



Unit 10 - Graphing Quadratic Functions

PREREQUISITE SKILLS:

- students should be able to add, subtract and multiply polynomials
- students should be able to factor polynomials
- students should be able to identify and graph linear functions
- students should be able to apply transformations to functions

VOCABULARY:

- **Axis of symmetry** – A line that passes through a figure in such a way that the part of the figure on one side of the line is a mirror reflection of the part on the other side of the line.
- **Dependent Variable** – It is a variable in a function that has its value determined by the choice of values of another variable.
- **Dilation** – A transformation in which a figure is enlarged (stretched) or reduced (shrunk).
- **Domain** – The input values of the function. For a quadratic function, the domain is all real numbers.
- **Independent Variable** – A variable in a function that may have its value freely chosen without considering values of any other variables.
- **Maxima** – The largest value (highest point) of the function.
- **Minima** – The smallest value (lowest point) of the function.
- **Parabola** – The U-shaped graph of a quadratic function.
- **Parent Quadratic function** – The simplest quadratic function, $f(x) = x^2$.
- **Quadratic function** – A function where the highest exponent of the variable is a square.
- **Range** – The output values of a function.
- **Reflection** – A transformation in which every point of a figure is mapped to a corresponding image across a line of symmetry.
- **Transformation** – A change in the position, size, or shape of a figure or graph.
- **Translation** – A figure is moved from one location to another on the coordinate plane without changing its size, shape or orientation.
- **Vertex** – The point of intersection of a parabola and its line of symmetry.
- **X-intercept** – Where the graph of a function intersects the x-axis.
- **Y-intercept** – Where the graph of a function intersects the y-axis.

SKILLS:

- Graph parabolas using translations and identify translations of quadratic functions from a graph and/or table.
- Show the maximum and minimum (vertex) on the graph of a quadratic function.
- Show intercepts and the axis of symmetry on the graphs of quadratic functions.
- Graph parabolas in different forms with and without technology.
- Compare multiple representations of functions.

**STANDARDS:**

- F.IF.C.7a** Graph linear and quadratic functions and show intercepts, maxima, and minima. *(Modeling Standard)
- F.IF.C.8a** Use the process of factoring and completing the square in a quadratic function to show zeros, extreme values, and symmetry of the graph, and interpret these in terms of a context.
- F.BF.B.3-1** Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $kf(x)$, $f(kx)$, $f(x + k)$ for special values of k (both positive and negative) and find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. Include linear, exponential, quadratic, and absolute value functions.
- F.IF.B.4-1** For a linear, exponential, or quadratic function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; and end behavior. *(Modeling Standard)
- F.IF.C.9-1** Compare properties of two linear, quadratic, and/or exponential functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). For example, given a graph of one quadratic function and an algebraic expression for another, say which has the larger maximum.
- F.LE.B.5** Interpret the parameters in a linear or exponential function in terms of a context. *(Modeling Standard)
- F.LE.A.1** Distinguish between situations that can be modeled with linear functions and with exponential functions. *(Modeling Standard)

LEARNING TARGETS:

- **10.1** To graph quadratic functions in standard form.
- **10.2** To graph quadratic functions in factored form.
- **10.3** To graph quadratic functions in vertex form.
- **10.4** To transform the graphs of quadratic equations.
- **10.5** To compare properties of two or more functions represented in different ways.

ESSENTIAL QUESTIONS:

- How does multiplying by and/or adding a constant to a function change the graph?
- What are the key features of the graphs of quadratic functions? How can the zeros, vertex, and axis of symmetry of a quadratic function be located?
- How are functions different and how are they similar?



BIG IDEAS:

The location and value of a constant within a function affects its graph. Quadratic equations can be written in vertex form $y = a(x - h)^2 + k$, where the vertex of the graph of the equation is at (h, k) and the axis of symmetry is the line $x = h$. The value of k determines the graph's vertical translation. The value of h determines the graph's horizontal translation. The value of a determines the direction of opening of the graph and the shape of the parabola.

A quadratic function is represented by a U-shaped curve, called a parabola, intercepts one or both axes and has one maximum or minimum value. Different forms of a quadratic function reveal different characteristics of the function.

It is important to compare linear, quadratic and exponential functions in various forms to see how they are different or similar and assess their properties. It is necessary to recognize these differences and similarities to draw conclusions about these models in real-world situations.

Notes, Examples and Exam Questions

Unit 10.1 To graph quadratic functions in standard form.

Parent Function: The most basic quadratic function is $f(x) = x^2$. This function is often called the parent quadratic function. Once you understand the parent function you can shift, reflect, and stretch the parent graph to get graphs of other quadratic functions.

The graph of the function $y = f(x)$ is the set of all points $(x, f(x))$, x is in the domain of f . We match domain values along the x -axis with their range values along the y -axis to get the ordered pairs that yield the graph of $y = f(x)$.

Graphing a Parabola Using a Table

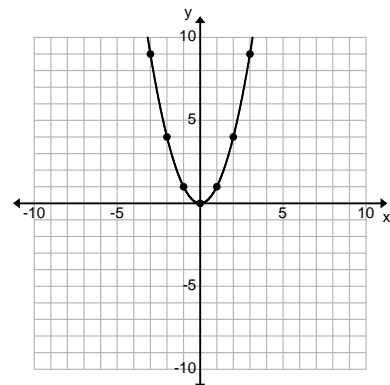
Ex 1: Graph the **parent quadratic function** $y = x^2$.

Step One: Make a table of values (t-chart).

x	-3	-2	-1	0	1	2	3
y	9	4	1	0	1	4	9

Step Two: Plot the points on a coordinate grid and connect to draw the parabola.

Note: The vertex is a minimum at $(0, 0)$, and the axis of symmetry (the vertical line that passes through the vertex) is $x = 0$.





What is the domain (the input values) of $f(x) = x^2$? {all real numbers}

What is the range (the output values)? $\{y \mid y \geq 0\}$

For what value of x is $f(x) = x^2$ increasing? For what values of x is it decreasing?

$f(x) = x^2$ is increasing for $x > 0$ and decreasing for $x < 0$.

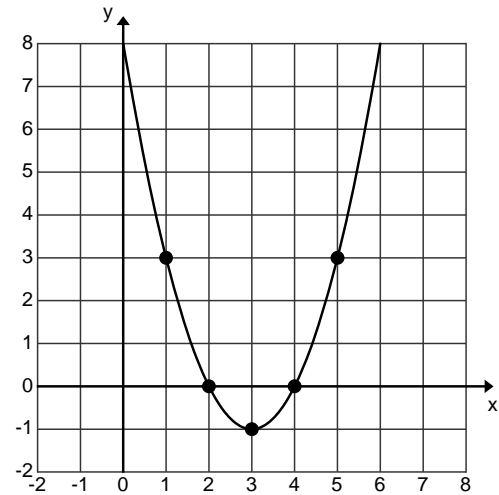
Why is $(0, 0)$ the vertex of the graph of $f(x) = x^2$? It is the lowest point on the parabola.

Graphing Quadratic Functions Using a Table

Ex 2: Graph $f(x) = x^2 - 6x + 8$ by using a table.

Make a table. Plot enough ordered pairs to see both sides of the curve.

x	$f(x) = x^2 - 6x + 8$	$(x, f(x))$
1	$f(1) = 1^2 - 6(1) + 8 = 3$	(1, 3)
2	$f(2) = 2^2 - 6(2) + 8 = 0$	(2, 0)
3	$f(3) = 3^2 - 6(3) + 8 = -1$	(3, -1)
4	$f(4) = 4^2 - 6(4) + 8 = 4$	(4, 0)
5	$f(5) = 5^2 - 6(5) + 8 = 3$	(5, 3)



Vertex: $(3, -1)$ **Axis of Symmetry:** $x = 3$

Domain: All real numbers **Range:** $y \geq -1$

Ex 3: Graph the parabola $y = -x^2$

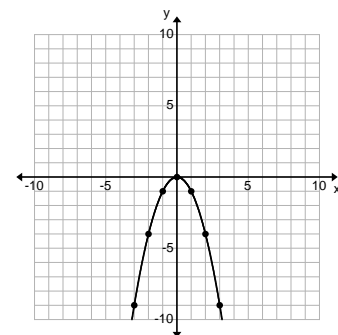
x	-3	-2	-1	0	1	2	3
y	-9	-4	-1	0	-1	-4	-9

Step One: Make a table of values (t-chart).

