Unit 11 - Solving Quadratic Functions – PART TWO

PREREQUISITE SKILLS:

- students should be able to add, subtract and multiply polynomials
- students should be able to factor polynomials
- students should be able to solve linear systems of equations
- students should be able to graph quadratic functions

VOCABULARY:

1. real number: any rational or irrational value
2. imaginary number: a number that when squared gives a negative result
3. quadratic equation: an equation where the highest exponent of the variable is a square
4. zeros of a function: also called a root of the function, it is the x value(s) that produce a function value of zero
5. zero product property: It states that if the product of two factors is zero, then one of the factors must be equal to zero
6. completing the square: a technique used to solve quadratic equations when factoring does not work
7. quadratic formula: a method used to find the roots of a quadratic equation by substituting in the coefficients of the quadratic equation
8. discriminant: the expression located under the radical in the quadratic formula that gives us information about the number of real roots of the quadratic equation
9. system of equations: a collection of two or more equations with a same set of unknowns

SKILLS:

- Solve quadratic equations graphically and algebraically
- Make decisions about which method would be best to use to solve a quadratic equation
- Solve a simple system with one linear and one quadratic equation
- Model real world situations with quadratic equations

STANDARDS:

A.REI.B.4b Solve quadratic equations by inspection (e.g., for $x^2 = 49$), taking square roots, completing the square, the quadratic formula and factoring, as appropriate.

8.NS.A.1 Know that numbers that are not rational are called irrational. Understand informally that every number has a decimal expansion; for rational numbers show that the decimal expansion repeats eventually, and convert a decimal expansion which repeats eventually into a rational number.

N.RN.B.3 Explain why the sum or product of two rational numbers is rational; that the sum of a rational number and an irrational number is irrational; and that the product of a nonzero rational number and an irrational number is irrational.

F.IF.C.7c Graph polynomial functions, identifying zeros when suitable factorizations are available, and showing end behavior. *(Modeling Standard)

A.SSE.B.3a Factor a quadratic expression to reveal the zeros of the function it defines. *(Modeling Standard)

A.SSE.A.2-1 Use the structure of a linear, exponential, or quadratic expression to identify ways to rewrite it. For example, see x^4 - y^4 as (x^2)^2 - (y^2)^2, thus recognizing it as a difference of squares that can be factored as (x^2 - y^2)(x^2 + y^2).

A.REI.B.4a Use the method of completing the square to transform any quadratic equation in x into an equation of the form (x - p)^2 = q that has the same solutions. Derive the quadratic formula from this form.

F.IF.C.8a Use the process of factoring and completing the square in a quadratic function to show zeros, extreme values, and symmetry of the graph, and interpret these in terms of a context.

A.SSE.B.3b Complete the square in a quadratic expression to reveal the maximum or minimum value of the function it defines. *(Modeling Standard)

N.Q.A.1 Use units as a way to understand problems and to guide the solution of multi-step problems; choose and interpret units consistently in formulas; choose and interpret the scale and the origin in graphs and data displays.

A.CED.A.2-1 Create linear, exponential, and quadratic equations in two or more variables to represent relationships between quantities and graph equations on coordinate axes with labels and scales. Limit exponentials to have integer inputs only. *(Modeling Standard)

F.IF.B.4-1 For a linear, exponential, or quadratic function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; and end behavior. *(Modeling Standard)

F.IF.B.5-1 Relate the domain of a linear, exponential, or quadratic function to its graph and, where applicable, to the quantitative relationship it describes. For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function. *(Modeling Standard)

A.REI.C.7 Solve a simple system consisting of a linear equation and a quadratic equation in two variables algebraically and graphically. For example, find the points of intersection between the line $y = -3x$ and the circle $x^2 + y^2 = 3$.

F.IF.C.7a Graph linear and quadratic functions and show intercepts, maxima, and minima. *(Modeling Standard)

LEARNING TARGETS:

- 11.1 To review number sets and introduce the existence of imaginary numbers.
- 11.2 To use the graph of a quadratic function to solve its related quadratic equation.
- 11.3 To solve quadratic equations by taking the square root.
- 11.4 To solve quadratic equations by factoring.
- 11.5 To solve quadratic equations by completing the square.
- 11.6 To solve quadratic equations by using the quadratic formula.
- 11.7 To choose the best algebraic method to solve a quadratic equation.
- 11.8 To model real world situations with quadratic equations.
- 11.9 To solve a simple system involving a linear and quadratic equation, both algebraically and graphically.

ESSENTIAL QUESTIONS:

- What makes an equation a quadratic equation?
- How can algebra be used to solve quadratic equations?
- How many solutions can quadratic equations have? Are all solutions always real numbers?
- How can I decide which method to use to solve quadratic equations?

BIG IDEAS:

Solving quadratics can be done by different techniques (graphing, by inspection, factoring, completing the square, quadratic formula), each more efficient than the others based on the characteristics of the quadratic function. Some quadratics do not have real solutions. The discriminant gives us good information about the nature of the roots of a quadratic equation. Completing the square is a good method to use when a quadratic equation is difficult to factor or cannot be factored. It can also be used to derive the quadratic formula. The solutions to some real-world problems can be found my modeling them with quadratic equations and graphs. Systems of equations may be solved using graphs, tables, eliminations or substitution. The solution to a linear and quadratic system, if it exists, is an ordered pair or two ordered pairs.

Notes, Examples and Exam Questions

Unit 11.5 To solve quadratic equations by completing the square.

