



## Unit 12 – Special Functions

### PREREQUISITE SKILLS:

- students should be able to describe a relation and a function
- students should be able to identify the domain and range of a function
- students should be able to graph linear functions
- students should be able to solve absolute value equations

### VOCABULARY:

1. domain: the input values of a relation
2. inverse function: two functions are inverse functions if the domain of the original function matches the range of the second function.
3. range: the output values of a relation
4. vertical line test: a relation is a function if and only if no vertical line intersects the graph of the function more than once
5. piecewise-defined function: a function that is written using two or more expressions
6. piecewise-linear function: a function in which the equation for each interval is linear
7. step function: a function whose graph is a series of line segments
8. absolute value function: a function whose rule contains an absolute value expression
9. axis of symmetry: a line that divides the graph into two symmetrical halves
10. vertex: the “corner” point on a graph

### SKILLS:

- Find the inverse of a linear function
- Graph the inverse of a function by reflecting the graph over the line  $y = x$
- Solve for the independent variable in terms of the dependent variable in real-world situations
- Determine when a situation requiring a step function will be necessary
- Identify all key features for the types of functions that make up a piecewise function
- Graph piecewise functions including step and absolute value functions

### STANDARDS:

**A.CED.A.4** Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations, limiting variables of interest to linear and quadratic. For example, rearrange Ohm’s law  $V = IR$  to highlight resistance  $R$ . \*(Modeling Standard)

**F.BF.B.4a-1** Solve an equation of the form  $f(x) = c$  for simple linear and quadratic functions  $f$  that have an inverse and write an expression for the inverse. For example,  $f(x) = 2x^3$  or  $f(x) = (x + 1)/(x - 1)$  for  $x \neq 1$ .



- F.BF.B.4c** Read values of an inverse function from a graph or a table, given that the function has an inverse.
- F.IF.C.7a** Graph linear and quadratic functions and show intercepts, maxima, and minima. \*(Modeling Standard)
- F.IF.C.7b-1** Graph piecewise-defined functions, including step functions and absolute value functions. \*(Modeling Standard)
- F.BF.B.3-1** Identify the effect on the graph of replacing  $f(x)$  by  $f(x) + k$ ,  $kf(x)$ ,  $f(kx)$ , and  $f(x + k)$  for specific values of  $k$  (both positive and negative) and find the value of  $k$  given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. Include linear, exponential, quadratic, and absolute value functions.
- F.IF.B.5-1** Relate the domain of a linear, exponential, or quadratic function to its graph and, where applicable, to the quantitative relationship it describes. For example, if the function  $h(n)$  gives the number of person-hours it takes to assemble  $n$  engines in a factory, then the positive integers would be an appropriate domain for the function. \*(Modeling Standard)
- F.IF.C.7b** Graph piecewise functions, including step functions and absolute value functions. \*(Modeling Standard)

### LEARNING TARGETS:

- 12.1 To find the inverse of a linear function.
- 12.2 To graph the inverse of a linear function.
- 12.3 To graph linear piecewise functions including absolute value and step functions.
- 12.4 To define a piecewise function represented by a graph or real world function.

### ESSENTIAL QUESTIONS:

- How is the inverse of a function related to the function?
- How do you find the inverse function of a given function?
- How do we model real-world situations when they cannot be described with a single function?

### BIG IDEAS:

The inverse of a function is “created” by reflecting the ordered pairs about the line  $y = x$ . Using Algebra properties, we can find the inverse function for any linear function.

Piecewise functions have two or more parts, which may be any type of linear or non-linear function and are used when a single function does not define a real-world situation well.

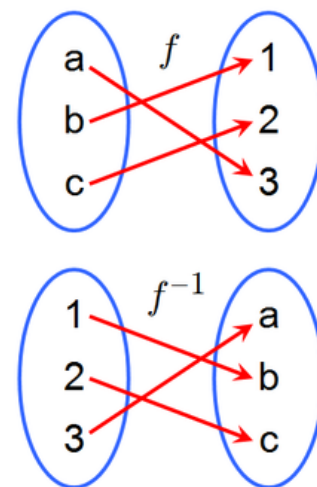
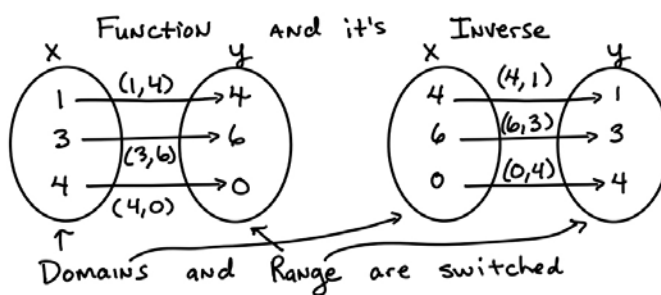


Notes, Examples and Exam Questions

**12.1, 12.2 To find the inverse of a linear function and To graph the inverse of a linear function.**

These learning targets will help us find the inverse of a linear function and see how the original function is related to its inverse. Let's take a look at a mapping diagram. The one on the right top shows our original function,  $f$ . The mapping on the bottom shows the inverse of  $f$ ,  $f^{-1}$ , the original outputs become the inputs, and the original inputs become the outputs.

The notation  $f^{-1}(x)$  indicates the inverse of a function  $f(x)$ . The domain of  $f^{-1}(x)$  is the range of  $f(x)$ , and the range of  $f^{-1}(x)$  is the domain of  $f(x)$ .

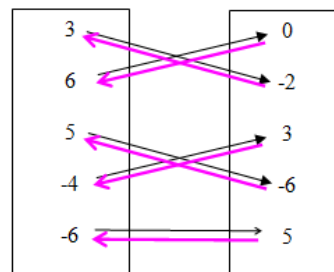
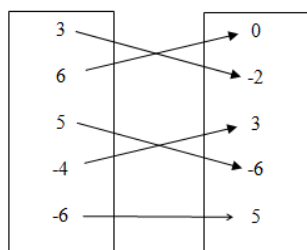


\*\*Note: The symbol -1 in  $f^{-1}$  is not to be interpreted as an exponent. In other words,  $f^{-1}(x) \neq \frac{1}{f(x)}$ .

The procedure for finding the inverse of a linear function is fairly basic.

- 1) Substitute y for f(x)
- 2) Switch the "x" and the "y" in the equation
- 3) Solve for y. You have now solved for the inverse of the function.

**Ex 1:** Find the ordered pairs for the function and then find the ordered pairs for the inverse function. Show the inverse mapping on the figure below as well.



**Ans: Function:  $\{(3, -2), (6, 0), (5, -6), (-4, 3), (-6, 5)\}$  Inverse:  $\{(-2, 3), (0, 6), (-6, 5), (3, -4), (5, -6)\}$**

**Inverse mapping is in pink.**

**Ex 2:**

Find the inverse of  $f(x) = 3x - 2$ .

If you need to find the domain and range, look at the original function and its graph and since it is a linear function, the domain was “all real numbers” and the range is “all real numbers”. To find the domain and range of the inverse, just swap the domain and range from the original function. For linear functions, domain and range will always be “all real numbers” for the original function and the inverse function.

Here's the original function:	$y = 3x - 2$
Interchange the x and the y:	$x = 3y - 2$
Solve for y:	$x + 2 = 3y$ $\frac{x + 2}{3} = y$
Rewrite in function notation:	$f^{-1}(x) = \frac{x + 2}{3}$

**Ex 3:**

Find the inverse of the given function.  $f(x) = -\frac{2}{3}x + 4$

Substitute y for  $f(x)$ :  $y = -\frac{2}{3}x + 4$

Switch the x and y variables:  $x = -\frac{2}{3}y + 4$

$$x - 4 = -\frac{2}{3}y$$

Solve for y:  $-\frac{3}{2}(x - 4) = y$

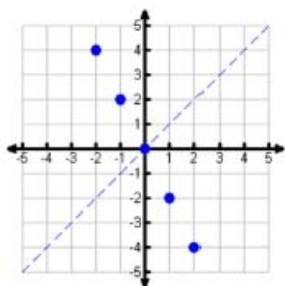
$$\frac{-3x + 4}{2} = y$$

$$\text{Ans: } f^{-1}(x) = \frac{4 - 3x}{2}$$



The graph of an inverse relation is a *reflection* of the graph of the original function. The line of reflection is  $y = x$ . Graph the original function, graph the line  $y = x$ , and reflect the figure over that line.