Step Two: Plot the points on a coordinate grid and connect

with a smooth curve to draw the parabola.

Note: The vertex is a maximum at $(0, 0)$, and the axis of symmetry is $x = 0$.





What is the domain of $f(x) = x^2$? {all real numbers}

What is the range? $\{y \mid y \geq 0\}$

For what value of x is $f(x) = x^2$ increasing? For what values of x is it decreasing?

$f(x) = x^2$ is increasing for $x < 0$ and decreasing for $x > 0$.

Why is $(0, 0)$ the vertex of the graph of $f(x) = -x^2$? It is the highest point on the parabola.

Graphing a Quadratic Function in Standard Form

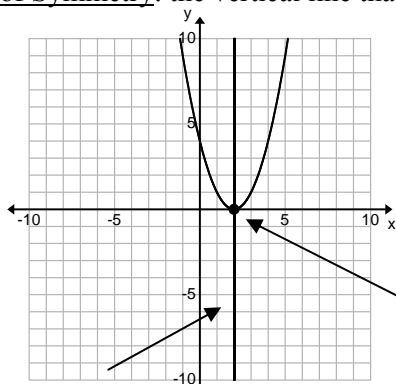
Standard Form of a Quadratic Function $y = ax^2 + bx + c$, when $a \neq 0$; a , b , and c are real numbers

Vertex: the vertex is the point $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$ Axis of Symmetry: $x = -\frac{b}{2a}$ y-intercept: c

Minimum Value: When a parabola opens upward, the y-value of the vertex is the minimum value.

Maximum Value: When a parabola opens downward the y-value of the vertex is the maximum value.

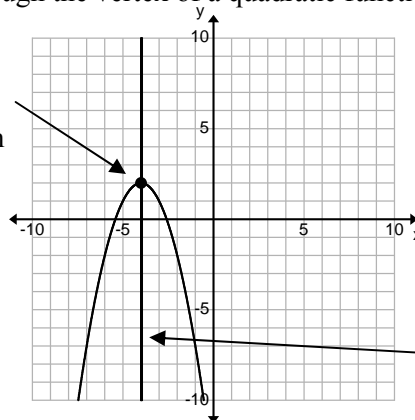
Axis of Symmetry: the vertical line that passes through the vertex of a quadratic function.



Axis of
Symmetry

$D: \{x \mid x \in \mathbb{R}\}$ The domain is all real numbers

$R: \{y \mid y \geq k\}$ The range is all values greater than or equal to the minimum.



Axis of
Symmetry

$D: \{x \mid x \in \mathbb{R}\}$ The domain is all real numbers

$R: \{y \mid y \leq k\}$ The range is all values less than or equal to the maximum.

Ex 4: Graph the quadratic function $y = x^2 - 6x - 1$. State the vertex, axis of symmetry, domain, and range.

Step One: Find the x -coordinate of the vertex. $a = 1$, $b = -6$ $x = -\frac{-6}{2(1)} = 3$

Find the y -coordinate of the vertex: $y = (3)^2 - 6(3) - 1 = 9 - 18 - 1 = -10$



Step Two: Make a table of values. When choosing x -values for the table, use the vertex, as well as a few values to the left of the vertex, and a few values to the right of the vertex.

x	0	1	2	3	4	5	6
y	-1	-6	-9	-10	-9	-6	-1

Note: When calculating the y -coordinate of points to the right and left of the vertex, notice the symmetry.

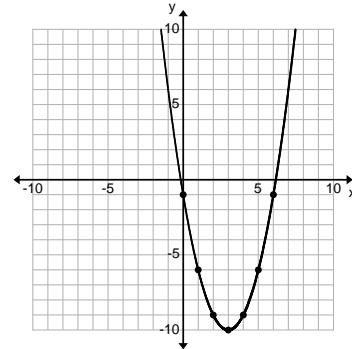
Step Three: Plot the points from the table and draw the parabola.

Vertex: $(3, -10)$

Axis of Symmetry: $x = 3$

Domain: all real numbers

Range: $y \geq -10$



Ex 5: Graph the quadratic function $y = 2x^2 - 4x - 5$. State the vertex and axis of symmetry.

Step One: Determine whether the graph opens upward or downward. If a is positive the graph opens upward. If a is negative the graph opens downward. Since $a = 2$ the parabola opens upward.

Step Two: Find the axis of symmetry. (the x -coordinate of the vertex) $x = -\frac{b}{2a}$

$$a = 2, b = -4 \quad x = -\frac{-4}{2(2)} = 1 \quad \text{The axis of symmetry is the line } x = 1$$

Step Three: Find the vertex.

The vertex lies on the axis of symmetry, so the x -coordinate is 1. The y -coordinate is the value of the function at this x -value, or $f(1)$.

$$f(1) = 2(1)^2 - 4(1) - 5 = -7 \quad \text{The vertex is } (1, -7)$$

Step Four: Find the y -intercept.

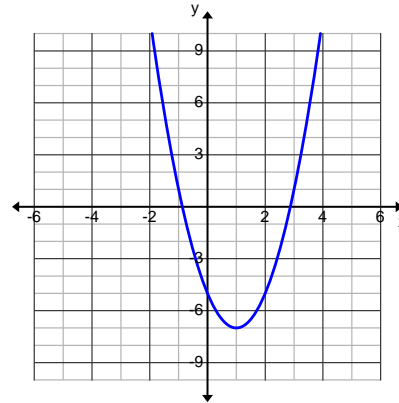
Because $c = -5$, the y -intercept is -5 .



Step Five: Graph by sketching the axis of symmetry and then plotting the vertex and the intercept point $(0, -5)$. Use the axis of symmetry to find another point on the parabola. Notice that $(0, -5)$ is 1 unit left of the axis of symmetry. The point on the parabola symmetrical to $(0, -5)$ is 1 unit to the right of the axis at $(2, -5)$. Connect points with a smooth curve to draw the parabola.

Vertex: $(1, -7)$

Axis of Symmetry: $x = 1$



Optional: To check, make a table of values. When choosing x -values for your T-table, use the vertex, a few values to the left of the vertex, and a few values to the right of the vertex.

x	-2	-1	0	1	2	3	4
y	-11	-6	-5	-7	-5	-6	-11

Note: When calculating the y -coordinate of points to the right and left of the vertex, notice the symmetry.

Ex 6:

Find the minimum or maximum value of $f(x) = 2x^2 - 2x + 5$. State the domain and range and graph the function.

Step One: Determine whether the function has a minimum or maximum value.

Because a is positive, the graph opens upward and has a minimum value.

Step Two: Find the x -value of the vertex.

Step Three: Then find the y -value of the vertex, $f\left(\frac{-b}{2a}\right)$.

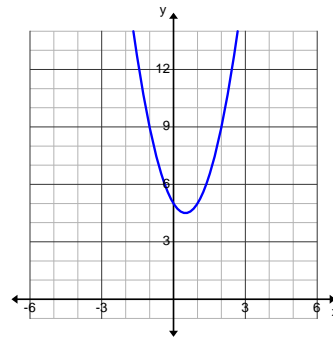
$$x = \frac{-b}{2a} = \frac{-(-2)}{2(2)} = \frac{2}{4} = \frac{1}{2}$$

$$f\left(\frac{1}{2}\right) = 2\left(\frac{1}{2}\right)^2 - 2\left(\frac{1}{2}\right) + 5 = \frac{9}{2} = 4\frac{1}{2}$$

Minimum Value: $4\frac{1}{2}$ or 4.5.



Domain: all real numbers, \mathbb{R} . Range: all real numbers greater than or equal to $4\frac{1}{2}$ $\{y \mid y \geq 4.5\}$ or $[4.5, \infty)$



Check with a Graphing Calculator. Graph $f(x) = 2x^2 - 2x + 5$.