Review: Factoring a Perfect Square Trinomial
$$a^2+2ab+b^2=(a+b)^2$$
$$a^2-2ab+b^2=(a-b)^2$$

Completing the Square: Writing an expression of the form x^2+bx as a perfect square trinomial in order to factor it as a binomial squared

Notice the pattern in the following perfect square trinomials:

$$x^2+6x+9=(x+3)^2 \qquad x^2-4x+4=(x-2)^2$$
$$x^2-8x+16=(x-4)^2 \qquad x^2+2x+1=(x+1)^2$$

Note: $9=\left(\frac{1}{2}\cdot 6\right)^2$ $\qquad 4=\left(\frac{1}{2}\cdot(-4)\right)^2$

$16=\left(\frac{1}{2}\cdot(-8)\right)^2$ $\qquad 1=\left(\frac{1}{2}\cdot 2\right)^2$

To complete the square of x^2+bx, we must add $\left(\frac{b}{2}\right)^2$.

Teacher Note: Algebra Tiles work well to illustrate completing the square.

Ex 1: Find the value of *c* such that $x^2-10x+c$ is a perfect square trinomial.

$b=-10$, therefore we must add $\left(\frac{-10}{2}\right)^2=(-5)^2 \rightarrow \boxed{c=25}$ to complete the square.

Note: $x^2-10x+25=(x-5)^2$

Solving a Quadratic Equation by Completing the Square:
This method can be used when the quadratic equation cannot be factored or is difficult to factor. By completing the square, we make the quadratic factorable and then can use the property of square roots to solve.

Ex 2: Solve $x^2+8x+7=0$ by completing the square.

Step One: Rewrite to make the lead coefficient 1. $\qquad x^2+8x+7=0$

Step Two: Take the constant term to the other side. $\qquad x^2+8x=-7$

Step Three: Complete the square (add $\left(\frac{b}{2}\right)^2$ to both sides).

$$x^2+8x+\boxed{\left(\frac{8}{2}\right)^2}=-7+\boxed{\left(\frac{8}{2}\right)^2}$$

$$x^2+8x+16=9$$

Step Four: Factor the perfect square trinomial.

$$(x+4)^2=9$$

Step Five: Take the square roots of both sides.

$$\sqrt{(x+4)^2}=\sqrt{9}$$

$$x+4=\pm 3$$

Step Six: Solve for the variable.

$$x+4=3 \quad x+4=-3$$

$$x=-1 \quad x=-7$$

The solution set is $\boxed{\{-7,-1\}}$. Check your answer by factoring.

Ex 3: Solve $2x^2+12x-4=0$ by completing the square.

Step One: Rewrite to make the lead coefficient 1.

$$\frac{2x^2}{2}+\frac{12x}{2}-\frac{4}{2}=\frac{0}{2}$$

$$x^2+6x-2=0$$

Step Two: Take the constant term to the other side.

$$x^2+6x=2$$

Step Three: Complete the square (add $\left(\frac{b}{2}\right)^2$ to both sides).

$$x^2+6x+\boxed{\left(\frac{6}{2}\right)^2}=2+\boxed{\left(\frac{6}{2}\right)^2}$$

$$x^2+6x+9=11$$

Step Four: Factor the perfect square trinomial.

$$(x+3)^2=11$$

Step Five: Take the square roots of both sides.

$$\sqrt{(x+3)^2}=\sqrt{11}$$

$$x+3=\pm\sqrt{11}$$

Step Six: Solve for the variable.

$$x+3=\sqrt{11} \quad x+3=-\sqrt{11}$$

$$x=-3+\sqrt{11} \quad x=-3-\sqrt{11}$$

The solution set is $\boxed{\{-3-\sqrt{11},-3+\sqrt{11}\}}$

Ex 4: Solve $x^2 - x - 3 = 0$ by completing the square.

Step One: Rewrite to make the lead coefficient 1. $x^2 - x - 3 = 0$

Step Two: Take the constant term to the other side. $x^2 - x = 3$

Step Three: Complete the square (add $\left(\frac{b}{2}\right)^2$ to both sides).

$$x^2 + -x + \boxed{\left(\frac{-1}{2}\right)^2} = 3 + \boxed{\left(\frac{-1}{2}\right)^2}$$

$$x^2 - x + \frac{1}{4} = \frac{13}{4}$$

Step Four: Factor the perfect square trinomial.

$$\left(x - \frac{1}{2}\right)^2 = \frac{13}{4}$$

Step Five: Take the square roots of both sides.

$$\sqrt{\left(x - \frac{1}{2}\right)^2} = \sqrt{\frac{13}{4}}$$

$$x - \frac{1}{2} = \pm\frac{\sqrt{13}}{2}$$

Step Six: Solve for the variable.

$$x - \frac{1}{2} = \frac{\sqrt{13}}{2} \qquad x - \frac{1}{2} = -\frac{\sqrt{13}}{2}$$

$$x = \frac{1+\sqrt{13}}{2} \qquad x = \frac{1-\sqrt{13}}{2}$$

The solutions are $\boxed{x = \frac{1 \pm \sqrt{13}}{2}}$.

<u>You Try</u>: Solve by completing the square. $4x^2 + 20x - 11 = 0$

<u>QOD</u>: Describe why adding $\left(\frac{b}{2}\right)^2$ to $x^2 + bx$ makes it a perfect square trinomial.

SAMPLE EXAM QUESTIONS

1. Given $4x^2 + 28x + c = (2x + q)^2$, where *c* and *q* are integers, what is the value of *c*?

A. 2

B. 7

C. 14

D. 49

ANS: D

2. **The quadratic equation $2x^2 - 16x - 15 = 0$ is rewritten as $(x-p)^2 = q$. What is the value of *q*?**

A. $\frac{47}{2}$

B. $\frac{15}{2}$

C. $\frac{143}{2}$

ANS: A

3. **What number should be added to both sides of the equation to complete the square in $x^2 + 8x = 17$?**

A. 4

B. 16

C. 29

D. 49

ANS: B

4. Find all solutions to the equation $x^2 - 10x + 25 = 81$. Show your work.

$x^2 - 10x + 25 = 81$

$(x-5)^2 = 81 \quad \rightarrow \quad \sqrt{(x-5)^2} = \sqrt{81}$

$x - 5 = \pm 9 \quad \rightarrow \quad x = 5 + 9 \text{ or } x = 5 - 9 \quad \rightarrow \quad \boxed{x = -4, 14}$

5. **What are the roots (solutions) of $x^2 + x - 1 = 0$?**

A. $\left\{-1-\sqrt{-3}, -1+\sqrt{-3}\right\}$

B. $\left\{-1-\sqrt{5}, -1+\sqrt{5}\right\}$

C. $\left\{\frac{-1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\right\}$

D. $\left\{\frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}\right\}$

ANS: D

Unit 11.6 To solve quadratic equations using the quadratic formula.

