations
Activity Sheet 2

In your group, determine the function rules for each of the following graphs. Write the function in vertex form, $y = a(x - h)^2 + k$. Be sure to remember to consider stretches and shrinks, horizontal shifts, vertical shifts, and reflections over the x-axis.
Note that each tick mark on the axes represents one unit.

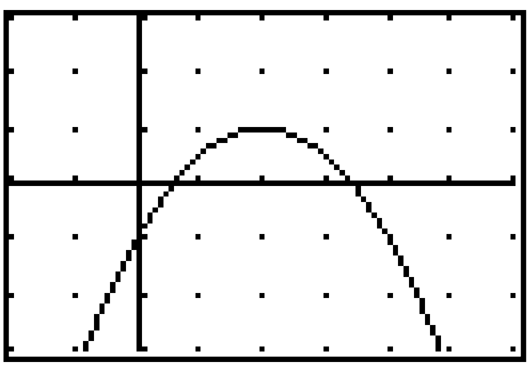
1. y = ________________



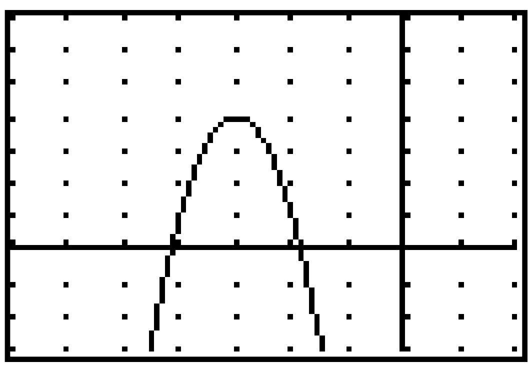
2. y = ________________



3. y = ________________

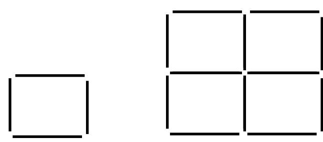


4. y = ________________



Toothpicks and Transformations

In your group, determine a mathematical model that gives the total number of toothpicks required to construct a square of any size that is subdivided into 1 x 1 squares of toothpicks. The 1 x 1 square and a 2 x 2 square are shown below.



1. Explore this pattern using your toothpicks to make larger squares, and complete the data table.

Number of Toothpicks on one side of the square	Total Number of toothpicks
0	
1	
2	
3	
4	
5	
6	
7	
8	

2. Do you see a pattern in the table of values that helps you generate the next entry in the table? Explain this pattern.

3. Use the pattern you have discovered to add some negative values to your table.

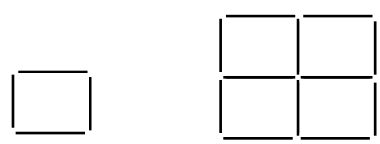
Number of Toothpicks on one side of the square	Total Number of toothpicks
-2	
-1	
0	
1	
2	
3	
4	
5	
6	
7	
8	

4. Using the data in your table, create a scatterplot and generate a mathematical model that represents the total number of toothpicks in a square with side length *n*. Find the same rule in as many other forms as you can. If your group needs a hint, see your teacher for one of the four hint sheets.

Toothpicks and Transformations
Hint # 1

In your group, determine a mathematical model that gives the total number of toothpicks required to construct a square of any size that is subdivided into 1 x 1 squares of toothpicks. The 1 x 1 square and a 2 x 2 square are shown below.

1. Our goal for this model is to determine the number of toothpicks that need to be added each time. How many toothpicks do you add to the first diagram in order to form the second diagram? How many do you add to the second figure, to make a third?