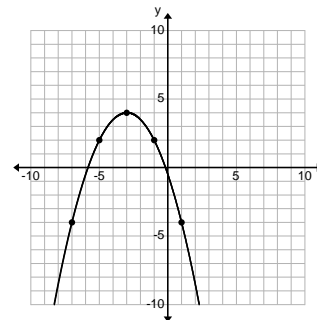
The graph and table should support the answer.

Ex 7: Graph the quadratic function $y = -\frac{1}{2}x^2 - 3x - \frac{1}{2}$. State the vertex and axis of symmetry.

Step One: Find the x -coordinate of the vertex. $a = -\frac{1}{2}, b = -3$ $x = -\frac{-(-3)}{2\left(-\frac{1}{2}\right)} = \frac{3}{-1} = -3$

Step Two: Make a table of values. When choosing x -values, use the vertex, a few values to the left of the vertex, and a few values to the right of the vertex. (Note: Because of the fraction, you may want to choose values that will guarantee whole numbers for the y -coordinates.)

x	-9	-7	-5	-3	-1	1	3
y	-14	-4	2	4	2	-4	-14



Step Three: Plot the points from the table and draw the parabola.

Vertex: $(-3, 4)$ Axis of Symmetry: $x = -3$



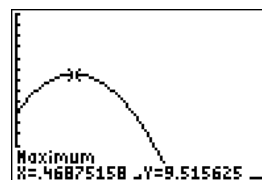
Using a Quadratic Model

Ex 8: A basketball's path can be modeled by $y = -16x^2 + 15x + 6$, where x represents time (in seconds) and y represents the height of the basketball (in feet). What is the maximum height that the basketball reaches?

Graph the function and find the maximum (in the CALC menu). The maximum is the vertex. The maximum height of the basketball is the y -coordinate of the vertex, which is approximately 9.5 ft .

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WINDOW
Xmin=0
Xmax=2
Xscl=1
Ymin=0
Ymax=15
Yscl=1
Xres=1
    