Deriving the Quadratic Formula by Completing the Square

Solve the quadratic equation $ax^2+bx+c=0$ by completing the square.

Step One: Rewrite so that the lead coefficient is 1.

$$\frac{ax^2}{a}+\frac{bx}{a}+\frac{c}{a}=\frac{0}{a}$$
$$x^2+\frac{b}{a}x+\frac{c}{a}=0$$

Step Two: Take the constant term to the other side.

$$x^2+\frac{b}{a}x=-\frac{c}{a}$$

Step Three: Complete the square (add $\left(\frac{b}{2}\right)^2$ to both sides).

$$x^2+\frac{b}{a}x+\boxed{\left(\frac{b}{2a}\right)^2}=-\frac{c}{a}+\boxed{\left(\frac{b}{2a}\right)^2}$$
$$x^2+\frac{b}{a}x+\frac{b^2}{4a^2}=\frac{-4ac+b^2}{4a^2}$$

Step Four: Factor the perfect square trinomial.

$$\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$$

Step Five: Take the square roots of both sides.

$$\sqrt{\left(x+\frac{b}{2a}\right)^2}=\sqrt{\frac{b^2-4ac}{4a^2}}$$
$$x+\frac{b}{2a}=\pm\sqrt{\frac{b^2-4ac}{4a^2}}$$

Step Six: Solve for the variable.

$$x+\frac{b}{2a}=\frac{\sqrt{b^2-4ac}}{2a} \qquad x+\frac{b}{2a}=\frac{-\sqrt{b^2-4ac}}{2a}$$
$$x=\frac{-b+\sqrt{b^2-4ac}}{2a} \qquad x=\frac{-b-\sqrt{b^2-4ac}}{2a}$$

The Quadratic Formula: To solve a quadratic equation in the form $ax^2+bx+c=0$,

use the formula $\boxed{x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}}$.

Note: To help memorize the quadratic formula, sing it to the tune of the song “Pop Goes the Weasel”.

Ex 5: Solve the quadratic equation $x^2-8x=-1$ using the quadratic formula.

Step One: Rewrite in standard form (if necessary). $x^2-8x+1=0$

Step Two: Identify *a*, *b*, and *c*. $a=1, b=-8, c=1$

Step Three: Substitute the values into the quadratic formula.

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$x=\frac{-(-8)\pm\sqrt{(-8)^2-4(1)(1)}}{2(1)}$$

Step Four: Simplify.

$$x=\frac{8\pm\sqrt{64-4}}{2}=\frac{8\pm\sqrt{60}}{2}=\frac{8\pm2\sqrt{15}}{2}=4\pm\sqrt{15}$$

The solution set is $\boxed{\left\{4-\sqrt{15},4+\sqrt{15}\right\}}$

Ex 6: Solve the quadratic equation $5x-1=-6x^2$ using the quadratic formula.

Step One: Rewrite in standard form (if necessary). $6x^2+5x-1=0$

Step Two: Identify *a*, *b*, and *c*. $a=6, b=5, c=-1$

Step Three: Substitute the values into the quadratic formula.

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$x=\frac{-(5)\pm\sqrt{(5)^2-4(6)(-1)}}{2(6)}$$

Step Four: Simplify.

$$x=\frac{-5\pm\sqrt{25+24}}{12}=\frac{-5\pm\sqrt{49}}{12}=\frac{-5\pm7}{12}$$

$$x=\frac{-5-7}{12}=\frac{-12}{12}=-1 \qquad x=\frac{-5+7}{12}=\frac{2}{12}=\frac{1}{6}$$

The solution set is $\boxed{\left\{-1,\frac{1}{6}\right\}}$. Check the answer by factoring.

Ex 7: Solve the quadratic equation $12x+4+9x^2=0$ using the quadratic formula.

Step One: Rewrite in standard form (if necessary). $9x^2+12x+4=0$

Step Two: Identify *a*, *b*, and *c*. $a=9, b=12, c=4$

Step Three: Substitute the values into the quadratic formula.

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$x=\frac{-(12)\pm\sqrt{(12)^2-4(9)(4)}}{2(9)}$$

Step Four: Simplify. $x=\dfrac{-12\pm\sqrt{144-144}}{18}=\dfrac{-12\pm\sqrt{0}}{18}=\dfrac{-12}{18}=-\dfrac{2}{3}$

The solution set is $\boxed{\left\{-\dfrac{2}{3}\right\}}$. Check the answer by factoring.

Ex 8: Solve the quadratic equation $2x^2-2x+3=0$ using the quadratic formula.

Step One: Rewrite in standard form (if necessary). $2x^2-2x+3=0$

Step Two: Identify *a*, *b*, and *c*. $a=2, b=-2, c=3$

Step Three: Substitute the values into the quadratic formula.

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$x=\frac{-(-2)\pm\sqrt{(-2)^2-4(2)(3)}}{2(2)}$$

Step Four: Simplify. $x=\dfrac{2\pm\sqrt{4-24}}{4}=\dfrac{2\pm\sqrt{-20}}{4}=\dfrac{2\pm\sqrt{-1}\cdot\sqrt{4}\cdot\sqrt{5}}{4}=\dfrac{2\pm2i\sqrt{5}}{4}$

There is **no real solution** to the quadratic equation only imaginary number solutions.

You Try: Solve the equation $6x-3=2x^2$ using the quadratic formula.