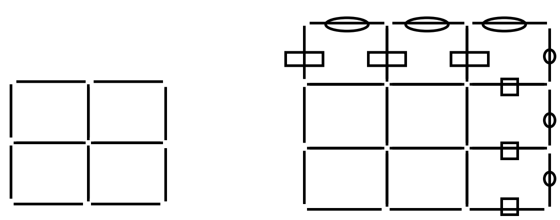


$t_1 = 4$

$t_2 = 4 +$ ________

$t_3 = t_2 +$ ________

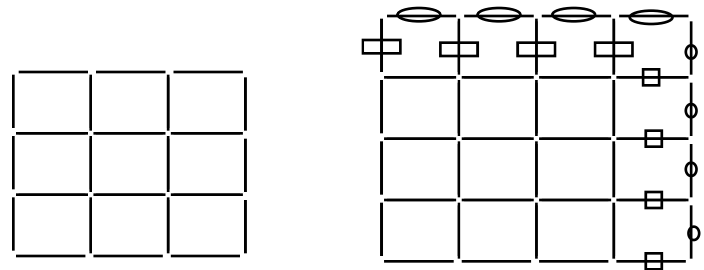
2. You need to generalize this now so that you will be able to write $t_n = t_{n-1} +$ ________ . Each time you increase the size of your square, you are adding four groups of toothpicks: horizontal outside (⬭), horizontal inside (□), vertical outside (o), and vertical inside (▭). Study this pattern.



n = 2 n = 3

To increase to a 3 x 3 square, how many toothpicks had to be added? How many toothpicks were added in each of the four categories: horizontal outside, horizontal inside, vertical outside and vertical inside?

3. Examine the pattern as you change from a 3 x 3 to a 4 x 4 square. Each time you increase the size of your square, you are adding four groups of toothpicks: horizontal outside ( ), horizontal inside ( ), vertical outside ( ), and vertical inside ( ). Study this pattern. How many toothpicks are in each group of four as you change from a 3 x 3 to a 4 x 4 square?



n = 3 n = 4

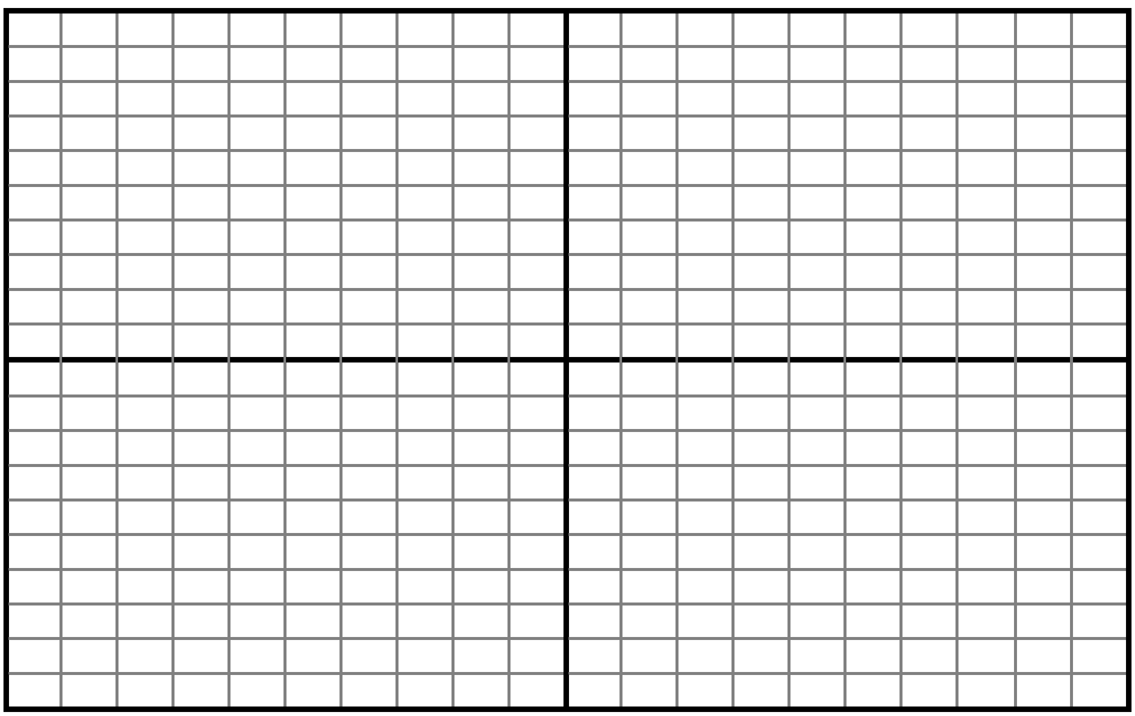
4. As you move from an n x n square to an (n + 1) x (n + 1) square, how many toothpicks must you add?