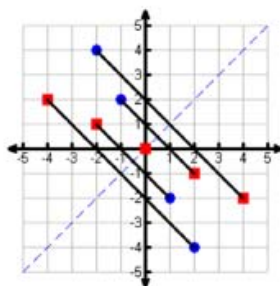
The original relation is the set of ordered pairs:  $\{(-2, 1), (-1, 2), (0, 0), (1, -2), (2, -4)\}$ . The inverse relation is the set of ordered pairs:  $\{(1, -2), (2, -1), (0, 0), (-2, 1), (-4, 2)\}$ . Notice that for the inverse relation the domain (x) and the range (y) reverse positions.



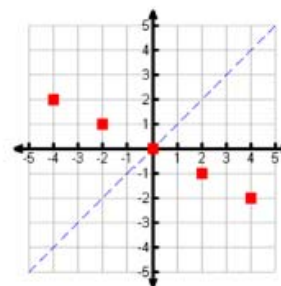
Original Relation

Domain:  $\{-2, -1, 0, 1, 2\}$

Range:  $\{-4, -2, 0, 1, 2\}$



The points are reflected over the line  $y = x$ . Notice that each point is the same distance away from the line, but on the opposite side of the line.



Inverse Relation

Domain:  $\{-4, -2, 0, 1, 2\}$

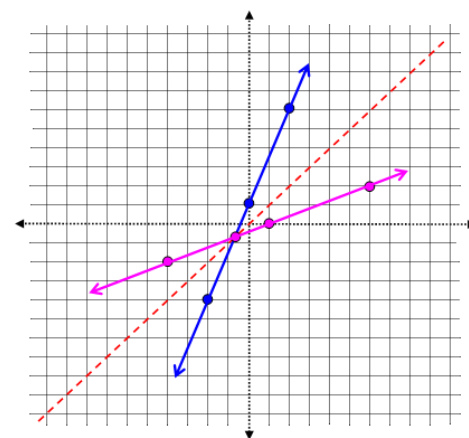
Range:  $\{-2, -1, 0, 1, 2\}$

If no vertical line intersects the graph of a function  $f$  more than once, then  $f$  is a function. This is called the **vertical line test**. If no horizontal line intersects the graph of a function  $f$  more than once, then the inverse of  $f$  is itself a function. This is called the **horizontal line test**. A function is a **one-to-one function** if and only if each second element corresponds to one and only one first element. In order for the inverse of a function to be a function, the original function must be a one-to-one function and meet the criteria for the vertical and horizontal line tests. In Algebra I, the students will only study the inverses of one-to-one linear functions.

**Ex 4:**

Graph the inverse of  $f(x) = \frac{5}{2}x + 1$ .

To graph the function, three ordered pairs were found:  $(0, 1)$ ,  $(2, 6)$ , and  $(-2, -4)$ . The original function is in blue. The red dashed line is the reflection line  $y = x$  and the ordered pairs were then reflected over this line:  $(1, 0)$ ,  $(6, 2)$ ,  $(-4, -2)$ . The pink line represents the inverse graph of  $f(x)$ .



**Ex 5:**

The formula  $F = f(C) = 1.8C + 32$  converts temperatures in degrees Celsius,  $C$ , to degrees Fahrenheit,  $F$ .

- a) What is the *input* to the function  $f$ ? What is the *output* ?

Ans: input: temperature in degrees Celsius, output: temperature in degrees Fahrenheit

- b) Find a formula for the inverse function giving Celsius as a function of Fahrenheit.

$$F = 1.8C + 32 \Rightarrow F - 32 = 1.8C \Rightarrow \frac{F - 32}{1.8} = C$$

\*\*\*Note that you do not switch the variables when you are finding inverses of models. This would be confusing because the letters are chosen to remind you of the real-life quantities they represent.

- c) Use inverse notation to write your formula.

$$f^{-1}(F) = \frac{F - 32}{1.8}$$

- d) What is the *input* to the function  $f^{-1}$ ? the *output* ?

Ans: input: temperature in degrees Fahrenheit, output: temperature in degrees Celsius

- e) Interpret the meaning of the notation:  $f(50) = 122$

Ans:

50° Celsius is 122° Fahrenheit

- f) Interpret the meaning of the notation:  $f^{-1}(200) = 93.3$

Ans:

200° Fahrenheit is 93.3° Celsius

**Ex 6:**

In bowling, a *handicap* is a change in score to adjust for differences in players' abilities. You belong to a bowling league in which each bowler's handicap,  $h$ , is determined by his or her average  $a$  using this formula:  $h = 0.9(200 - a)$ . (If the bowler's average is over 200, the handicap is 0).

- a) What is the input and output of the function?

input: average ( $a$ ), output: bowler's handicap,  $h$ .

- b) Find the inverse of the function.

$$h = 0.9(200 - a)$$

$$\frac{h}{0.9} = 200 - a$$

$$a = 200 - \frac{h}{0.9}$$



c) What is the input and output of the inverse function?

input: bowler's handicap,  $h$ , output: average ( $a$ ).

d) Find your average if your handicap is 27.

$$a = 200 - \frac{h}{0.9}$$

$$a = 200 - \frac{27}{0.9}$$

$$a = 200 - 30 = \boxed{70}$$

**Ex 7:**

You and a friend are playing a number-guessing game. You ask your friend to think of a positive number, square the number, multiply the result by 2, and then add 3. If your friend's final answer is 53, what was the original number chosen? Use an inverse function in your solution.

Find the original equation:  $y = 2x^2 + 3$

$$y = 2x^2 + 3$$

$$y - 3 = 2x^2$$

Find the inverse:

$$\frac{y-3}{2} = x^2$$

$$x = \sqrt{\frac{y-3}{2}}$$

Substitute  $y = 53$  and solve for  $x$ :

$$x = \sqrt{\frac{53-3}{2}} = \sqrt{25} = \boxed{5}$$

### SAMPLE EXAM QUESTIONS

1. What is the inverse of  $f(x) = 2x + 9$ ?

A.  $f^{-1}(x) = \frac{x}{2} + 9$

C.  $f^{-1}(x) = \frac{x-9}{2}$

B.  $f^{-1}(x) = \frac{1}{2x+9}$

D.  $f^{-1}(x) = \frac{2}{x-9}$

Ans: C



2. If  $f^{-1}(x) = \frac{4}{3}x + 8$ , what is  $f(x)$ ?

A.  $f(x) = \frac{3}{4}(x-8)$

C.  $f(x) = \frac{4}{3}x - 6$

B.  $f(x) = \frac{3}{4}x - 8$

D.  $f(x) = \frac{4}{3}(x-8)$

Ans: A

3. If the point  $(a, b)$  lies on the graph  $y = f(x)$ , the graph of  $y = f^{-1}(x)$  must contain point

A.  $(0, b)$

B.  $(a, 0)$

C.  $(b, a)$

D.  $(-a, -b)$

Ans: B

4. The inverse function of  $\{(2, 6), (-3, 4), (7, -5)\}$  is

A.  $\{(-2, 6), (3, 4), (-7, -5)\}$

C.  $\{(6, 2), (4, -3), (-5, 7)\}$

B.  $\{(2, -6), (-3, -4), (7, 5)\}$

D.  $\{(-6, -2), (-4, 3), (5, -7)\}$

Ans: C

5. Given the relation  $A: \{(3, 2), (5, 3), (6, 2), (7, 4)\}$

A. Both  $A$  and  $A^{-1}$  are functions.

C. Only  $A^{-1}$  is a function.

B. Neither  $A$  nor  $A^{-1}$  are functions.

D. Only  $A$  is a function.

Ans: D

6. The inverse of the function  $2x + 3y = 6$  is

A.  $y = -\frac{2}{3}x + 2$

C.  $y = \frac{3}{2}x + 2$

B.  $y = -\frac{3}{2}x + 3$

D.  $y = \frac{2}{3}x + 3$

Ans: B





7. A function  $f(x)$  takes values of  $x$  and applies the following:

Step 1) divide  $x$  by 5

Step 2) subtract 3 from the result in Step 1

Which of these describes the inverse function of  $f(x)$ ?

- A. Step 1) multiply  $x$  by 5  
Step 2) add 3 to the result in Step 1
- B. Step 1) subtract 3 from  $x$   
Step 2) divide the result in Step 1 by 5
- C. Step 1) add 3 to  $x$   
Step 2) multiply the result in Step 1 by 5
- D. Step 1) divide  $x$  by  $\frac{1}{5}$   
Step 2) subtract  $-3$  from the result in Step 1

Ans: C

**12.3, 12.4 To graph linear piecewise functions including absolute value and step functions and To define a piecewise function represented by a graph or real world function.**

**Absolute Value Function:** a function that can be written in the form  $f(x) = a|x - h| + k$

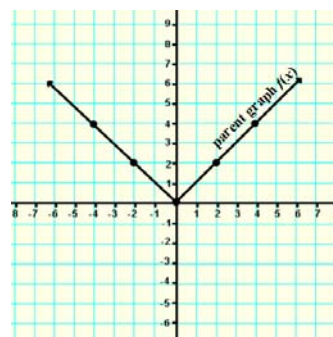
**Absolute value function** - A function written as  $f(x) = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -x & \text{if } x < 0 \end{cases}$ , for all values of  $x$ .