QOD: Write a conjecture about how the radicand in the quadratic formula relates to the number of solutions that a quadratic equation has.

SAMPLE EXAM QUESTIONS

1. **What are the roots (solutions) of $x^2+x-1=0$?**

A. $\left\{-1-\sqrt{-3},-1+\sqrt{-3}\right\}$

B. $\left\{-1-\sqrt{5},-1+\sqrt{5}\right\}$

C. $\left\{\dfrac{-1-\sqrt{5}}{2},\dfrac{-1+\sqrt{5}}{2}\right\}$

D. $\left\{\dfrac{1-\sqrt{5}}{2},\dfrac{1+\sqrt{5}}{2}\right\}$

ANS: C

2. Which of the following is the correct use of the quadratic formula to find the solutions of the equation $2x^2 - 7x = 5$?

A. $\left\{ \frac{7-\sqrt{(-7)^2-4(2)(5)}}{2(2)}, \frac{7+\sqrt{(-7)^2-4(2)(5)}}{2(2)} \right\}$

B. $\left\{ \frac{7-\sqrt{(-7)^2-4(2)(-5)}}{2(2)}, \frac{7+\sqrt{(-7)^2-4(2)(-5)}}{2(2)} \right\}$

C. $\left\{ \frac{-7-\sqrt{(-7)^2-4(2)(5)}}{2(2)}, \frac{-7+\sqrt{(-7)^2-4(2)(5)}}{2(2)} \right\}$

D. $\left\{ \frac{-7-\sqrt{(-7)^2-4(2)(-5)}}{2(2)}, \frac{-7+\sqrt{(-7)^2-4(2)(-5)}}{2(2)} \right\}$

ANS: B

3. Which shows the correct use of the quadratic formula to find the solutions of $8x^2 + 2x = 1$?

A. $x = \frac{2 \pm \sqrt{(2)^2 - 4(8)(1)}}{2(8)}$

B. $x = \frac{2 \pm \sqrt{(2)^2 - 4(8)(-1)}}{2(8)}$

C. $x = \frac{-2 \pm \sqrt{(2)^2 - 4(8)(1)}}{2(8)}$

D. $x = \frac{-2 \pm \sqrt{(2)^2 - 4(8)(-1)}}{2(8)}$

ANS: D

4. What are the solutions of $3x^2 - 6x = -2$**?**

A. $x = \frac{1 \pm \sqrt{3}}{3}$

B. $x = \frac{-1 \pm \sqrt{3}}{3}$

C. $x = 1 \pm \frac{\sqrt{3}}{3}$

D. $x = -1 \pm \frac{\sqrt{3}}{3}$

ANS: C

<u>Determining the Number of Real Solutions of a Quadratic Equation Using the Discriminant</u>

Teacher Note: Students should have discovered this on their own in the QOD in the previous section.

<u>Discriminant:</u> the *discriminant* of the quadratic equation $ax^2 + bx + c = 0$ is $b^2 - 4ac$

Note: The discriminant is the radicand of the quadratic formula!

If $b^2 - 4ac < 0$, then there are *no real solutions*.

If $b^2 - 4ac = 0$, then there is *one solution*.

If $b^2 - 4ac > 0$, then there are *two real solutions*.

Ex 9: Determine the number of real solutions that the equations have.

1. $3x^2 - 2x = -1$

Rewrite the equation in standard form. $3x^2 - 2x + 1 = 0$ $a = 3, b = -1, c = 1$

Find the discriminant. $b^2 - 4ac = (-1)^2 - 4(3)(1) = 1 - 12 = -11$

Determine the number of real solution(s). $b^2 - 4ac = -11 < 0$, so there are no real solutions.

2. $4-5x=x^2$

Rewrite the equation in standard form. $x^2+5x-4=0$ $a=1,\ b=5,\ c=-4$

Find the discriminant. $b^2-4ac=(5)^2-4(1)(-4)=25+16=41$

Determine the number of real solution(s). $b^2-4ac=41>0$, so there are two real solutions.

3. $9x^2-12x+4=0$

Rewrite the equation in standard form. $9x^2-12x+4=0$ $a=9,\ b=-12,\ c=4$

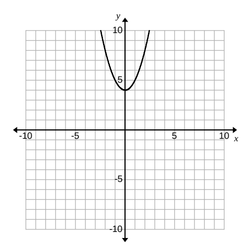
Find the discriminant. $b^2-4ac=(-12)^2-4(9)(4)=144-144=0$

Determine the number of real solution(s). $b^2-4ac=0$, so there is one real solution.

Determining the Number of *x*-Intercepts of a Quadratic Function Using the Discriminant

Because the *x*-intercepts of $y=ax^2+bx+c$ are the same as the zeros of the equation $ax^2+bx+c=0$, we can use the discriminant to determine the number of *x*-intercepts that a quadratic function has.

Ex 10: Sketch the graph of a quadratic function with a negative discriminant.



Because the discriminant, $b^2-4ac<0$, the function will have no *x*-intercept.

A sample answer is shown in the graph.

Note: Any parabola which does not intersect the x-axis is an acceptable answer.

Application Problem:

Ex 11: A baton twirler tosses a baton into the air. The baton leaves the twirler's hand 6 feet above the ground and has an initial vertical velocity of 45 feet per second. This can be modeled by the equation $h=-16t^2+45t+6$, where *h* is the height (in feet) and *t* is the time (in seconds). The twirler wants her baton to reach at least 40 feet. Will the baton reach that height?

Substitute $h=12$. $40=-16t^2+45t+6$

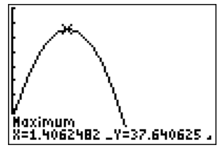
Write in standard form. $0=-16t^2+45t-34$ $a=-16,\ b=45,\ c=-34$

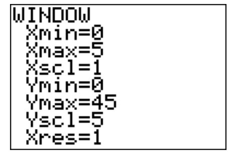
Find the discriminant. $b^2 - 4ac = (45)^2 - 4(-16)(-34) = 2025 - 2176 = -151$

Since the discriminant is less than 0, this equation has no real solution. Therefore, the baton could not reach 40 feet.