```





Using a Quadratic Model

Ex 9: A baseball is thrown with a vertical velocity of 50 ft/sec from an initial height of 6 ft. The height h in feet of the baseball can be modeled by $h(t) = -16t^2 + 50t + 6$, where t is the time in seconds since the ball was thrown.

Approximately how many seconds does it take the ball to reach its maximum height?

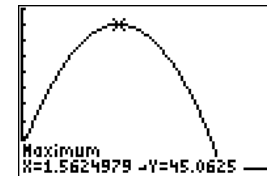
About 1.6 seconds

What is the maximum height that the ball reaches?

About 45 ft.

Plot1	Plot2	Plot3
$\sqrt{Y_1} = -16X^2 + 50X + 6$		
$\sqrt{Y_2} =$		
$\sqrt{Y_3} =$		
$\sqrt{Y_4} =$		
$\sqrt{Y_5} =$		

WINDOW
Xmin=0
Xmax=4
Xscl=1
Ymin=-1
Ymax=50
Yscl=5
Xres=█

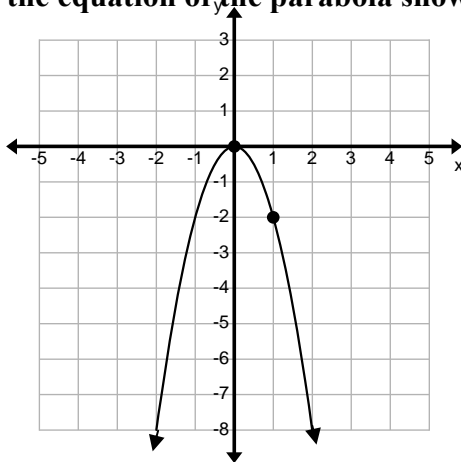


You Try: Find the vertex and axis of symmetry for the following quadratic function. Determine if the parabola will open up or down. Then graph the parabola. $y = -2x^2 - x + 2$

QOD: How many points does it take to determine a unique parabola?

Sample Exam Questions

1. What is the equation of the parabola shown?



- (A) $y = -4x^2$ (B) $y = -2x^2$
 (C) $y = -\frac{1}{2}x^2$ (D) $y = -\frac{1}{4}x^2$

Ans: B



2. Find the vertex of $y = x^2 + 2x - 3$ and state if it is a maximum or a minimum.

- (A) $(-1, -4)$; maximum
- (B) $(-1, -4)$; minimum
- (C) $(-4, -1)$; maximum
- (D) $(-4, -1)$; minimum

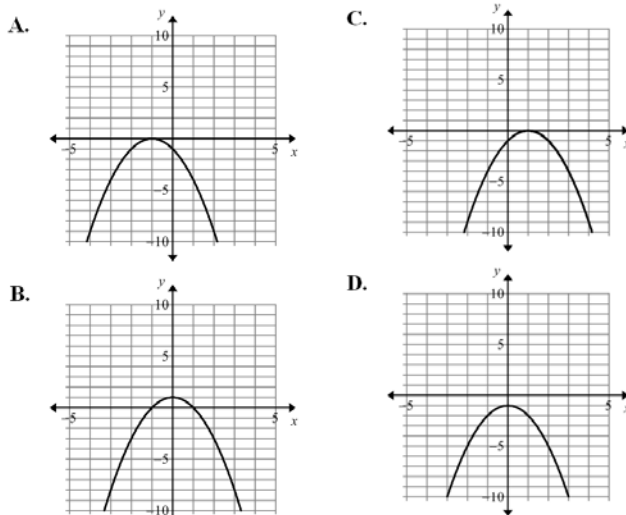
Ans: B

3. What is the maximum of the quadratic function $f(x) = -2x^2 + 4x + 6$?

- A. $f(x) = 1$
- B. $f(x) = 6$
- C. $f(x) = 3$
- D. $f(x) = 8$

Ans: D

4. Which graph represents $f(x) = -x^2 + 2x - 1$?



Ans: C

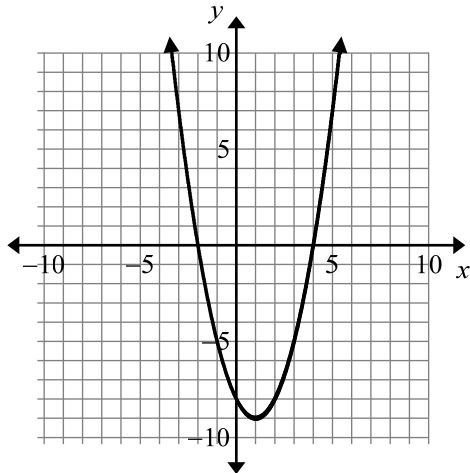
5. Consider $h(x) = 3x^2 + 6x + 2$. What are its vertex and y-intercept?

- a. vertex: $(-1, -1)$, y-intercept: $(0, 2)$
- b. vertex: $(-2, 2)$, y-intercept: $(0, -2)$
- c. vertex: $(1, 1)$, y-intercept: $(0, 2)$
- d. vertex: $(-2, 1)$, y-intercept: $(0, -2)$

Ans: A



6. Which equation *best* represents the following graph? Why?



- A. $y = x^2 - 2x + 8$
- B. $y = x^2 - 2x - 8$
- C. $y = x^2 + 2x - 8$
- D. $y = x^2 + 2x + 8$

Ans: B

7. Which of the following are true statements about the graph of $y = -x^2 + 8x - 4$?

- I. Opens Up
- II. Opens Down
- III. Axis of symmetry $x = -4$
- IV. Axis of symmetry $x = 4$

- A. I and III only
- B. I and IV only
- C. II and III only
- D. II and IV only

Ans: D

8. Find the vertex of the parabola:

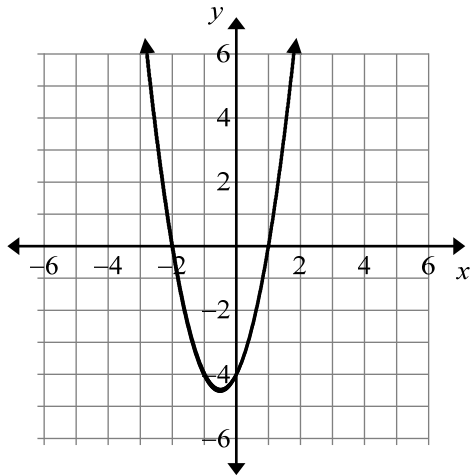
$$y = 2x^2 + 12x + 7$$

- A. $(-6, 7)$
- B. $(-3, 61)$
- C. $(-3, -11)$
- D. $(6, 151)$

Ans: B



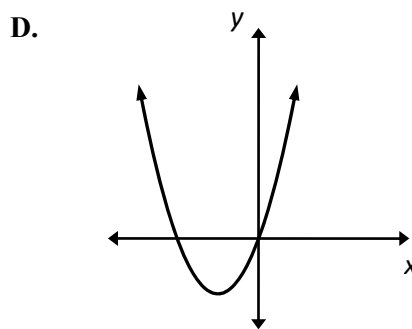
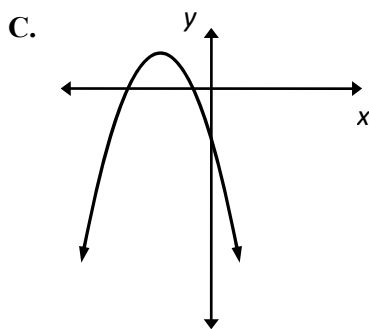
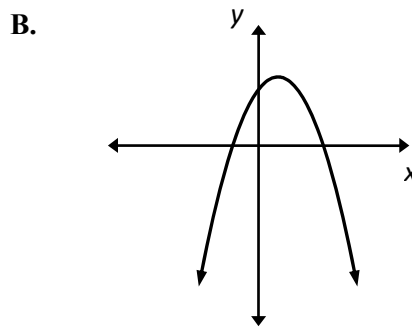
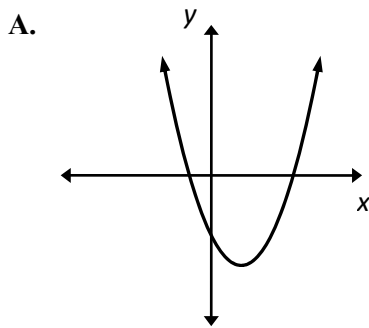
9. What is the domain and range of the function $y = 2x^2 + 2x - 4$ shown in the graph below?



- A. Domain: all real numbers
Range: all real numbers
- B. Domain: $-2 \leq x \leq 2$
Range: $y \geq -4.5$
- C. Domain: all real numbers
Range: $y \geq -4.5$
- D. Domain: $-2 \leq x \leq 2$
Range: all real numbers

Ans: C

10. A quadratic function is given by $h(x) = ax^2 + bx + c$, where a and c are negative real numbers. Which of these could be the graph of $y = h(x)$?



Ans: C



Unit 10.2 To graph quadratic functions in factored form.

Zero(s) of Quadratic Functions: the x -value(s) where the function intersects the x -axis

To find the zero(s), factor the quadratic and set each factor equal to 0.

**Note: We can graph quadratic functions when the function is in factored form.

Step One: First factor the quadratic, if not already factored

Step Two: Find the zeros of the function. Plot these values on the x -axis.

Step Three: Find the vertex of the function. Since the x -coordinate of the vertex of a parabola is exactly the midpoint of the x -intercepts, use the midpoint formula to find the x -coordinate of the vertex. After finding the x -coordinate, substitute in to find the y -coordinate of the vertex.

Step Four: Note the value of a to decide whether the graph opens up or down. Connect the three points to sketch the parabola.

Ex 10: Find the zero(s) of the quadratic function $y = -x^2 - 2x + 3$ and graph the parabola.

Step One: Factor the quadratic polynomial.

$$y = -(x^2 + 2x - 3)$$

$$y = -(x + 3)(x - 1)$$

Step Two: Set each factor equal to 0 and solve.

$$x + 3 = 0 \quad x - 1 = 0$$

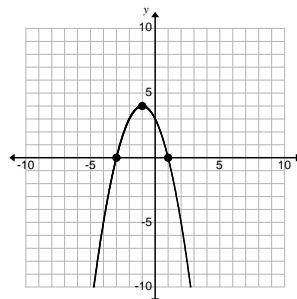
$$x = -3 \quad x = 1$$

Step Three: Find the coordinates of the vertex.

$$x = \frac{-3 + 1}{2} = \frac{-2}{2} = -1$$

$$y = -(-1)^2 - 2(-1) + 3 = -1 + 2 + 3 = 4$$

Step Four: Plot the points and sketch the parabola.





Ex 11: Find the zero(s) of the quadratic function $y = 2x^2 - 2x - 12$ and graph the parabola.

Step One: Factor the quadratic polynomial.

$$y = 2(x^2 - x - 6)$$

$$y = 2(x - 3)(x + 2)$$

Step Two: Set each factor equal to 0 and solve.

$$x - 3 = 0 \quad x + 2 = 0$$

$$x = 3 \quad x = -2$$

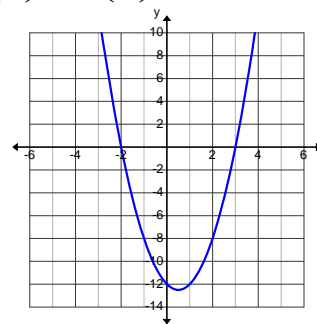
Step Three: Find the coordinates of the vertex.

$$x = \frac{-2 + 3}{2} = \frac{1}{2}$$

$$y = 2\left(\frac{1}{2}\right)^2 - 2\left(\frac{1}{2}\right) - 12 = \frac{1}{2} - 1 - 12 = -12\frac{1}{2}$$

Step Four: Plot the points and sketch the parabola.

** a is positive, so the parabola opens up



Using a Quadratic Model

Ex 12: A science class designed a ball launcher and tested it by shooting a tennis ball straight up from the top of a 15-story building. They determined that the motion of the ball could be described the function:

$h(t) = -16t^2 + 144t + 160$, where t represents the time the ball is in the air in seconds and $h(t)$ represents the height, in feet, of the ball above the ground at time, t . What is the maximum height of the ball? At what time will the ball hit the ground?

a. Find the time it takes for the ball to hit the ground.

Factor: $h(t) = -16(t^2 - 9t - 10)$

$$h(t) = -16(t + 1)(t - 10)$$

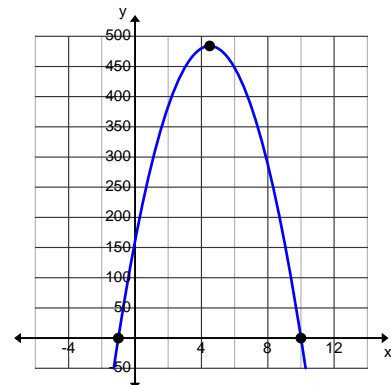
Solve for t : $t = 10$ seconds as time cannot be negative.

It takes **10 seconds** to hit the ground.

Find the vertex:

$$x = \frac{-1 + 10}{2} = 4.5$$

$$h(4.5) = -16(4.5)^2 + 144(4.5) + 160 = \boxed{484 \text{ ft}}$$





Sample Exam Questions

1. Where is the axis of symmetry in the quadratic $f(x) = 3(x-9)(x+5)$?

- A. $x = 4$ C. $x = 2$
 B. $x = 6$ D. $x = -2$

Ans: C

In questions 2 and 3, consider a quadratic $y = f(x)$ that has x -intercepts at $(r, 0)$ and $(s, 0)$, and a y -intercept at $(0, c)$.

2. The function $y = f(x)$ has an axis of symmetry at $x = \frac{r+s}{2}$.

- A. True B. False

Ans: A

3. The function $y = f(x+2)$ has x -intercepts at $(r+2, 0)$ and $(s+2, 0)$.

- A. True B. False

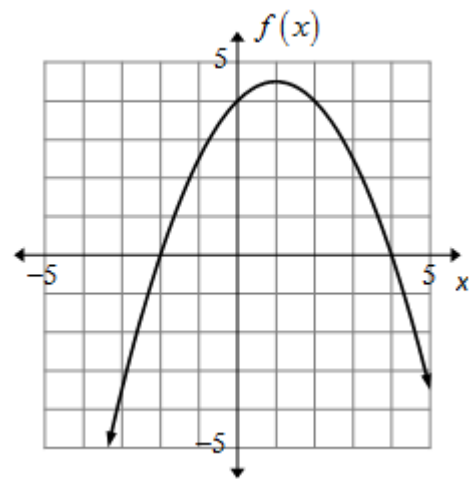
Ans: B

4. Look at the graph of the quadratic $f(x)$ below.

The graph of $g(x) = 3x^2 + bx - 24$ has the same x -intercepts.

What is the value of b ?

- A. -6
 B. -2
 C. 1
 D. 14

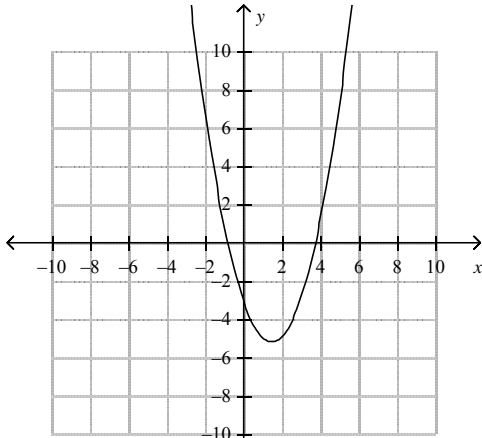


Ans: A

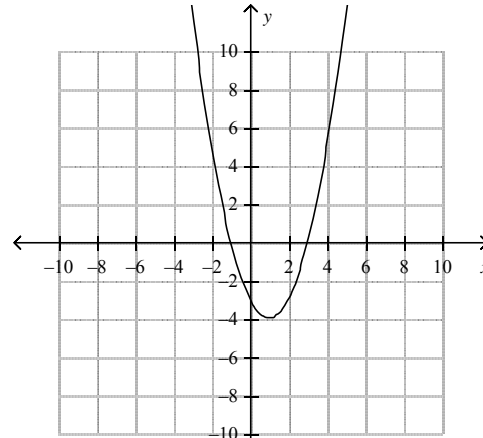


5. Use the quadratic function $f(x) = (x + 3)(x - 1)$, to graph the function.

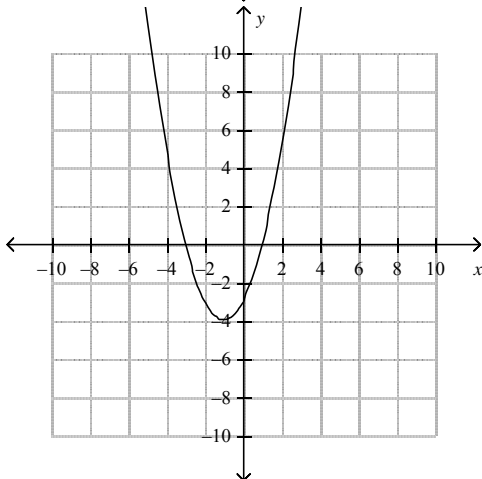
A.



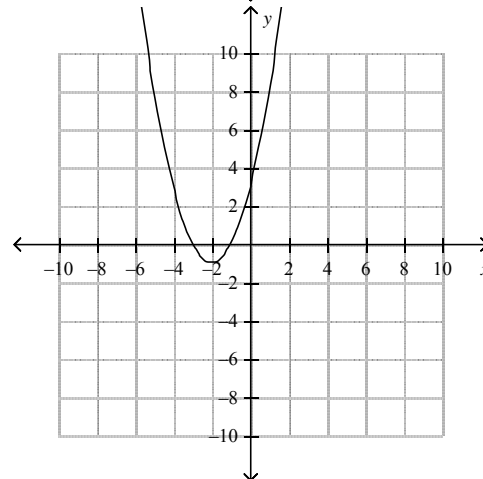
C.



B.



D.



Ans: B

Unit 10.3 To graph quadratic functions in vertex form.

If the quadratic is written in the form $y = a(x - h)^2 + k$, then it is written in what is called vertex form. In this form, the vertex coordinates can be easily found because the vertex is (h, k) . This point is very helpful when graphing the parabola. The a tells us whether it opens up (positive) or down (negative) and also indicates whether it will be "skinny" (value larger than one) or "fat" (value less than one). To find other points for the graph, substitute zero into the equation for x and solve for y . Using symmetry, find another point on the opposite side of the axis of symmetry.

Ex 13: Graph the quadratic function $y = -\frac{1}{2}(x + 3)^2 + 4$. State the vertex and axis of symmetry.

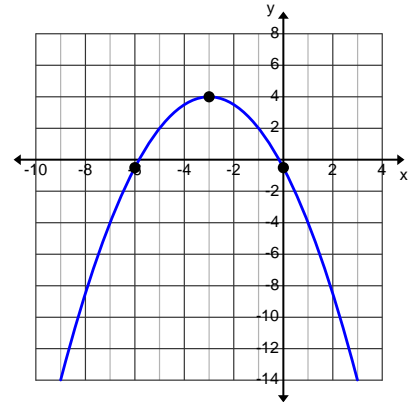
Step One: Determine if the graph opens up or opens down. Because a is $-\frac{1}{2}$ the graph opens down.



Step Two: Identify the vertex and axis of symmetry. Note: Another way of writing the function is

$$y = -\frac{1}{2}(x - (-3))^2 + 4. \text{ So the vertex is } (-3, 4) \text{ and the axis of symmetry is } x = -3.$$

Step Three: Find the y-intercept. $y = -\frac{1}{2}(0 + 3)^2 + 4 \rightarrow y = -\frac{1}{2}$



Optional: Make a table of values. When choosing x -values for the T-table, use the vertex, a few values to the left of the vertex, and a few values to the right of the vertex. (Note: Because of the fraction, you may want to choose values that will guarantee whole numbers for the y -coordinates.) Five to seven points will give a nice graph of the parabola.

x	-9	-7	-5	-3	-1	1	3
y	-14	-4	2	4	2	-4	-14

Ex 14: Graph the quadratic function $y = 2(x + 2)^2 - 3$. State the vertex and axis of symmetry.

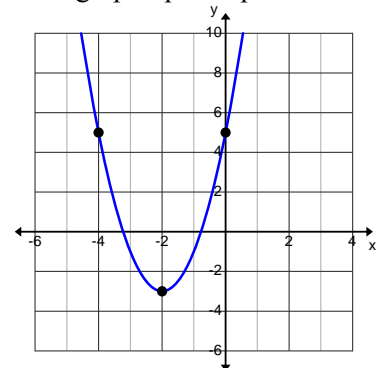
Step One: Determine if the graph opens up or opens down. Because a is 2 the graph opens up.

Step Two: Identify the vertex and axis of symmetry.

$$\text{The vertex is } (-2, -3) \text{ and the axis of symmetry is } x = -2.$$

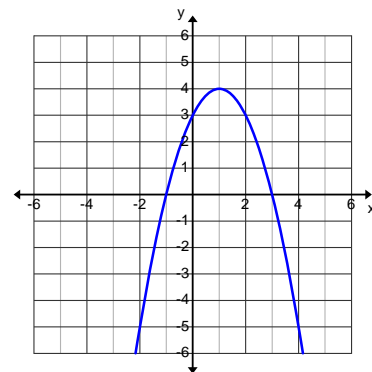
Step Three: Find the y-intercept. $y = 2(0 + 2)^2 - 3 \rightarrow y = 5$

Step Four: Using symmetry, another point is $(-4, 5)$.



Ex 15: Writing the Equation of a Quadratic Function in Vertex Form

Write an equation for the parabola in vertex form given the graph.





Step One: Find the vertex from the graph. The vertex is at $(1, 4)$. So, the vertex form of the equation is $y = a(x-1)^2 + 4$.

Step Two: To solve for a , we will choose a point on the parabola and substitute it into the equation for (x, y) .

$$\begin{aligned} 0 &= a(3-1)^2 + 4 \\ \text{Choose } (3, 0). \quad 0 &= 4a + 4 \\ -4 &= 4a \\ a &= -1 \end{aligned}$$

Step Three: Substitute a and the vertex form of the equation is $y = -(x-1)^2 + 4$.

Ex 16: Automotive Application

The minimum braking distance d in feet for a vehicle on dry concrete is approximated by the function $d(v) = .035v^2$, where v is the vehicle's speed in miles per hour. If the vehicle's tires are in poor condition, the braking-distance function is $d_p(v) = .065v^2$. What kind of transformation describes this change and what does the transformation mean?

Examine both functions in vertex form.

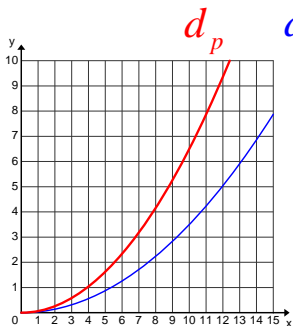
$$d(v) = .035(v-0)^2 + 0$$

$$d_p(v) = .065(v-0)^2 + 0$$

The value a has increased from .035 to .065. The increase indicates a vertical stretch.

Find the stretch factor by comparing the new a -value to the old a value:

$$\frac{d_p(v)}{d(v)} = \frac{0.065}{0.035} \approx 1.857$$



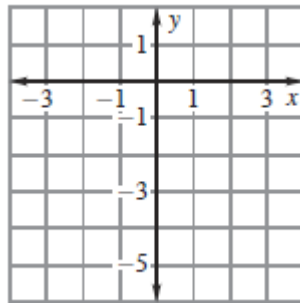