**Ex 1:** Use a table of values to graph the function  $f(x) = |x|$ .

vertex

$x$	-4	-2	0	2	4
$f(x)$	4	2	0	2	4

$f(x) = |x|$  is the parent function. Notice that the parent graph is V-shaped, opens up and the vertex of the graph is on the origin.



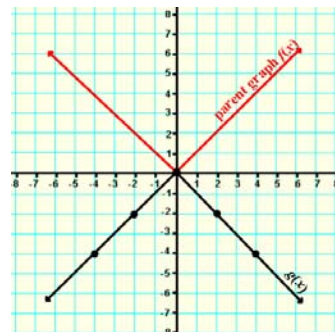


**Ex 2:** Use a table of values to graph the function  $g(x) = -|x|$

vertex

$x$	-4	-2	0	2	4
$g(x)$	-4	-2	0	-2	-4

Notice that the only difference between this function rule and the parent function is that  $a = -1$  (the opposite of the parent function). The difference between this graph and the parent graph is that it has been reflected across the  $x$ -axis, so it opens down.



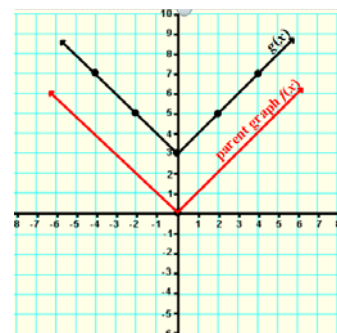
**Note:** It is important to make sure to find the vertex of an absolute-value equation to avoid graphing one side of the function only. To do this, make sure to start the table of values in the middle with the  $x$  value that gives 0 for the expression inside the absolute value symbols. Then, assign  $x$  values on each side of that first value.

**Ex 3:** Use a table of values to graph the function  $g(x) = |x| + 3$ .

vertex

$x$	-4	-2	0	2	4
$g(x)$	7	5	3	5	7

Notice that the only difference between this function rule and the parent function is that 3 is being added to the absolute value expression. The difference between this graph and the parent graph is that it has been translated up 3 units.

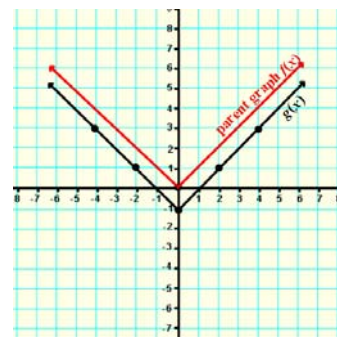


**Ex 4:** Use a table of values to graph the function  $g(x) = |x| - 1$ .

vertex

$x$	-4	-2	0	2	4
$g(x)$	3	1	-1	1	3

Notice that the only difference between this function rule and the parent function is that 1 is being subtracted from the absolute value expression. The difference between this graph and the parent graph is that it has been translated down 1 unit.



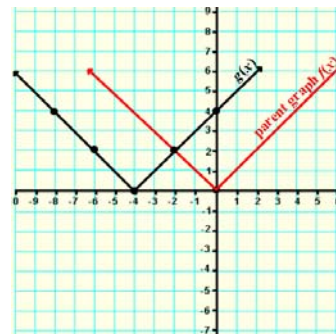


**Ex 5:** Use a table of values to graph the function  $g(x) = |x + 4|$ .

vertex

$x$	-8	-6	-4	-2	0
$g(x)$	4	2	0	2	4

Notice that the only difference between this function rule and the parent function is that 4 is being added to the variable inside of the absolute value expression. The difference between this graph and the parent graph is that it has been translated left 4 units.

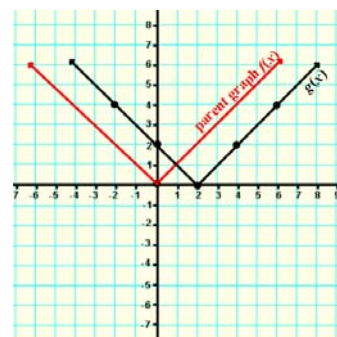


**Ex 6:** Use a table of values to graph the function  $g(x) = |x - 2|$

vertex

$x$	-2	0	2	4	6
$g(x)$	4	2	0	2	4

Notice that the only difference between this function rule and the parent function is that 2 is being subtracted from the variable inside of the absolute value expression. The difference between this graph and the parent graph is that it has been translated right 2 units.

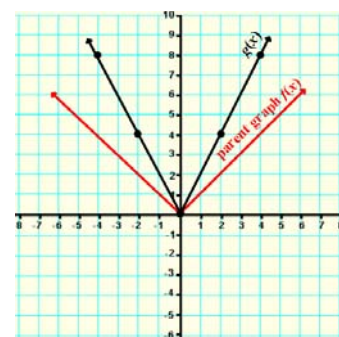


**Ex 7:** Use a table of values to graph the function  $g(x) = 2|x|$

vertex

$x$	-4	-2	0	2	4
$g(x)$	8	4	0	4	8

Notice that the only difference between this function rule and the parent function is that  $a = 2$  (the  $a$  value has doubled). The difference between this graph and the parent graph is that it has been stretched (narrower); the rate of change from each domain value to the next has doubled.



\* Since  $a = 2$ , the slope is  $-2$  to the left of the vertex and  $2$  to the right of the vertex.

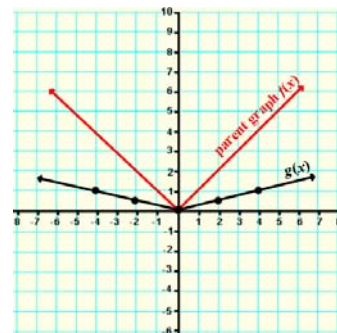


**Ex 8:** Use a table of values to graph the function  $g(x) = \frac{1}{4}|x|$

vertex

$x$	-4	-2	0	2	4
$g(x)$	1	0.5	0	0.5	1

Notice that the only difference between this function rule and the parent function is that  $a = \frac{1}{4}$  (the  $a$  value is one quarter the size). The difference between this graph and the parent graph is that it has been shrunk (wider); the rate of change from each domain value to the next is one quarter the amount.



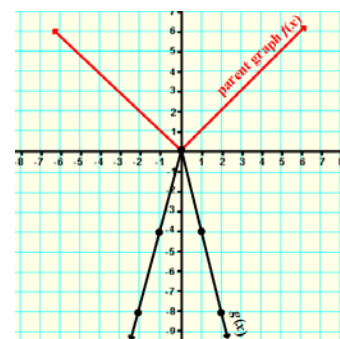
\* Since  $a = \frac{1}{4}$ , the slope is  $-\frac{1}{4}$  to the left of the vertex and  $\frac{1}{4}$  to the right of the vertex.

**Ex 9:** Use a table of values to graph the function  $g(x) = -4|x|$

vertex

$x$	-2	-1	0	1	2
$g(x)$	-8	-4	0	-4	-8

Notice that the only difference between this function rule and the parent function is that  $a = -4$  (the  $a$  value is the opposite of four times the size). The difference between this graph and the parent graph is that it has been reflected across the  $x$ -axis and it has stretched (narrower); the rate of change from each domain value to the next is the opposite of four times the amount.



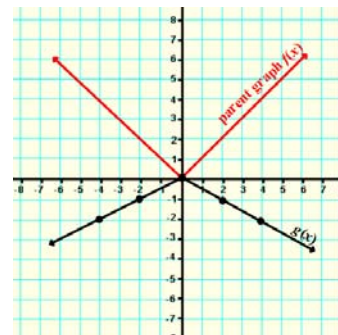
\* Since  $a = -4$ , the slope is 4 to the left of the vertex and  $-4$  to the right of the vertex.



**Ex 10:** Use a table of values to graph the function  $g(x) = -\frac{1}{2}|x|$

vertex

$x$	-4	-2	0	2	4
$g(x)$	-2	-1	0	-1	-2



Notice that the only difference between this function rule and the parent function is that  $a = -\frac{1}{2}$  (the  $a$  value is the opposite of half of the size). The difference between this graph and the parent graph is that it has been reflected across the  $x$ -axis and it has shrunk (wider); the rate of change from each domain value to the next is the opposite of half of the amount.