How high will the baton reach?

Graph the function $h = -16t^2 + 45t + 6$.





Find the maximum (vertex).

The baton will reach approximately $\boxed{37.64 \text{ ft}}$.

You Try: Find values for c so that the equation will have no real solution, one real solution, and two real solutions. $x^2 + 3x + c = 0$

QOD: Write a quadratic equation which can be factored. Find its discriminant.

Teacher Note: Have students share their answers to the QOD and allow students to make a conjecture for how to determine if a quadratic polynomial is factorable using the discriminant. (It must be a perfect square.)

SAMPLE EXAM QUESTIONS

1. The graph of $y = -x^2 + x + 12$ has how many x-intercepts?

A. 12
B. 2
C. 1
D. 0

ANS: B

For question 2, the quadratic equation $f(x) = 2x^2 - 3x + c = 0$ has exactly one real solution.

2. $c = \frac{9}{8}$

A. True
B. False

ANS: A

3. How many real solutions does the equation $x^2 + 4 = 0$ have?

A. 0

B. 1

C. 2

ANS: A

4. How many real solutions does the equation $3y^2 = 0$ have?

A. 0

B. 1

C. 2

ANS: B

5. A quadratic expression has two factors. One factor is $(2x-3)$.

In each part below, find another factor of the quadratic, if possible. If the situation described is not possible, explain why.

a) The quadratic has no real zeros.
It is NOT possible for the quadratic to have no real zeros since we know it already has one. If the quadratic has a factor of $2x - 3$, then it has a zero at $x = \frac{3}{2}$.

b) The quadratic has only one real zero.
To have only one real zero, the other factor must be the same, so the expression is $(2x-3)^2$.

c) The quadratic has two distinct real zeros.
To have two real zeros, the other factor can be <u>any</u> linear expression, for example $(2x-3)(x+2)$.

6. Given $f(x) = x^2 - 2x + 9$.

a) Complete the square for $f(x)$.

$$\begin{aligned} f(x) &= x^2 - 2x + 9 \\ &= (x^2 - 2x + 1) + 9 + 1 \\ &= (x+1)^2 + 10 \end{aligned}$$

b) Using the quadratic formula, explain why the graph of $y = f(x)$ has no x-intercepts.

$$x = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(9)}}{2(1)}$$
$$= \frac{2 \pm \sqrt{-34}}{2}$$

Because the discriminant is negative, there are no real zeros for this quadratic.

For question 7, the quadratic equation $f(x) = 2x^2 - 3x + c = 0$ has exactly one real solution.

7. $f(x)$ **can be written as a difference of squares.**

A. True

B. False

ANS: B

8. **The graph of** $y = x^2 - 3x + 6$ **has how many** ***x*****-intercepts?**

A. 0

B. 1

C. 2

D. 6

ANS: A

Units 11.7 To choose the best algebraic method to solve a quadratic equation.

Some students believe that since the quadratic formula method can be used on ALL quadratic equations that it is the "best" method for solving ALL quadratic equations. However, other methods may actually be simpler and quicker, so why do extra work?

If the problem says, "Solve by _____ method", make sure to solve by the method indicated. If a method is not stated, use the suggestions below to decide which algebraic method to use.

When solving a quadratic equation, follow these steps (in this order) to decide on a method:

1. Try first to solve the equation by factoring. Be sure that your equation is in standard form $(ax^2 + bx + c = 0)$ before you start your factoring attempt. Don't waste a lot of time trying to factor your equation; if you can't get it factored in less than 60 seconds, move on to another method.

2. Next, look at the side of the equation containing the variable. Is that side a perfect square? If it is, then you can solve the equation by taking the square root of both sides of the equation. Don't forget to include a ± sign in your equation once you have taken the square root.
3. Next, if the coefficient of the squared term is 1 and the coefficient of the linear (middle) term is even, completing the square is a good method to use.
4. Finally, the quadratic formula will work on any quadratic equation. However, if using the formula results in awkwardly large numbers under the radical sign, another method of solving may be a better choice.

A note about the graphing method: Using a graph to determine the roots (x-intercepts) of a quadratic equation may prove to be a difficult process, especially if graphing by hand. If the x-intercepts are not integer values, exact values will not be found. If approximate answers are acceptable, a graphing calculator could be used to find the zeros.

Ex 12: Choose the best method for solving the quadratic equations.

a. $x^2 - 12x = -27$

Once the equation is in standard, form it can readily be factored, so BY FACTORING would be the best method.

b. $(2x-2)^2 = 8$

Since the left side of the equation is a perfect square, BY SQUARE ROOT METHOD would be the best method.

c. $2x^2 - x + 5 = 0$

The equation is not factorable and completing the square would not be convenient, BY QUADRATIC FORMULA would be the best method.

d. $x^2 + 6x - 11 = 0$

This equation is not factorable, and the side containing the variable is not a perfect square. The coefficient of the squared term is one and the coefficient of the linear term is even, so BY COMPLETING THE SQUARE would be the best method.

SAMPLE EXAM QUESTIONS

1. **What is the solution set for the equation** $x^2 + 8x + 16 = 49$**?**

A. $\{4, 7\}$ **C.** $\{-7, -4\}$

B. $\{-11, 3\}$ **D.** $\{-3, 11\}$

ANS: B

2. What is the solution set of $-4x^2 = 5x + 9$ **?**

A. $\left\{-1, -\frac{1}{4}\right\}$

B. $\left\{-1, \frac{9}{4}\right\}$

C. $\left\{\frac{-5+\sqrt{119}}{4}, \frac{-5-\sqrt{119}}{4}\right\}$

D. There are no real solutions.

ANS: D

3. What is the solution set of the equation $36x^2 - 25 = 0$ **?**

A. $\left\{\frac{5}{6}\right\}$

B. $\left\{\frac{25}{36}\right\}$

C. $\left\{\frac{-5}{6}, \frac{5}{6}\right\}$

D. $\left\{\frac{-25}{36}, \frac{25}{36}\right\}$

ANS: C

11.8 To model real world situations with quadratic equations.

Quadratic equations lend themselves to modeling situations that happen in real life, such as the rise and fall of profits from selling goods, the decrease and increase in the amount of time it takes to run a mile based on your age, throwing a ball into the air, and so on.