The function d_p represents a vertical stretch of d by a factor of approximately 1.9. Because the value of each function approximates braking distance, a vehicle with tires in poor condition takes about 1.9 times as many feet to stop as a vehicle with good tires does.

Graph: Graph both functions on a graphing calculator. The graph of d_p appears to be vertically stretched compared with the graph of d .

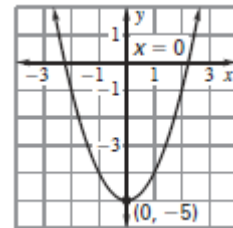


Sample Exam Questions

1. Graph the function $y = x^2 - 5$. Label the vertex and axis of symmetry.



Ans:



2. A quadratic function is defined as $y = (x + 4)^2 - 7$. Which statement is true?

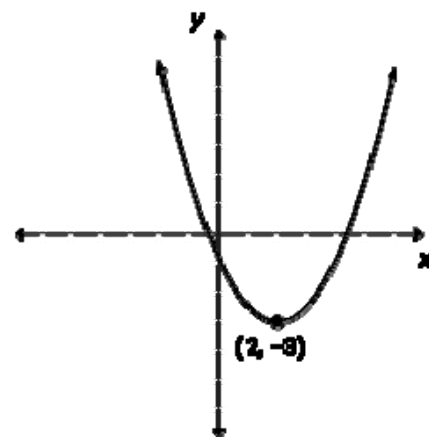
- A. The parabola has a maximum value of -7 .
- B. The parabola has a minimum value of -7 .
- C. The parabola has a maximum value of -4 .
- D. The parabola has a minimum value of -4 .

Ans: B

3. Use the graph below.

Which equation could define the given parabola,
where a is a positive real number?

- A. $f(x) = a(x - 2)^2 - 3$
- B. $f(x) = a(x + 2)^2 - 3$
- C. $f(x) = a(x - 2)^2 + 3$
- D. $f(x) = a(x + 2)^2 + 3$



Ans: A



4. A parabola is defined as $f(x) = a(x-3)^2 + 10$, where a is a positive real number. As a increases, what happens to the y -coordinate of the parabola's vertex?

- A. it decreases
- B. it increases
- C. it does not change

Ans: C

5. A parabola is defined as $f(x) = a(x-3)^2 + 10$, where a is a positive real number. As a increases, what happens to the y -coordinate of the parabola's y -intercept?

- A. it decreases
- B. it increases
- C. it does not change

Ans: C

Unit 10.4 To transform the graphs of quadratic equations.

Reflection, Shrinking and Stretching, Vertical and Horizontal Translations

Comparing $y = x^2$ and $y = -x^2$: The vertex is $(0,0)$, and the axis of symmetry is $x = 0$ for both graphs. When a is **positive**, the parabola opens **up** and its vertex is a **minimum**; when a is **negative**, the parabola opens **down** and its vertex is a **maximum**.

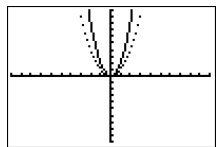
DISCOVERY:



Calculator Exploration: Transformations with $y = x^2$ and $y = -x^2$.

Use the graphing calculator or Desmos to investigate the graphs of quadratic functions. Describe the effect on the graphs of $y = x^2$ and $y = -x^2$. (Note: In the calculator graphs shown, $y = x^2$ or $y = -x^2$ is graphed as a dotted line.)

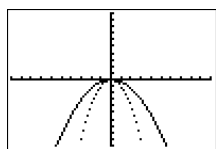
1. $y = 2x^2$



Compare to $y = x^2$

Vertex: Same $(0,0)$ Opens *narrower* than $y = x^2$

2. $y = -\frac{1}{3}x^2$

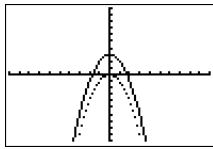


Compare to $y = -x^2$

Vertex: Same $(0,0)$ Opens *wider* than $y = -x^2$



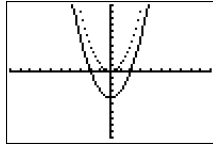
3. $y = -x^2 + 3$



Compare to $y = -x^2$

Vertex: *Up* 3 (0,3) Opens *the same as* $y = -x^2$

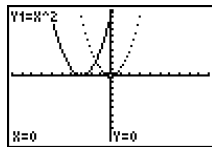
4. $y = x^2 - 4$



Compare to $y = x^2$

Vertex: *Down* 4 (0,-4) Opens *the same as* $y = x^2$

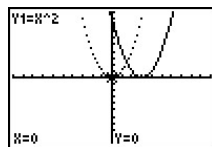
5. $y = (x + 3)^2$



Compare to $y = x^2$

Vertex: *left* 3 (-3,0) Opens *the same as* $y = x^2$

6. $y = (x - 3)^2$



Compare to $y = x^2$

Vertex: *right* 3 (3,0) Opens *the same as* $y = x^2$

Transformations of Quadratic Functions

$$y = a(x - h)^2 + k$$

a indicates a reflection across the x-axis and/or a vertical stretch or compression.

h indicates a horizontal translation

k indicates a vertical translation

Vertical Stretch: $|a| > 1$

Vertical Shrink: $0 < |a| < 1$

Reflection over x-axis: $-a$

Reflection over y-axis: $-x$

Horizontal Translation: $h < 0$ moves left

$h > 0$ moves right

Vertical Translation: $k < 0$ moves down

$k > 0$ moves up

Other: Axis of Symmetry $x = h$

Vertex is (h, k)

If *a* is positive the parabola opens up. If *a* is negative the parabola opens down.



Ex 17: Describe the shift, reflect and stretch of the parent function. $f(x) = x^2$

a. $f(x) = \frac{1}{4}(x+5)^2$

- Vertical Shrink by a factor of $\frac{1}{4}$ (wider)
- Horizontal Shift 5 units left.

b. $f(x) = (x-2)^2 - 1$

- Horizontal Shift 2 units right
- Vertical Shift 1 units down

c. $f(x) = -3(x-5)^2 + 7$

- Reflect across the x axis
- Vertical Stretch by a factor of 3
- Horizontal Shift 5 units right
- Vertical Shift 7 units up

Ex 18: Graph the function below using knowledge of transformations. Graph the parent function on the same graph.

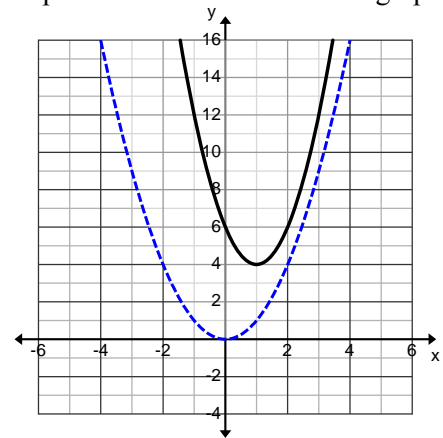
$$f(x) = 2(x-1)^2 + 4$$

Step One: Graph the parent function: $f(x) = x^2$ (dashed in blue)

Step Two: Describe the transformations from the parent graph

- Vertical stretch by a factor of 2
- Horizontal shift 1 unit right
- Vertical shift 4 units up

Step Three: Graph the transformed function



Ex 19: Graph the function using transformations. Find the vertex, axis of symmetry, domain and range.

$$f(x) = -(x+3)^2 - 2$$

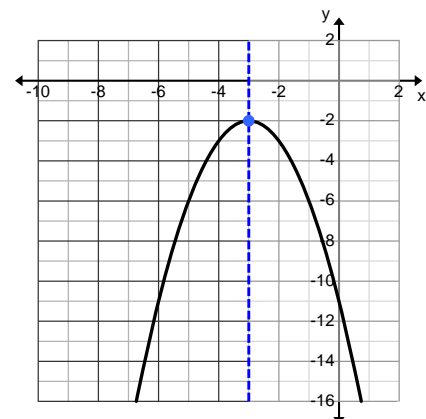
Step One: Describe the transformations from the parent graph

- Reflect across the x-axis
- Horizontal shift 3 units left
- Vertical shift 2 units down

Step Two: Graph using transformations

vertex: $(-3, -2)$ axis: $x = -3$

Domain: \mathcal{R} (all real numbers) Range: $y \leq -2$





Explore: The height h in feet of a baseball on Earth after t seconds can be modeled by the function $h(t) = -16(t - 1.5)^2 + 36$ where -16 is a constant in ft / s^2 due to Earth's gravity.