\* Since  $a = -\frac{1}{2}$ , the slope is  $\frac{1}{2}$  to the left of the vertex and  $-\frac{1}{2}$  to the right of the vertex.

**Exploration Activity:** Discover the effects the variables  $a, h,$  and  $k$  have on the graph. Graph the following functions on the graphing calculator and compare each graph to the graph of  $y = |x|$ .

Note: On the graphing calculator, use the command ABS for absolute value. This is found in the MATH, NUM menu. To enter the first function below, use the keystrokes:



Comparison to  $y = |x|$

- |                          |                                                                               |
|--------------------------|-------------------------------------------------------------------------------|
| 1. $y =  x + 3 $         | Shift left 3; Vertex: $(-3, 0)$                                               |
| 2. $y =  x - 5 $         | Shift right 5; Vertex: $(5, 0)$                                               |
| 3. $y =  x  + 4$         | Shift up 4; Vertex: $(0, 4)$                                                  |
| 4. $y =  x  - 1$         | Shift down 1; Vertex: $(0, -1)$                                               |
| 5. $y = 2 x $            | Narrower, slopes are $-2$ and $2$ on each side of the vertex                  |
| 6. $y = \frac{1}{3} x $  | Wider; slopes are $-\frac{1}{3}$ and $\frac{1}{3}$ on each side of the vertex |
| 7. $y = -4 x $           | Narrower and flipped                                                          |
| 8. $y = -\frac{1}{2} x $ | Wider and flipped                                                             |

**Conclusions:**

Describe the effects of  $a$ ,  $h$ , and  $k$  in the equation  $f(x) = a|x - h| + k$  on the graph of  $f(x) = |x|$  the “Parent Function”.

- $a$ : if,  $a < 0$  then, the graph opens-down V (reflected across  $x$ -axis)  
 if,  $|a| < 1$  then, the graph shrinks, it's wider than  $f(x) = |x|$   
 if,  $|a| > 1$  then, the graph stretches, it's narrower than  $f(x) = |x|$
- $h$ : if,  $h < 0$  then, the graph shifts  $h$  units to the left ( $x$ -coordinate of vertex =  $h$ )  
 if,  $h > 0$  then, the graph shifts  $h$  units to the right ( $x$ -coordinate of vertex =  $h$ )
- $k$ : if,  $k < 0$  then, the graph shifts  $k$  units down ( $y$ -coordinate of vertex =  $k$ )  
 if,  $k > 0$  then, the graph shifts  $k$  units up ( $y$ -coordinate of vertex =  $k$ )

\*\*\*\*Note:  $(h, k)$  is always the vertex of the graph of the function.

**Ex 11:** Without graphing, find the vertex of the absolute value function  $f(x) = 2|x - 5| - 2$ .

$h = 5$ , so the  $x$ -coordinate of the vertex is 5.  $k = -2$ , so the  $y$ -coordinate of the vertex is  $-2$ .

The vertex is  $(5, -2)$ .

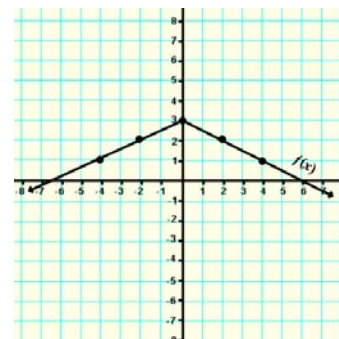
**Ex 12:** Graph the absolute value function  $f(x) = -\frac{1}{2}|x| + 3$

Transformations on  $f(x) = |x|$ : reflect over  $x$ -axis, (opens down)

wider, since  $|a| < 1$

graph shifted up 3

Find the vertex:  $h = 0$  and  $k = 3$ , so the vertex is  $(0, 3)$



\*Since  $a = -\frac{1}{2}$ , the slope is  $\frac{1}{2}$  to the left of the vertex and  $-\frac{1}{2}$  to the right of the vertex.

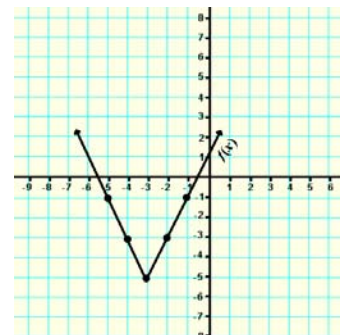
**Ex 13:** Graph the absolute value function  $y = 2|x + 3| - 5$ .

Transformations on  $f(x) = |x|$ : narrower, since  $|a| > 1$

graph shifted left 3

graph shifted down 5

Find the vertex:  $h = -3$  and  $k = -5$ , so the vertex is  $(-3, -5)$



\* Since  $a = 2$ , the slope is  $-2$  to the left of the vertex and  $2$  to the right of the vertex.





### Solving Absolute Value Equations Graphically

When we solve any function for  $x$ , what we are really solving for are the  $x$ -intercepts of the function's graph (roots).

**Ex 14:** Solve the equation  $3|x-2|=9$  graphically and algebraically.

Graphically

Step 1: Set the equation equal to zero and write the related function rule.

$$3|x-2| - 9 = 9 - 9$$

$$3|x-2| - 9 = 0$$

$$f(x) = 3|x-2| - 9$$

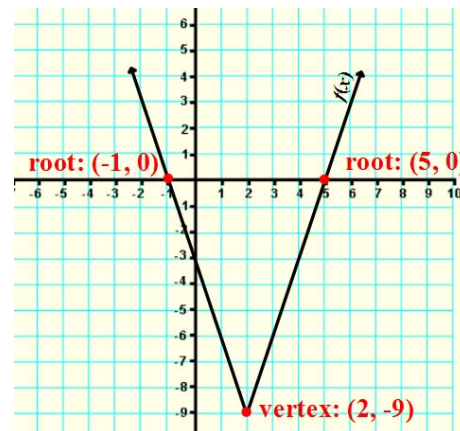
Step 2: Graph the absolute value function.

vertex is at  $(2, -9)$

$a = 3$ , so slope is  $-3$  to the left of the vertex and  $3$  to the right of the vertex

Step 3: Find the  $x$ -intercepts (roots) of the graph.

Solutions:  $x = -1, 5$



Algebraically:

Step 1: Isolate the absolute value expression

$$\frac{3|x-2|}{3} = \frac{9}{3}$$

$$|x-2| = 3$$

Step 2: Solve the absolute value equation.

$$x - 2 = 3 \quad \text{and} \quad x - 2 = -3$$

$$x = 5 \quad \text{and} \quad x = -1$$



## SAMPLE EXAM QUESTIONS

1. Graph the absolute value function  $f(x) = -\frac{2}{3}|x-4|-1$ .

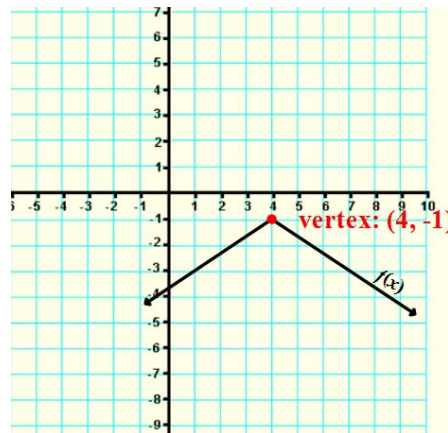
ANS: Step 1: graph the vertex.

$h = 4$  and  $k = -1$ , so the vertex is  $(4, -1)$

Step 2: graph the V-shaped graph

$a = -\frac{2}{3}$ , so the slope is  $\frac{2}{3}$  to the left of the

vertex and  $-\frac{2}{3}$  to the right of the vertex



2. Describe graphically why an absolute value equation can have two, one, or no solutions.

ANS: It depends on whether the graph opens up or down and where the vertex lies with respect to the  $x$ -axis.

\*There are two ways that the function would have two solutions:

If the graph opens up and the vertex lies below the  $x$ -axis, then the graph will cross the  $x$ -axis in two places and the function would have two solutions. Similarly, if the graph opens down and the vertex lies above the  $x$ -axis, the graph will cross the  $x$ -axis in two places and the function would have two solutions.

\*There is one way that the function would have exactly one solution:

If the vertex falls on the  $x$ -axis the function would have one solution.

\*There are two ways that the function would have no solutions:

If the graph opens up and the vertex lies above the  $x$ -axis, then the graph will never cross the  $x$ -axis and the function would have no solutions. Similarly, if the graph opens down and the vertex lies below the  $x$ -axis, the graph will never cross the  $x$ -axis and the function would have no solutions.