The wonderful part of having something that can be modeled by a quadratic is that you can easily solve the equation when set equal to zero and predict the patterns in the function values.

The vertex and *x*-intercepts are especially useful. These intercepts tell you where numbers change from positive to negative or negative to positive, so you know, for instance, where the ground is located in a physics problem or when you'd start making a profit or losing money in a business venture.

The vertex tells you where you can find the absolute maximum or minimum cost, profit, speed, height, time, or whatever you're modeling.

Ex 13: The distance traveled by a dropped object (ignoring air resistance) equals $\frac{1}{2}gt^2$, where g is the acceleration of the object due to gravity and t is the time since it was dropped. If acceleration due to gravity is about 10 m/s^2, how much time does it take an object to fall 80 meters?

Step One: Input the values into the quadratic equation $80 = \frac{1}{2}(10)t^2$

Step Two: Solve for time using the square root method

$$80 = 5t^2$$
$$16 = t^2$$
$$t = \pm 4$$

Step Three: Make sure the solution makes since in the context of the problem. Since time cannot be negative, there is only one solution. It takes four seconds to fall 80 meters.

Ex 14: The braking distance d, in feet, for a car can be modeled by $d = \frac{3(s^2 + 10s)}{40}$. where s is the speed of the car in miles per hour. What is the fastest speed that a car can be moving so that braking distance does not exceed 150 feet?

Step One: Input the value into the quadratic equation $150 = \frac{3(s^2 + 10s)}{40}$

Step Two: Simplify

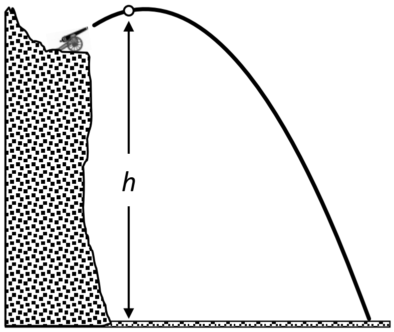
$$6000 = 3(s^2 + 10s)$$
$$2000 = s^2 + 10s$$

Step Three: Solve by factoring

$$s^2 + 10s - 2000 = 0$$
$$(s + 50)(s - 40) = 0$$
$$s = -50, 40$$

Step Four: Use common sense when answering the question! Speed cannot be a negative quantity so our solution is the speed cannot exceed 40 miles per hour.

Ex 15: Use the diagram and scenario below.



A cannonball is shot from the top of an ocean cliff as shown. The height (in meters) of the cannonball above the water is given by $h(t) = -5t^2 + 15t + 8$, where t is the number of seconds after the shot.

What does the constant, 8, in the equation represent?

It tells us that the cannon is 8 meters above the water.

How long does it take for the cannonball to reach its maximum height after it's shot.

To find the maximum height, find the x-coordinate of the vertex of the function.

$$x = \frac{-b}{2a} \rightarrow x = \frac{-15}{2(-5)} = -\frac{3}{2}$$

The cannonball reaches its maximum height at 1.5 seconds after it is shot.

How long until the cannonball hits the water?

The cannonball will hit the water when the function is equal to zero. Set the quadratic function equal to zero and solve.

$$-5t^2 + 15t + 8 = 0$$

$$x = \frac{-15 \pm \sqrt{(15)^2 - 4(-5)(8)}}{2(-5)}$$

$$x = \frac{-15 \pm \sqrt{385}}{-10}$$

$$x \approx -0.462, 3.462$$

The cannonball will hit the water about 3.462 seconds after it is shot.

Ex 16: A highway underpass is parabolic in shape. If the curve of the underpass can be modeled by $h(x) = 50 - 0.02x^2$, where x and $h(x)$ are in feet, then how high is the highest point of the underpass, and how wide is it?

$$x = \frac{-b}{2a} \rightarrow x = \frac{0}{2(-0.02)} = 0$$

The highest point occurs at the vertex: $50 - 0.02(0) = 50$

The vertex is (0, 50) so the highest point of the underpass is 50 feet.

To find the width of the underpass, find the distance between the two x-intercepts.

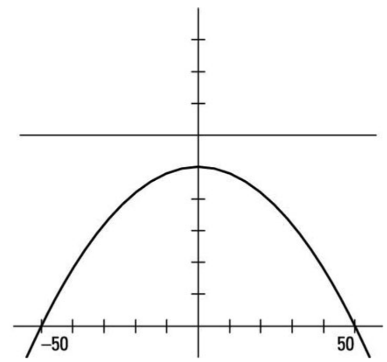
$0 = 50 - 0.02x^2$

$-50 = -0.02x^2$

$2500 = x^2$

$x = \pm 50$

The distance between the two intercepts is the width of the underpass, which is 100 feet.



Ex 17: The height of Carl, the human cannonball, is given by $h(t) = -16t^2 + 56t + 40$ where h is in feet and t is in seconds after the launch.

a) What was his height at the launch?
At the launch, *t = 0*, so *h(0)=-16(0)²+56(0)+40, the height is 40 feet.*

b) What is his maximum height?
The maximum height is the y-coordinate of the vertex.

$$x = \frac{-b}{2a} = \frac{-56}{2(-16)} = \frac{7}{4} = 1.75$$

$h(1.75) = -16(1.75)^2 + 56(1.75) + 40 =$ 89 feet

c) How long before he lands in the safety net, 8 feet above the ground?
Find *h(t)* = 8.