- What if...? The gravity on Mars is only 0.38 times that on Earth. If the same baseball were thrown on Mars, it would reach a maximum height 59 feet higher and 2.5 seconds later than on Earth. Describe the transformations that must be applied to make the function model the height of the baseball on Mars.
- Write a height function for the baseball thrown on Mars.

QOD: Describe the shift, reflect, and stretch of a quadratic equation in vertex form.

Write About It: Describe the graph of $f(x) = 999,999(x + 5)^2 + 5$ without graphing it.

The graph would be a very narrow parabola opening upward with its vertex at $(-5, 5)$

Closure: What type of graph would a function of the form $y = a(x - h)^2 + k$ have if $a = 0$? What type of function would it be?

Closure: Show equivalent quadratic functions in both standard form and vertex form. Review how to use each form to determine the y-intercept, axis of symmetry, vertex, and maximum/minimum value. Use a graph to check.

$$f(x) = 2x^2 - 4x + 6$$

$$f(x) = 2(x - 1)^2 + 4$$

y-intercept: 6	Axis of Symmetry: $x = 1$
Vertex: (1, 4)	Minimum Value: 4

SAMPLE EXAM QUESTIONS

- How would the graph of the function $y = x^2 - 3$ be affected if the function were changed to $y = x^2 - 2$?

- The graph would shift 2 units up.
- The graph would shift 5 units up.
- The graph would shift 2 units to the right.
- The graph would shift 5 units down.

Ans: B



2. How would you translate the graph of $y = -x^2$ to produce the graph of $y = -x^2 - 4$?

- A. translate the graph of $y = -x^2$ down 4 units
- B. translate the graph of $y = -x^2$ up 4 units
- C. translate the graph of $y = -x^2$ left 4 units
- D. translate the graph of $y = -x^2$ right 4 units

Ans: A

3. Which transformation from the graph of a function $f(x)$ describes the graph of $\frac{1}{3}f(x)$?

- A. horizontal shift left $\frac{1}{3}$ unit
- B. vertical shift up $\frac{1}{3}$ unit
- C. vertical compression by a factor of $\frac{1}{3}$
- D. vertical shift down $\frac{1}{3}$ unit

Ans: C

4. Identify the vertex of $g(x) = (x + 14)^2 - 8$.

- A. $(-14, -8)$
- B. $(-14, 8)$
- C. $(14, -8)$
- D. $(14, 8)$

Ans: A

5. Use this description to write the quadratic function in vertex form:

The parent function $f(x) = x^2$ is vertically stretched by a factor of 3 and translated 8 units right and 1 unit down.

- A. $g(x) = \frac{1}{3}(x - 8)^2 - 1$
- B. $g(x) = \frac{1}{3}(x + 8)^2 - 1$
- C. $g(x) = 3(x - 8)^2 - 1$
- D. $g(x) = 3(x + 8)^2 - 1$

Ans: C

6. Use this description to write the quadratic function in vertex form:

The parent function $f(x) = x^2$ is vertically compressed by a factor of $\frac{1}{2}$ and translated 11 units left and 5 units down.

- A. $g(x) = \frac{1}{2}(x - 11)^2 - 5$
- B. $g(x) = \frac{1}{2}(x + 11)^2 - 5$
- C. $g(x) = 2(x - 11)^2 - 5$
- D. $g(x) = 2(x + 11)^2 - 5$

Ans: B



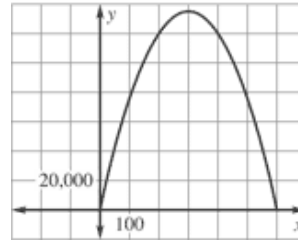
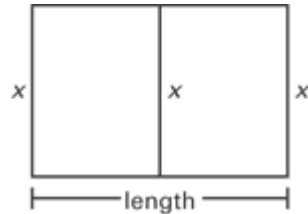
7. Rick uses 1800 feet of fencing to build a rectangular pen. He divides the pen into two sections that have the same area. Let x be the width (in feet) of the pen, as shown in the drawing.

Part A: Write an expression to represent the length of the pen in terms of x . Justify your work.

Part B: Write an equation for the area y of the pen in terms of x . Graph the equation using a graphing calculator.

Part C: Does the function in **Part B** have a maximum or a minimum value? Explain.

Part D: Rick wants the pen to have the largest possible area. What width x should he use? What is the area of the pen with the largest area?



Ans:

Part A: $900 - 1.5x$; Rick has 1800 feet of fencing. He uses $3x$ feet to make two sides and a divider for the pen. So, the amount of fencing he has to make the other two sides of the fence is $1800 - 3x$. Divide $1800 - 3x$ by 2 to get the length of one of the two sides: $900 - 1.5x$.

Part B: $y = x(900 - 1.5x)$, or $y = -1.5x^2 + 900x$; see graph below. $y = -1.5x^2 + 900x$; use x -scale $0 \leq x \leq 700$ with intervals of 50, and y -scale $0 \leq y \leq 160,000$ with intervals of 20,000.]

Part C: The function has a maximum value because the value of the coefficient of x^2 is negative.

Part D: 300 feet; 135,000 square feet

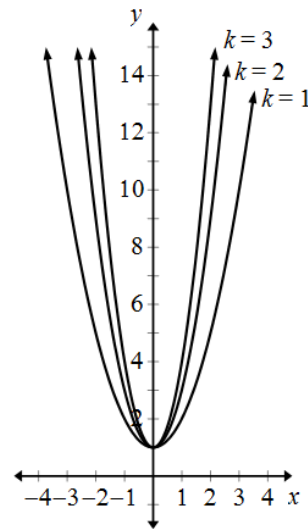
8. Use the graph.

Which equation defines this set of parabolas?

A. $y = kx^2 + 1$

B. $y = \frac{1}{k}x^2 + 1$

C. $y = x^2 + k$



Ans: A

In questions 9-10, consider a quadratic $y = f(x)$ that has x -intercepts at $(r, 0)$ and $(s, 0)$, and a y -intercept at $(0, c)$.

9. The function $y = f(x) - 2$ has a y -intercept at $(0, c - 2)$.

A. True

B. False

Ans: A



10. If $y = f(x)$ opens upward, then $y = -f(x)$ opens downward.

- A. True
B. False

Ans: A

11. Define and sketch the three quadratic functions that have the following characteristics.

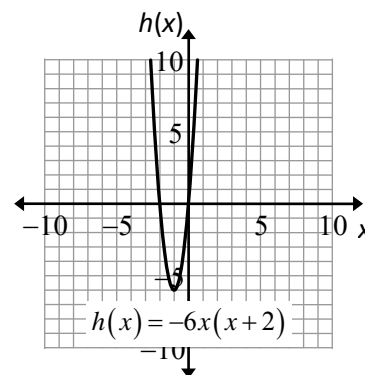
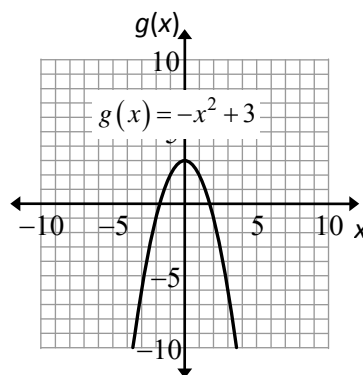
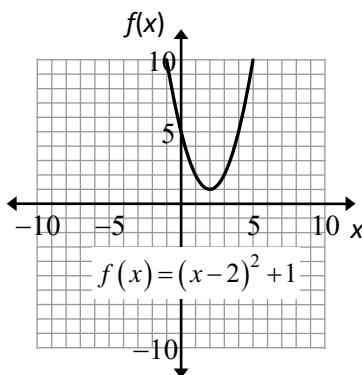
- (a) f has an axis of symmetry at $x = 2$ and no x -intercepts.
 (b) g has a y -intercept at 3 and opens downward.
 (c) h has a zero at $x = -2$ and a minimum value of -6 .

Answers will vary.

(a) f should be of the form, or equivalent to, $f(x) = a(x-2)^2 + k$ where $a > 0$ and $k > 0$.

(b) g should be of the form, or equivalent to, $g(x) = ax^2 + bx + 3$ where $a < 0$.