5. Use the pattern that you have developed to complete the rule: $t_n = t_{n-1} +$ ________ .

Toothpicks and Transformations
Hint # 2

In your group, determine a mathematical model that gives the total number of toothpicks required to construct a square of any size that is subdivided into 1 x 1 squares of toothpicks.

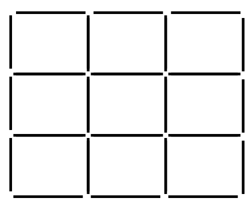
On the grid below make a scatterplot using the data points from your table of values from *Toothpicks and Transformations* problem # 3. Sketch a curve and determine the equation for that curve using what you have studied in this lesson.



Toothpicks and Transformations
Hint # 3

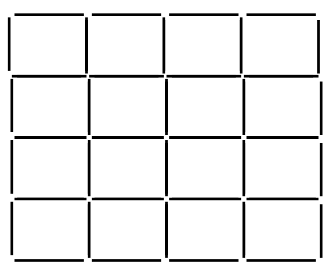
In your group, determine a mathematical model that gives the total number of toothpicks required to construct a square of any size that is subdivided into 1 x 1 squares of toothpicks.

For this model, you will examine the pattern and determine the number of toothpicks required for an n x n square in terms of n. Study the patterns below.



n = 3

1. How many columns of vertical toothpicks are in the figure?
2. How many rows of horizontal toothpicks are in the figure?
3. How many toothpicks are in each row?
4. Combine this information to give the total number of toothpicks.



n = 4

5. How many columns of vertical toothpicks are in the figure?
6. How many rows of horizontal toothpicks are in the figure?
7. How many toothpicks are in each row?
8. Combine this information to give the total number of toothpicks.

Generalize this pattern to answer these same four questions for an n x n square.

9. How many columns of vertical toothpicks would you have for an n x n square?
10. How many rows of horizontal toothpicks would you have for an n x n square?
11. How many toothpicks are in each row?
12. Combine this information to give the total number of toothpicks.
13. State the mathematical model that gives the total number of toothpicks for an n x n square.

Toothpicks and Transformations
Hint # 4

In your group, determine a mathematical model that gives the total number of toothpicks required to construct a square of any size that is subdivided into 1 x 1 squares of toothpicks.

Use your graphing calculator to help you determine a mathematical model for this relationship.

1. Enter the data from "Toothpicks and Transformations" problem # 3 into your calculator.
2. Use your calculator to create a scatterplot.
3. Identify the type of function you think this scatterplot represents, and then use the calculator to determine the best model for your data. What type of function do you have? Give the equation you determined using your calculator.
4. Plot this model on your scatterplot to see how well it fits your data.

Transformations Activity Sheet 1

Answers

1. $y = \frac{1}{5}(x-3)^2 - 2$ *(c)*
2. $y = -5(x + 3)^2 - 2$ *(a)*
3. $y = 5(x - 3)^2 - 2$ *(b)*
4. $y = -(x - 2)^2 + 3$ *(d)*

Transformations Activity Sheet 2

Answers

1. $y = 2(x + 1)^2 - 2$
2. $y = 2(x - 1)^2 - 2$
3. $y = \frac{1}{2}(x - 1)^2 + 1$
4. $y = -3(x + 3)^2 + 4$

Toothpicks and Transformations

Selected Answers

1. Explore this pattern using your toothpicks to make larger squares, and complete the data table.

Number of Toothpicks on one side of the square	Total Number of toothpicks
0	0
1	4
2	12
3	24
4	40
5	60
6	84
7	112
8	144