$-16t^2 + 56t + 40 = 8 \rightarrow -16t^2 + 56t + 32 = 0$

$-8(2t^2 - 7t - 4) = 0$

$-8(2t + 1)(t - 4) = 0$

$t = 4$ or $t = -\frac{1}{2}$

Since *t* represents time, the only answer is the positive root, so **4 seconds.**

Ex 18: Abigail tosses a coin off a bridge into the stream below. The distance, in feet, the coin is above the water is modeled by the equation $h(t) = -16t^2 + 96t + 112$, where t represents time in seconds.

a) What is the greatest height of the coin? Find the y-coordinate of the vertex.
256 feet

b) How much time will it take for the coin to hit the water?

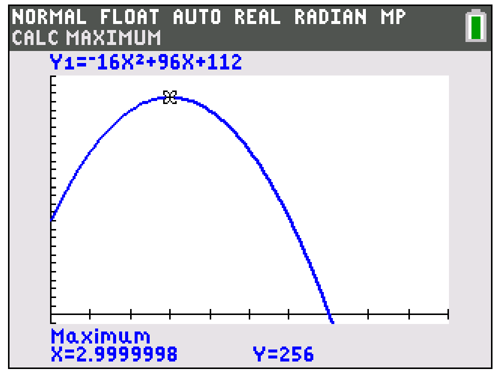
$h(t) = -16t^2 + 96t + 112$

$0 = -16(t^2 - 6t - 7)$

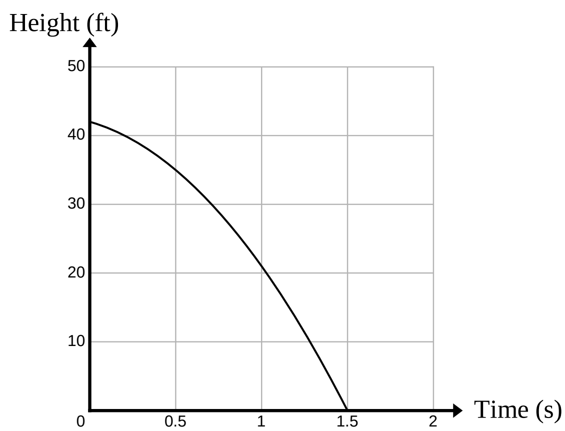
$0 = -16(t - 7)(t + 1)$

$t = 7$ or $t = -1$

It will take 7 seconds.



Ex 19: Use the graph provided to choose the best description of what the graph represents.



A. A ball is dropped from a height of 42 feet and lands on the ground after 3 seconds.

B. A ball is dropped from a height of 42 feet and lands on the ground after 1.5 seconds.

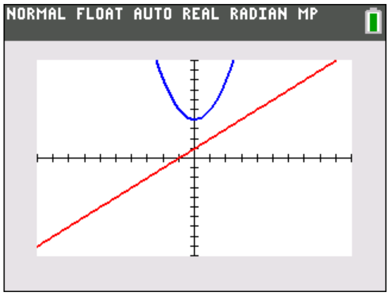
C. A ball is shot up in the air and reaches a height of 42 feet after 1 second.

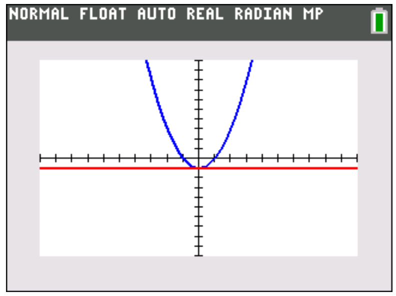
D. A ball is shot up in the air, reaches a height of 42 feet, and lands on the ground after 1.5 seconds.

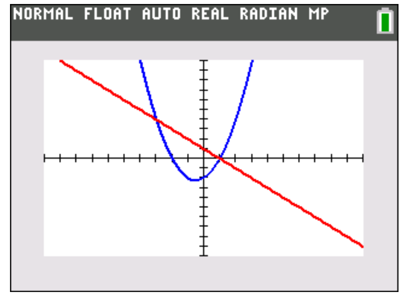
ANS: B

11.9 To solve a simple system involving a linear and quadratic equation, both algebraically and graphically.

Students are familiar with solving linear systems, but now will solve a system involving one linear equation and one quadratic. This can be done with graphing (with or without technology) or using algebraic methods – elimination and the substitution method. Students need to recognize that this type of system can have no solutions, one solution or two solutions.







$y = x^2 + 4$
$y = x + 1$
No Solutions

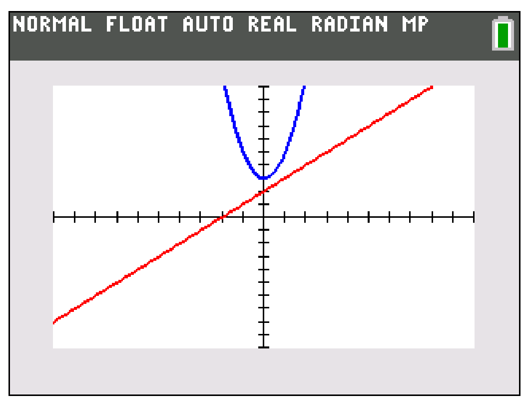
$y = x^2 - 1$
$y = -1$
One Solution

$y = x^2 + x - 2$
$y = -x + 1$
Two Solutions

Ex 20: Find the number of solutions for the system by graphing. $y = 2x^2 + 3$
$y = x + 2$

Step One: The linear equation has a slope of 1 and a *y*-intercept of 2.
Step Two: The quadratic is in vertex form. The vertex is at (0, 3) and opens up.
Step Three: Graph both equations on the same coordinate plane.

There are no points of intersection, so there is no solution to the system of equations.