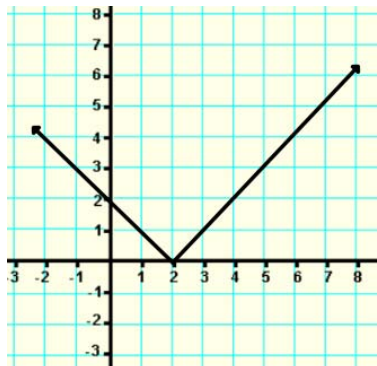
3. Determine if the function  $f(x) = -3|x+2|-5$  opens up or down and find the vertex.
- opens up; vertex is at  $(2, -5)$
  - opens up; vertex is at  $(-2, -5)$
  - opens down; vertex is at  $(2, -5)$
  - opens down; vertex is at  $(-2, -5)$

ANS: D





4. Use the graph below.



What is the equation of the function?

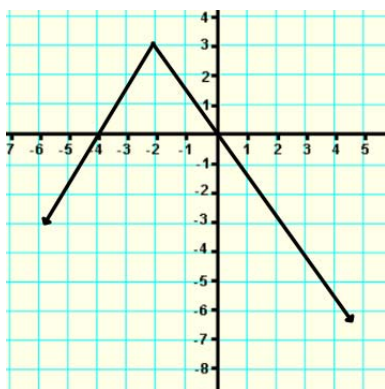
- A.  $y = |x + 2|$
- B.  $y = |x - 2|$
- C.  $y = |x| + 2$
- D.  $y = |x| - 2$

ANS: B

5. If an absolute value function's graph is narrower than that of the parent graph, what do you know about the function?
- A. the graph opens down
  - B.  $h$  is negative
  - C.  $|a| > 1$
  - D.  $|a| < 1$

ANS: C

6. State the domain and the range of the function, given its graph.



ANS: domain: all real numbers  
 range:  $-\infty \leq y \leq 3$   
 The vertex is the maximum point on the graph.



7. Which function below has a graph that has been translated right from the parent function's graph?

A.  $f(x) = -2|x + 4| - 1$

B.  $g(x) = \frac{1}{2}|x| + 3$

C.  $h(x) = -|x| - 6$

D.  $k(x) = |x - 5| + 1$

ANS: D

8. Which function below has a graph that has been narrowed and reflected across the  $x$ -axis when compared to the parent function's graph?

A.  $f(x) = -2|x + 4| - 1$

B.  $g(x) = \frac{1}{2}|x| + 3$

C.  $h(x) = -|x| - 6$

D.  $k(x) = |x - 5| + 1$

ANS: A

9. What is the axis of symmetry of the graph of  $f(x) = -4|x| + 1$ ?

A.  $x = -4$

B.  $x = 0$

C.  $x = 1$

D.  $x = -1$

ANS: B

10. Is the vertex of the graph of  $h(x) = \frac{2}{3}|x + 2| - 4$  the minimum or maximum point on the graph. Justify your answer.

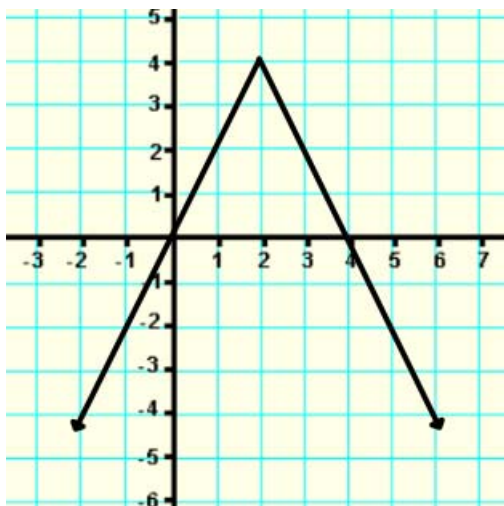
ANS: It's the minimum point of the graph.  
The vertex is at  $(-2, -4)$ , which is below the  $x$ -axis and since  $a$  is positive we know that graph opens up.

11. If you know the vertex of an absolute value function is  $(1, 4)$  and  $a > 0$ , state the range of the graph of the function.

ANS: Range:  $y \geq 4$



12. Use the graph below.



What is the equation of the function?

- A.  $f(x) = -2|x - 2| + 4$
- B.  $f(x) = -2|x + 2| + 4$
- C.  $f(x) = -2|x - 4| + 2$
- D.  $f(x) = -2|x + 4| + 2$

ANS: A

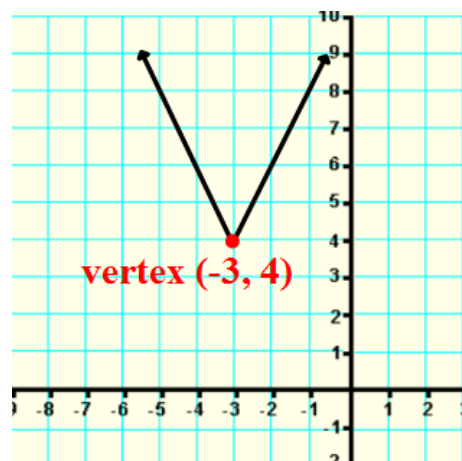
13. Graph the absolute value function  $f(x) = 2|x + 3| + 4$ .

ANS: Step 1: graph the vertex.

$h = -3$  and  $k = 4$ , so the vertex is  $(-3, 4)$

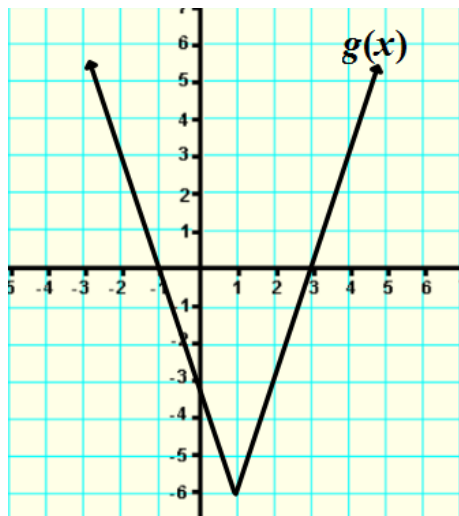
Step 2: graph the V-shaped graph

$a = 2$ , so the slope is  $-2$  to the left of the vertex and  $2$  to the right of the vertex





14. Use the graph below to solve the absolute value function  $g(x) = 3|x - 1| - 6$ .



ANS:  $x = -1, 3$

15. For the absolute value function  $h(x) = -|x| + 1$ , identify the function's domain and range; the vertex of the function's graph; the axis of symmetry of the function's graph; and the function's minimum or maximum value.

ANS: domain: all real numbers  
 range:  $y \leq 1$   
 vertex:  $(0, 1)$   
 axis of symmetry:  $x = 0$   
 maximum value: 1

**Piecewise Function:** a function that is represented by a combination of equations, each corresponding to a part of the domain

**Evaluating a Piecewise Function:**

Ex 15: Find  $f(-3)$  and  $f(2)$  for the function  $f(x) = \begin{cases} x^2 - 1, & x < 0 \\ 4x + 1 & x \geq 0 \end{cases}$ .

Step One: Determine which equation to use based upon the value of  $x$ .

$f(-3)$ :  $x = -3 < 0$ , so we will use the first equation.

$f(2)$ :  $x = 2 \geq 0$ , so we will use the second equation.



Step Two: Substitute the value of  $x$  into the appropriate equation.

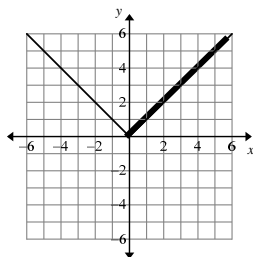
$$f(-3) = (-3)^2 - 1 = 9 - 1 = 8$$

$$f(2) = 4(2) + 1 = 9$$

### Graphing a Piecewise Function:

**Ex 16:** Graph the piecewise function:  $f(x) = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$

Graph each piece. The bolded portion is the first “piece”.

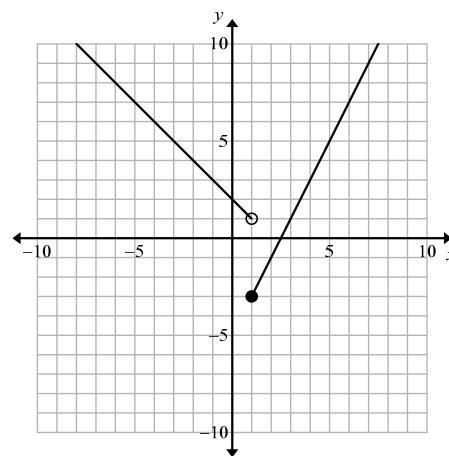


Note: This is the Absolute Value Function!