(c) h should be of the form, or equivalent to, $h(x) = a(x+2)(x-r)$ where $a(r+2)^2 = 24$.



Unit 10.5 To compare properties of two or more functions represented in different ways.

In this unit, students should be able to compare quadratic functions that are in different forms – table form, graph form, vertex form, standard form and factored form. We can compare where the vertices of different functions are (where the highest maximum or lowest minimum occurs), compare which quadratic has the highest average rate of change within a given interval, or compare a certain output value of the functions.

The average rate of change of a function over a specified interval gives us a sense of whether the function values are mostly increasing or mostly decreasing on the interval. The Average Rate of Change of a function

$f(x)$ from $x = a$ to $x = b$ is: $\frac{f(b) - f(a)}{b - a}$. Hopefully this formula will look familiar to the students!



Ex 20: Consider the quadratic function $f(x) = (x-3)^2 - 4$ whose graph is shown.

a) Find the average rate of change from $x = 1$ to $x = 3$.

$$\frac{f(b) - f(a)}{b - a} = \frac{f(3) - f(1)}{3 - 1} = \frac{-4 - 0}{2} = \frac{-4}{2} = -2$$

Since the average rate of change is negative, we know that the function values are mostly decreasing on this interval.

b) Find the average rate of change from $x = 2$ to $x = 5$.

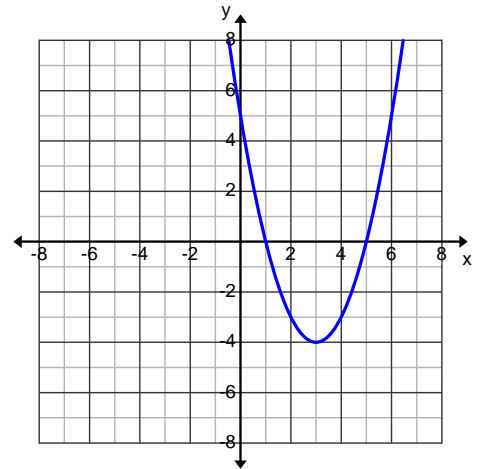
$$\frac{f(5) - f(2)}{5 - 2} = \frac{0 - (-3)}{3} = \frac{3}{3} = 1$$

Since the average rate of change is positive, we know that the function values are mostly increasing on this interval.

c) Find the average rate of change from $x = 2$ to $x = 4$.

$$\frac{f(4) - f(2)}{4 - 2} = \frac{-3 - (-3)}{2} = \frac{0}{2} = 0$$

Since the average rate of change is equal to zero, the quadratic function increases and decreases over the interval but does so equally, as you can see by looking at the graph.



Ex 21: John is standing on a platform 300 ft in the air and throws a ball straight up at 20 feet per second. This situation can be modeled by: $h(t) = -16t^2 + 20t + 300$. John's friend, Frank, is standing on a different platform 200 ft in the air and throws a ball straight up at 40 feet per second. The situation can be modeled with the equation:

$$h(t) = -16t^2 + 40t + 200.$$

a) Which ball reaches the highest altitude? Find the maximum value by finding the y-coordinate of the vertex for John and Frank.

$$\text{John: } x = -\frac{20}{2(-16)} = \frac{5}{8} \rightarrow h(t) = -16\left(\frac{5}{8}\right)^2 + 20\left(\frac{5}{8}\right) + 300 = 306.25$$

$$\text{Frank: } x = -\frac{40}{2(-16)} = \frac{5}{4} \rightarrow h(t) = -16\left(\frac{5}{4}\right)^2 + 40\left(\frac{5}{4}\right) + 200 = 225$$

John has the higher maximum value.



b) Which ball will hit the ground first? Factor both equations to find the zeros.

$$\text{John: } 0 = -4(4t^2 - 5t - 75) \rightarrow 0 = -4(4t + 15)(t - 5) \quad t = 5 \text{ or } -3.75$$

$$\text{Frank: } 0 = -8(2t^2 - 5t - 25) \rightarrow 0 = -8(2t + 5)(t - 5) \quad t = 5 \text{ or } -2.5$$

Since time cannot be negative, both balls hit the ground at 5 seconds. **It is a tie!**

Ex. 22 Sally and Sam are testing out their new potato shooters from their tree houses which are at different heights. The chart to the right shows the time, t , in seconds and height, h , in meters of the potato pieces shot from Sam's shooter. The time, T , and height, H , of Sally's potato shooter can be represented by the following equation:

$$\text{Sally's Shooter: } H = -t^2 + 4t + 5$$

Use the information provided to answer the following questions:

a) Whose potato pieces went higher?

Find the vertex for Sam and Sally's Shooter.

$$\text{Sally: } x = -\frac{4}{2(-1)} = 2 \quad -(2)^2 + 4(2) + 5 = 9 \quad (2, 9). \text{ Since Sally's starts at 5 meters}$$

(when $x = 0$), Sally's travels 4 meters up.

Sam: Looking at the symmetry in the table, the vertex occurs at 1.25 seconds. Using the rate of change for this parabola, the vertex would be at (1.25, 10.5). It would travel to 10.5 meters. However, Sam's shooter starts at 7m, so it only travels 10.5 meters up. **Sally's potato pieces went higher.**

b) Whose potato pieces stayed in the air longer, Sally's or Sam's?

Looking at the table under the *time* column, Sam's pieces hit the ground ($h=0$) at 3.5 seconds. For Sally, we can factor the equation to find the zeros: $-(t-5)(t+1)$. At $t = 5$ seconds, Sally's pieces will hit the ground, so again **Sally wins and her pieces stayed in the air longer.**

Sam's Shooter	
t	h
0	7
0.5	9
1	10
1.5	10
2	9
2.5	7
3	4
3.5	0



Sample Exam Questions

1. The table below is of a quadratic function, $g(x)$, where x is measured in seconds and $g(x)$ is measured in meters.

x	0	1	2	3	4
$g(x)$	2.3	-1.0	1.7	10.4	25.1

What is the approximate rate of change over the interval $0 \leq x \leq 4$?

- (A) 22.8 m/s
- (B) 8.7 m/s
- (C) 6.3 m/s
- (D) 5.7 m/s

Ans: D

2. The table below is of the quadratic $f(x)$.

x	-3	-2	-1	0	1	2
$f(x)$	0	-9	-12	-9	0	15

A second quadratic is defined as $g(x) = x^2 - 6x - 5$.

Which is true about the two functions' minimum values?

- (A) $f(x)$ has a smaller minimum value.
- (B) $g(x)$ has a smaller minimum value.
- (C) The minimum values of $f(x)$ and $g(x)$ are equal.
- (D) Which function has the smaller minimum cannot be determined from the information given.

Ans: B