Ex 21: Solve the system by using the elimination method: $y = x^2 - 11x - 36$, $y = -12x + 36$

Step One: Eliminate the common variable, the y.

$$\begin{array}{rl} y = & x^2 - 11x - 36 \\ -(y = & \quad\quad -12x + 36) \\ \hline 0 = & x^2 + x - 72 \end{array}$$

Step Two: Factor and solve for x.

$0 = x^2 + x - 72$

$0 = (x+9)(x-8)$

$x = -9$ or $x = 8$

Step Three: Find the corresponding y values.

Use either equation.

$y = -12(-9) + 36$ $\quad$ $y = -12(8) + 36$

$y = 144$ $\quad$ $y = -60$

Step Four: Write the solution as ordered pairs: $(-9, 144)$ and $(8, -60)$

Ex 22: Solve the system by using the substitution method: $y = x^2 - 6x + 9$, $y + x = 5$

Step One: Solve the linear equation for y.

$y = x^2 - 6x + 9$

$y = 5 - x$

Step Two: Write a single equation containing one variable. $\quad 5 - x = x^2 - 6x + 9$

Step Three: Factor and solve for x.

$0 = x^2 - 5x + 4$

$0 = (x-4)(x-1)$

$x = 4$ or $x = 1$

Step Four: Find the corresponding y values.

Use either equation.

$y = 5 - (4)$ $\quad$ $y = 5 - (1)$

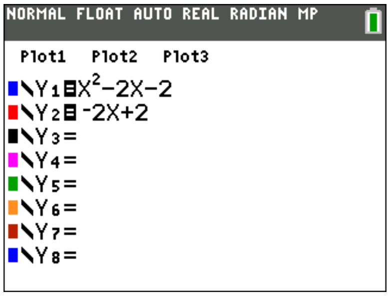
$y = 1$ $\quad$ $y = 4$

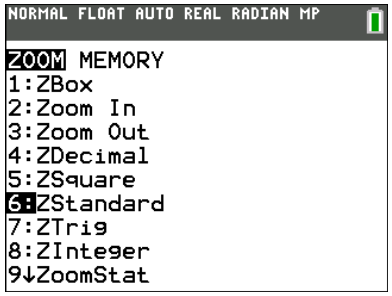
Step Five: Write the solution as ordered pairs: $(4, 1)$ and $(1, 4)$

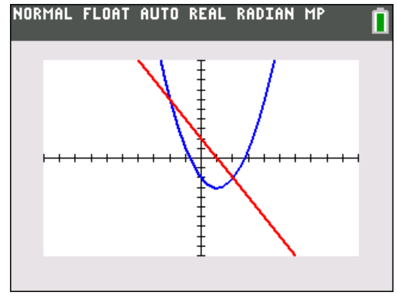
Ex 23: Solve the system using technology:

$$y = x^2 - 2x - 2$$
$$y = -2x + 2$$

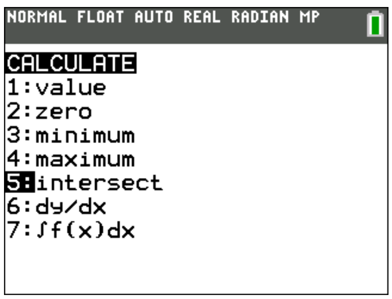
Step One: Enter the first equation in Y1 and the second equation in Y2. Press ZOOM 6 to get the standard window to view the graph.

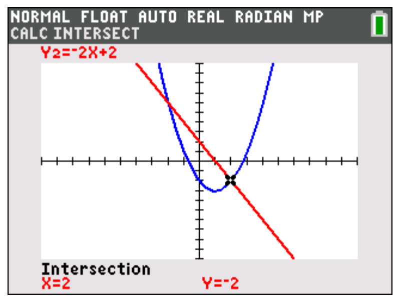




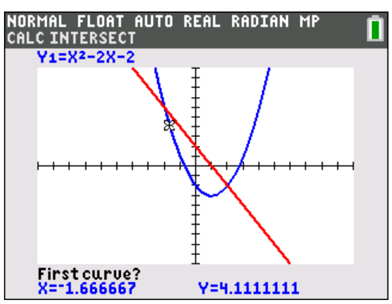


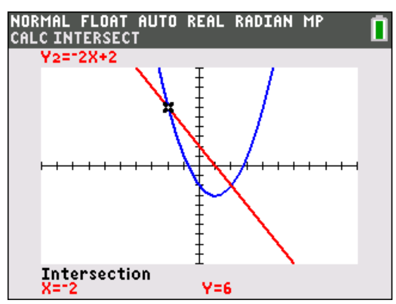
Step Two: Use the CALC feature and select 5: Intersect to find the first point of intersection. Place the cursor near one of the intersection points and hit "ENTER" three times.





Step Three: Find the second intersection point by repeating Step Two.





Step Four: The solutions of the system are: $(2,-2)$ and $(-2,6)$

SAMPLE EXAM QUESTIONS

1. Solve the system of equations.

$$\begin{cases} y = (x+4)^2 - 6 \\ -2x + y = 5 \end{cases}$$

A. (–4, 6) **C.** (0, 5)

B. (–5, –5) and (–1, 3) **D.** (–5, –5)

ANS: B

For questions 2-3, use the table below.

x	–4	–3	–2	–1	0	1
$f(x)$	–23	–10	–3	–2	–7	–18
$g(x)$	–13	–11.5	–10	–8.5	–7	–5.5

2. $f(x) = g(x)$ at (0, –7).

A. True

B. False

ANS: A

3. $f(x) = g(x)$ somewhere on the interval $-3 < x < -2$.

A. True

B. False

ANS: B

4. The parabola $y = x^2 - 9$ and the line $y = -8x$ intersect at two points. Which equation would be useful to find these points?

A. $(-8x)^2 - 9 = 0$

B. $-8(x^2 - 9) = 0$

C. $x^2 + 8x - 9 = 0$

D. $x^2 - 8x - 9 = 0$

ANS: C