**Ex 17:** Graph the piecewise function  $f(x) = \begin{cases} -x + 2, & x < 1 \\ 2x - 5, & x \geq 1 \end{cases}$

Step One: Graph the function  $y = -x + 2$  for all values of  $x$  less than 1. There should be an open circle on the point  $(1, 1)$ .

Step Two: Graph the function  $y = 2x - 5$  for all values of  $x$  greater than 1 in the same coordinate plane. There should be a closed circle on the point  $(1, -3)$ .



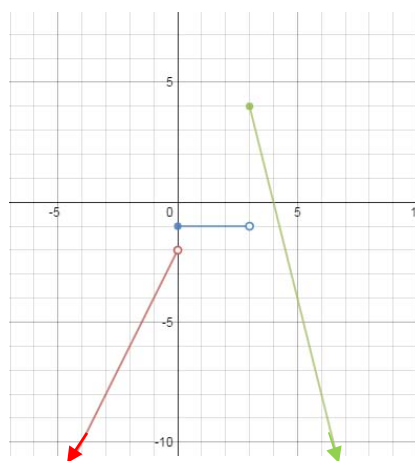


**Ex 18:** Graph the piecewise function  $f(x) = \begin{cases} 2x - 2, & x < 0 \\ -1, & 0 \leq x < 3 \\ -4x + 16, & x \geq 3 \end{cases}$

Step One: Graph the function  $y = 2x - 2$  for all values of  $x$  less than 0. There should be an open circle on the point  $(-2, 0)$ . (Function in red in the graph below)

Step Two: Graph the function  $y = -1$ , which is a horizontal line with a closed circle on the point  $(0, -1)$  and an open circle on the point  $(3, -1)$ . (Function in blue in the graph below)

Step Three: Graph the function  $y = -4x + 16$  for all values greater than and equal to 3. Plug 3 into the equation to find the starting point  $(3, 4)$  and then use the slope to find another point. (Function in green in the graph below)



### Writing the Equation of a Piecewise Function:

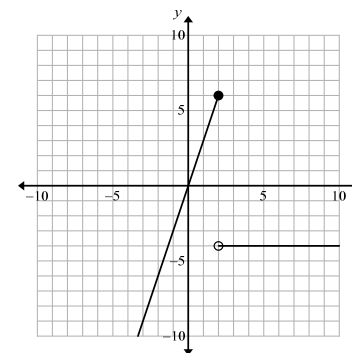
**Ex 19:** Write the equation of the piecewise function shown in the graph.

Step One: Find the equation for each portion

The first linear function has a  $y$ -intercept of 0 and has a slope of 3, so the linear function is  $y = 3x$ .

The second linear function is a horizontal line with a  $y$ -intercept of  $-4$ , so the linear function is  $y = -4$ .

Step Two: Write the function in set notation. Make sure to note the open and closed circles and the value of  $x$  where the models change.

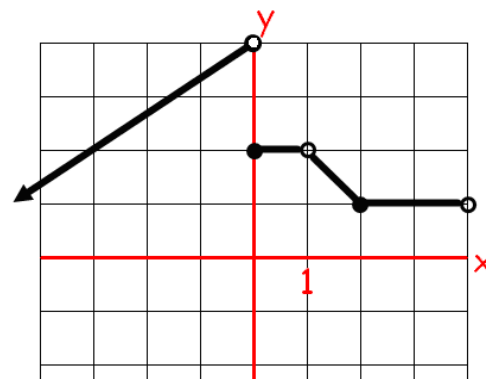


$$f(x) = \begin{cases} 3x, & x \leq 2 \\ -4, & x > 2 \end{cases}$$

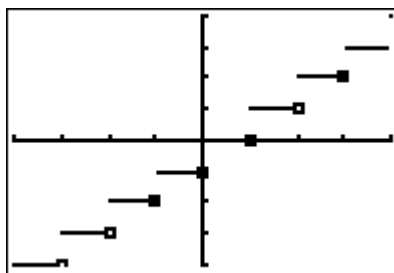


**Ex 20:** Write functions for the piecewise graph at the right.

Ans: 
$$f(x) = \begin{cases} \frac{2}{3}x + 4 & x < 0 \\ 2 & 0 \leq x < 1 \\ -x + 3 & 1 < x \leq 2 \\ 1 & 2 \leq x < 4 \end{cases}$$



The Greatest Integer Function:  $g(x) = \llbracket x \rrbracket$  For every real number  $x$ ,  $g(x)$  is the greatest integer less than or equal to  $x$ . The greatest integer function is an example of a **step function**. Note: All points seen on the graph to the right are solid points. The left endpoints of each segment are open circles.



**Modeling:** Piecewise functions can be used in real-world applications and model real-life situations.

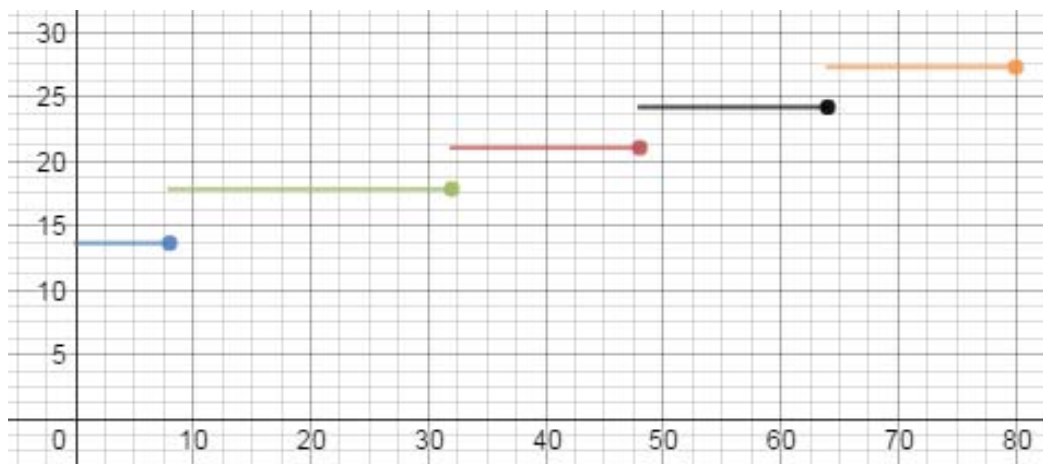
**Ex 21:**

In 2005, the cost  $C$  (in dollars) to send U.S. Postal Service Express Mail up to 5 pounds depended on the weight  $w$  (in ounces) according to the following function:

$$C(w) = \begin{cases} 13.65, & \text{if } 0 < w \leq 8 \\ 17.85, & \text{if } 8 < w \leq 32 \\ 21.05, & \text{if } 32 < w \leq 48 \\ 24.20, & \text{if } 48 < w \leq 64 \\ 27.30, & \text{if } 64 < w \leq 80 \end{cases}$$



Graph the function.



**Ex 22:**

A parent drives from home to the grocery store at 0.9 mile per minute for 4 minutes, stops to buy snacks for the team for 2 minutes, and then drives to the soccer field at a speed of 0.7 mile per minute for 3 minutes. Write a piecewise function for the parent's distance from home to the soccer field during this time.

Let  $d(t)$  represent the distance traveled in time,  $t$ . Use the formula distance equals rate times time.

$$\text{Ans: } d(t) = \begin{cases} 0.9t & 0 \leq t \leq 4 \\ 3.6 & 4 < t \leq 6 \\ 3.6 + 0.7(t - 6) & 6 < t \leq 9 \end{cases}$$

← Parent travels at 0.9 mi/min for 4 minutes.  
 ← Distance traveled is constant for 2 minutes.  
 ← Add the distance traveled at 0.7 mi/min to the distance already traveled.

**QOD:** In a piecewise function, why must one part of the graph have an open circle as an endpoint, and the other have a closed circle as an endpoint?