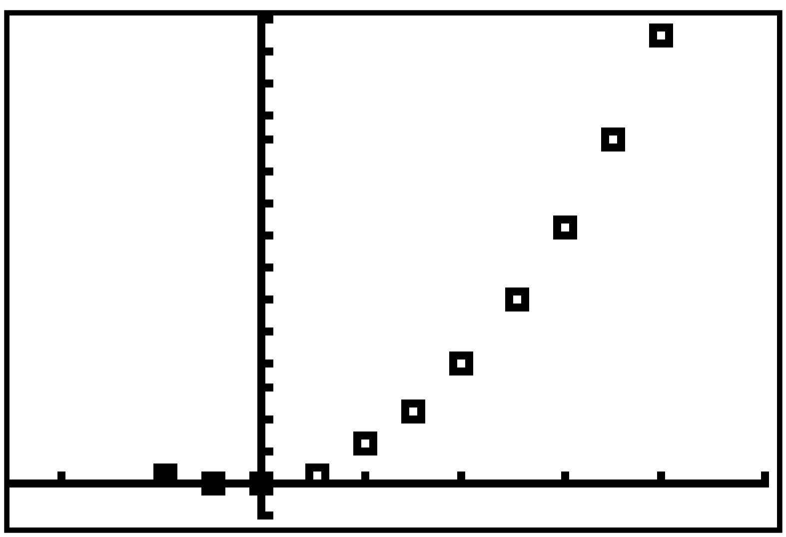
2. Do you see a pattern in the table of values that helps you generate the next entry in the table? Explain this pattern.

   Some students may notice that you are adding multiples of 4: first add 4, then 8, 12, 16, 20, 24, 28, and finally 32.

3. Use the pattern you have discovered to add some negative values to your table.

Number of Toothpicks on one side of the square	Total Number of toothpicks
-2	4
-1	0
0	0
1	4
2	12
3	24
4	40
5	60
6	84
7	112
8	144

4. Using the data in your table, create a scatterplot and generate a mathematical model that represents the total number of toothpicks in a square with side length *n*. Find the same rule in as many other forms as you can. If your group needs a hint, see your teacher for one of the four hint sheets.



Toothpicks and Transformations
Hint # 1
Answers

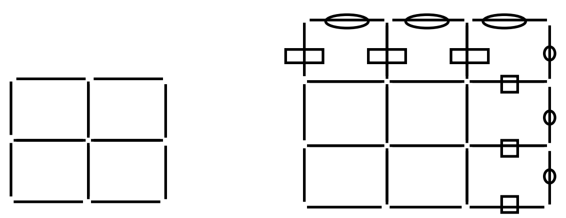
1. Our goal for this model is to determine the number of toothpicks that need to be added each time. How many toothpicks do you add to the first diagram in order to form the second diagram? *8* How many do you add to the second figure, to make a third? *12*

   $t_1 = 4$

   $t_2 = 4 + \mathit{8}$

   $t_3 = t_2 + \mathit{12}$

2. You need to generalize this now so that you will be able to write $t_n = t_{n-1} +$ ________ . Each time you increase the size of your square, you are adding four groups of toothpicks: horizontal outside ( ), horizontal inside ( ), vertical outside ( ), and vertical inside ( ). Study this pattern.