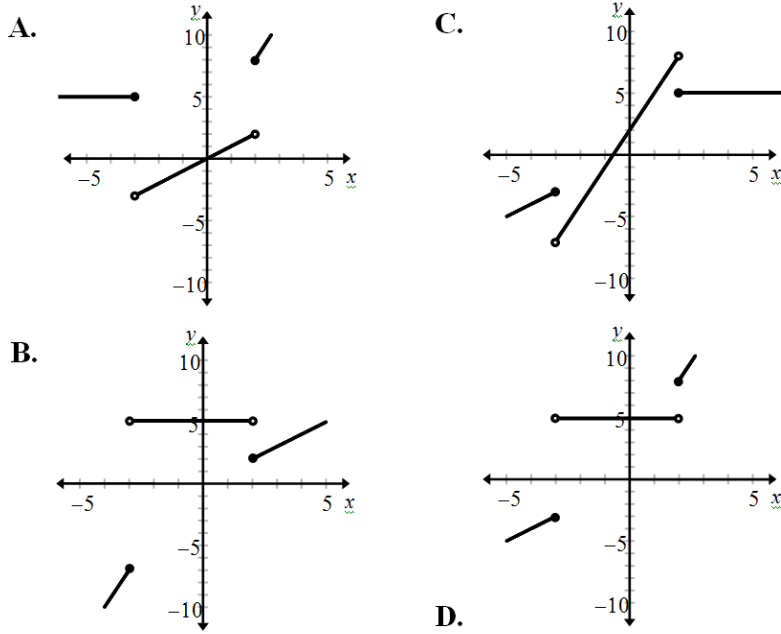




SAMPLE EXAM QUESTIONS

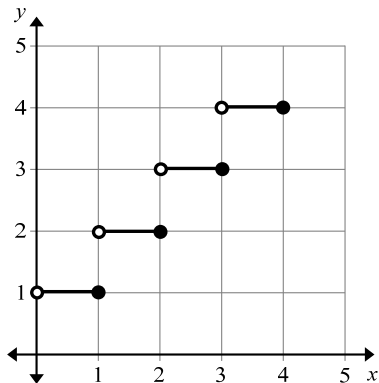
1. Which graph represents the piecewise function below?

$$f(x) = \begin{cases} x, & \text{if } x \geq 2 \\ 5, & \text{if } -3 < x < 2 \\ 3x + 2, & \text{if } x \leq -3 \end{cases}$$



ANS: B

2. Which function represents the graph below?





$$\text{A. } f(x) = \begin{cases} x+1, & \text{if } 0 < x < 4 \\ x, & \text{if } 0 \leq x \leq 4 \end{cases}$$

$$\text{B. } f(x) = \begin{cases} x, & \text{if } x \in \{1, 2, 3, 4\} \\ x+1, & \text{if } x \in \{0, 1, 2, 3\} \end{cases}$$

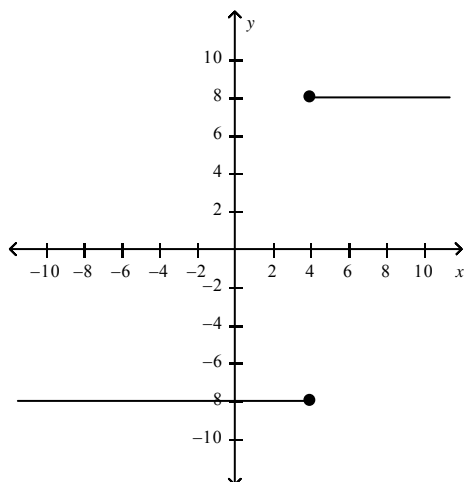
$$\text{C. } f(x) = \begin{cases} 1, & \text{if } 0 \leq x < 1 \\ 2, & \text{if } 1 \leq x < 2 \\ 3, & \text{if } 2 \leq x < 3 \\ 4, & \text{if } 3 \leq x < 4 \end{cases}$$

$$\text{D. } f(x) = \begin{cases} 1, & \text{if } 0 < x \leq 1 \\ 2, & \text{if } 1 < x \leq 2 \\ 3, & \text{if } 2 < x \leq 3 \\ 4, & \text{if } 3 < x \leq 4 \end{cases}$$

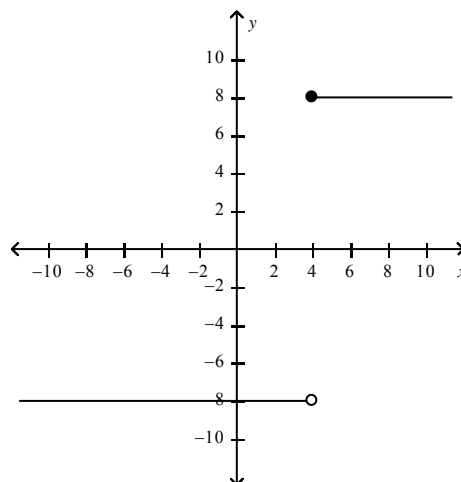
ANS: D

$$3. \quad \text{Graph the function: } y = \begin{cases} -8, & \text{if } x < 4 \\ 8, & \text{if } x \geq 4 \end{cases}$$

a.

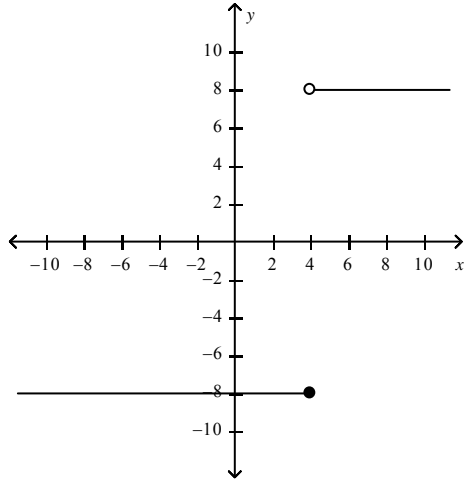


c.

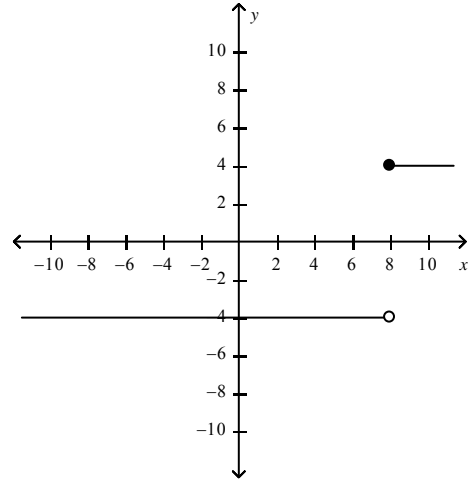




b.



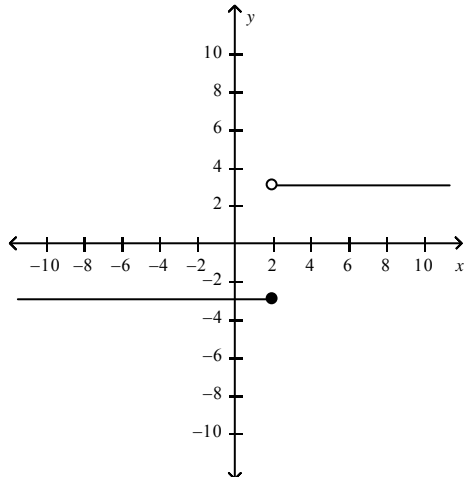
d.



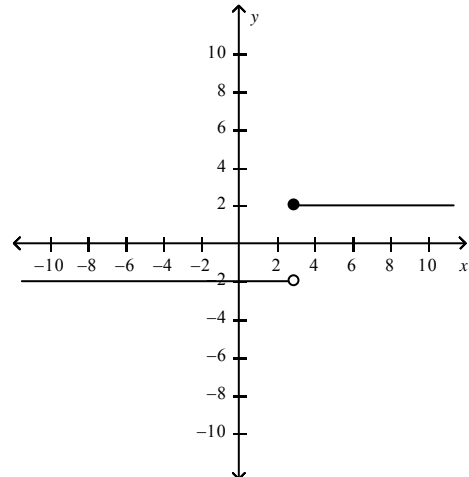
ANS: C

4. Graph  $g(x) = \begin{cases} -3 & \text{if } x < 2 \\ 3 & \text{if } x \geq 2 \end{cases}$ .

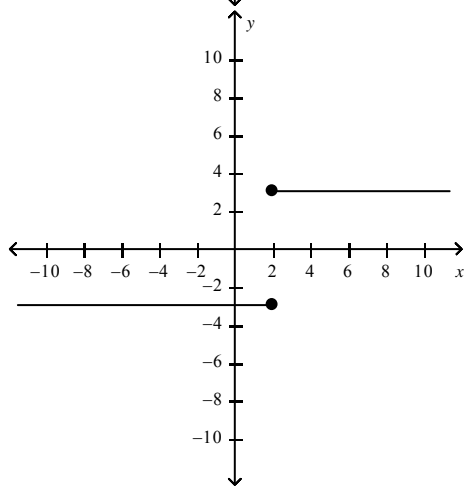
a.



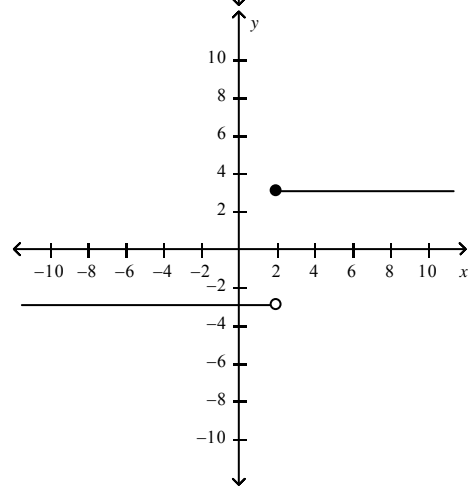
c.



b.



d.

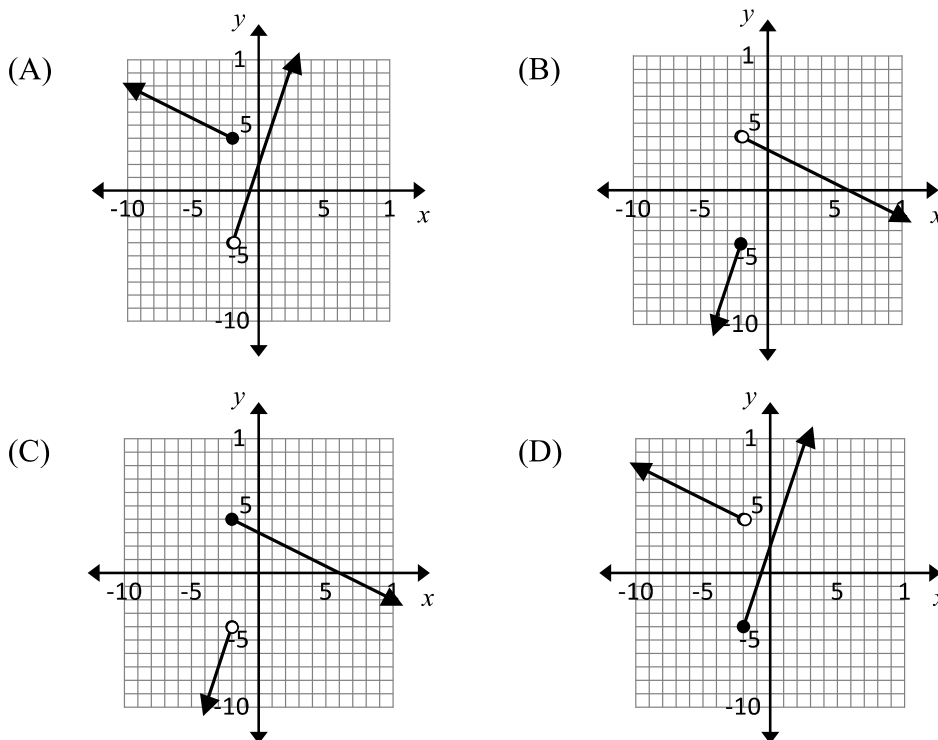


ANS: D



5. Which graph represents the piecewise function?

$$f(x) = \begin{cases} 3x + 2, & x < -2 \\ -\frac{1}{2}x + 3, & x \geq -2 \end{cases}$$



ANS: C

6. An online retailer charges shipping based on the following table.

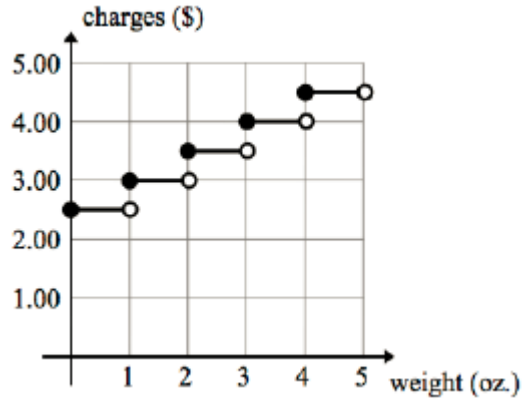
Weight of Order	Shipping Charge
0 lb. – less than 1 lb.	\$2.50
1 lb. – less than 2 lb.	\$3.00
2 lb. – less than 3 lb.	\$3.50
3 lb. – less than 4 lb.	\$4.00
etc.	etc.

(a) Write an equation that describes shipping as a function of weight.

$$\text{charges} = \$2.50 + \frac{\$0.50}{\text{pound}} \llbracket \text{weight} \rrbracket \quad f(x) = \begin{cases} 2.50 & \text{if } 0 \leq x < 1 \\ 3.00 & \text{if } 1 \leq x < 2 \\ 3.50 & \text{if } 2 \leq x < 3 \\ 4.00 & \text{if } 3 \leq x < 4 \\ 4.50 & \text{if } 4 \leq x < 5 \end{cases}$$



(b) Sketch the function.

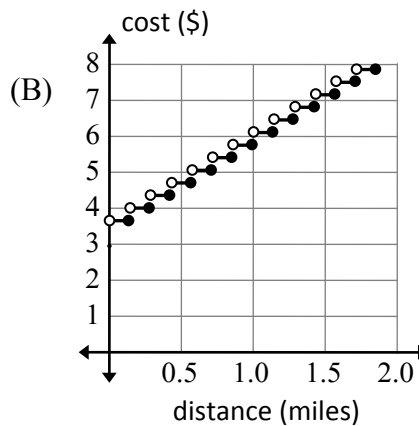
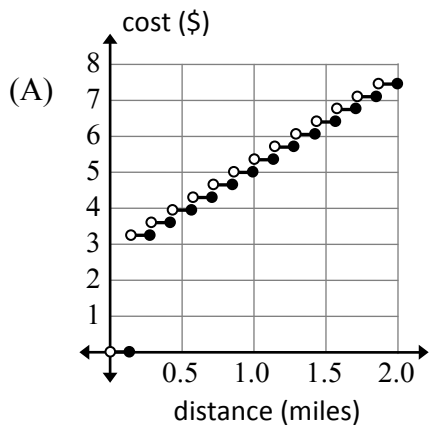


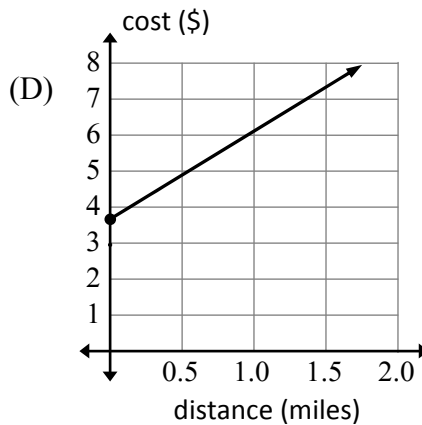
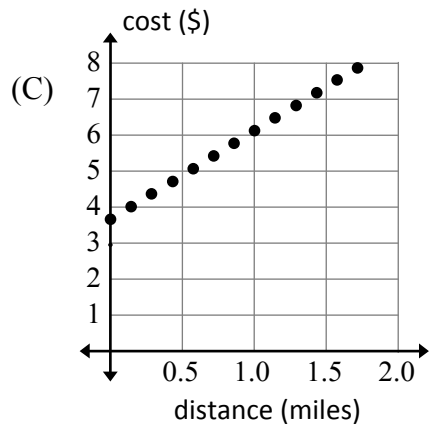
7. A piecewise function is defined as  $f(x) = \begin{cases} x-2, & \text{for } x \geq 0 \\ -x-2, & \text{for } x < 0 \end{cases}$ . Which is another way of defining this function?

- (A)  $f(x) = |x-2|$
- (B)  $f(x) = |x|-2$
- (C)  $f(x) = -(x-2)$
- (D)  $f(x) = -x+2$

ANS: B

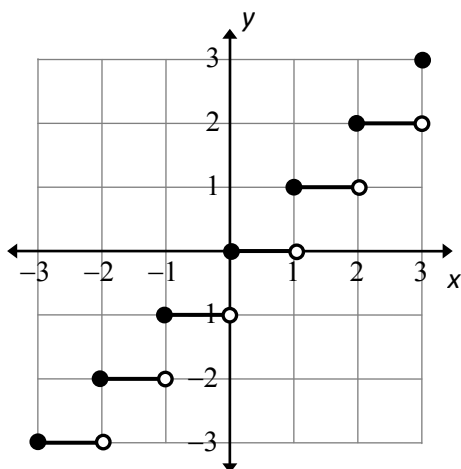
8. Taxi fare in Las Vegas is \$3.30 plus \$0.35 for every  $\frac{1}{7}$  of a mile or fraction thereof. Which graph shows the cost of a Las Vegas taxi ride of  $x$  miles?





ANS: B

9. Use the graph.



What is the equation of the function?

- (A)  $y = x$
- (B)  $y = |x|$
- (C)  $y = \lceil x \rceil$
- (D)  $y = x - 1$

ANS: C