n = 2 n = 3

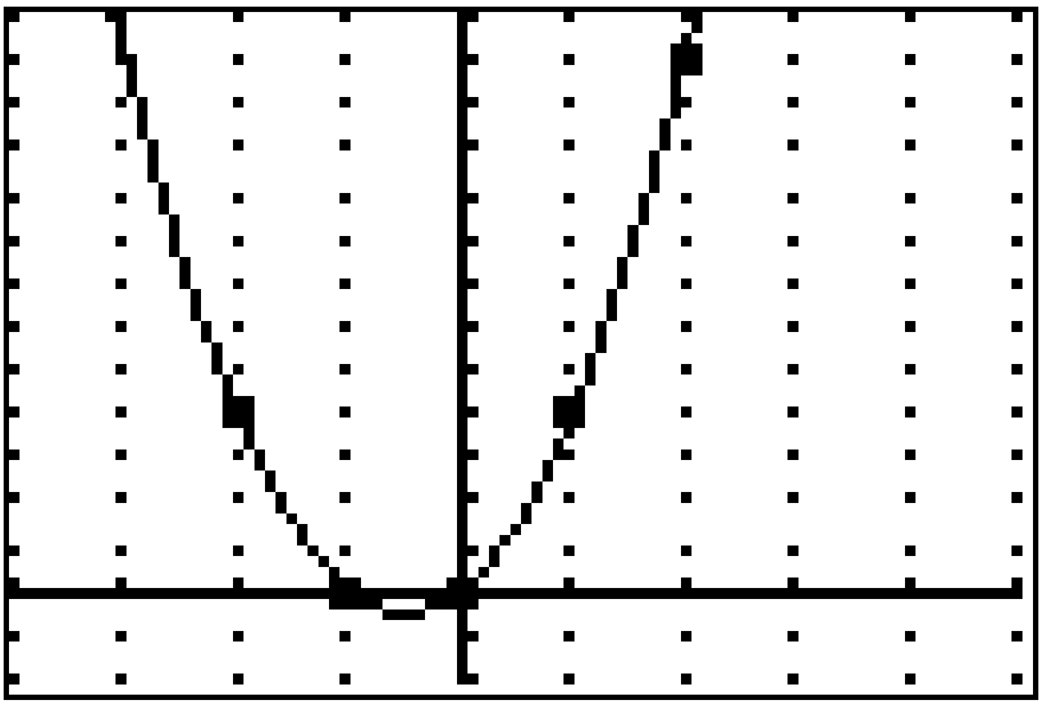
   To increase to a 3 x 3 square, how many toothpicks had to be added? *12* How many toothpicks were added in each of the four categories: horizontal outside, horizontal inside, vertical outside and vertical inside? *4 in each group*

3. Examine the pattern as you change from a 3 x 3 to a 4 x 4 square. Each time you increase the size of your square, you are adding four groups of toothpicks: horizontal outside ( ), horizontal inside ( ), vertical outside ( ), and vertical inside ( ). Study this pattern. How many toothpicks are in each group of four as you change from a 3 x 3 to a 4 x 4 square? *4 toothpicks are in each group of 4*

4. As you move from an n x n square to an (n + 1) x (n + 1) square, how many toothpicks must you add? *4(n + 1)*

5. Use the pattern that you have developed to complete the rule: $t_n = t_{n-1} + \mathit{4n}$.

Toothpicks and Transformations
Hint # 2
Answers



The vertex is (- 0.5, - 0.5). If you move over one unit from the vertex, you go up 2 units to hit the graph. This information gives the model: $y = 2(x + 0.5)^2 - 0.5$.

Toothpicks and Transformations
Hint # 3
Answers

For n = 3
1. How many columns of vertical toothpicks are in the figure? *4*
2. How many rows of horizontal toothpicks are in the figure? *4*
3. How many toothpicks are in each row? *3*
4. Combine this information to give the total number of toothpicks. *24*

For n = 4

5. How many columns of vertical toothpicks are in the figure? *5*
6. How many rows of horizontal toothpicks are in the figure? *5*
7. How many toothpicks are in each row? *4*
8. Combine this information to give the total number of toothpicks. *40*

For an n x n square

9. How many columns of vertical toothpicks would you have for an n x n square? $n + 1$
10. How many rows of horizontal toothpicks would you have for an n x n square? $n + 1$
11. How many toothpicks are in each row? n
12. Combine this information to give the total number of toothpicks. $2n(n + 1)$
13. State the mathematical model that gives the total number of toothpicks for an n x n square. $y = 2n(n + 1)$

Toothpicks and Transformations
Hint # 4
Selected Answers

3. *Students will probably identify this general shape as a parabola. Using the calculator to find the quadratic model, students will determine the model* $y = 2x^2 + 2x$

4. Plot this model on your scatterplot to see how well it fits your data.