Unit 11 - Solving Quadratic Functions – PART ONE

PREREQUISITE SKILLS:

- students should be able to add, subtract and multiply polynomials
- students should be able to factor polynomials
- students should be able to solve linear systems of equations
- students should be able to graph quadratic functions

VOCABULARY:

1. real number: any rational or irrational value
2. imaginary number: a number that when squared gives a negative result
3. quadratic equation: an equation where the highest exponent of the variable is a square
4. zeros of a function: also called a root of the function, it is the x value(s) that produce a function value of zero
5. zero product property: It states that if the product of two factors is zero, then one of the factors must be equal to zero
6. completing the square: a technique used to solve quadratic equations when factoring does not work
7. quadratic formula: a method used to find the roots of a quadratic equation by substituting in the coefficients of the quadratic equation
8. discriminant: the expression located under the radical in the quadratic formula that gives us information about the number of real roots of the quadratic equation
9. system of equations: a collection of two or more equations with a same set of unknowns

SKILLS:

- Solve quadratic equations graphically and algebraically
- Make decisions about which method would be best to use to solve a quadratic equation
- Solve a simple system with one linear and one quadratic equation
- Model real world situations with quadratic equations

STANDARDS:

A.REI.B.4b Solve quadratic equations by inspection (e.g., for $x^2 = 49$), taking square roots, completing the square, the quadratic formula and factoring, as appropriate.

8.NS.A.1 Know that numbers that are not rational are called irrational. Understand informally that every number has a decimal expansion; for rational numbers show that the decimal expansion repeats eventually, and convert a decimal expansion which repeats eventually into a rational number.

N.RN.B.3 Explain why the sum or product of two rational numbers is rational; that the sum of a rational number and an irrational number is irrational; and that the product of a nonzero rational number and an irrational number is irrational.

F.IF.C.7c Graph polynomial functions, identifying zeros when suitable factorizations are available, and showing end behavior. *(Modeling Standard)

A.SSE.B.3a Factor a quadratic expression to reveal the zeros of the function it defines. *(Modeling Standard)

A.SSE.A.2-1 Use the structure of a linear, exponential, or quadratic expression to identify ways to rewrite it. For example, see x^4 - y^4 as (x^2)^2 - (y^2)^2, thus recognizing it as a difference of squares that can be factored as (x^2 - y^2)(x^2 + y^2).

A.REI.B.4a Use the method of completing the square to transform any quadratic equation in x into an equation of the form (x - p)^2 = q that has the same solutions. Derive the quadratic formula from this form.

F.IF.C.8a Use the process of factoring and completing the square in a quadratic function to show zeros, extreme values, and symmetry of the graph, and interpret these in terms of a context.

A.SSE.B.3b Complete the square in a quadratic expression to reveal the maximum or minimum value of the function it defines. *(Modeling Standard)

N.Q.A.1 Use units as a way to understand problems and to guide the solution of multi-step problems; choose and interpret units consistently in formulas; choose and interpret the scale and the origin in graphs and data displays.

A.CED.A.2-1 Create linear, exponential, and quadratic equations in two or more variables to represent relationships between quantities and graph equations on coordinate axes with labels and scales. Limit exponentials to have integer inputs only. *(Modeling Standard)

F.IF.B.4-1 For a linear, exponential, or quadratic function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; and end behavior. *(Modeling Standard)

F.IF.B.5-1 Relate the domain of a linear, exponential, or quadratic function to its graph and, where applicable, to the quantitative relationship it describes. For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function. *(Modeling Standard)

A.REI.C.7 Solve a simple system consisting of a linear equation and a quadratic equation in two variables algebraically and graphically. For example, find the points of intersection between the line y = –3x and the circle x^2 + y^2 = 3.

F.IF.C.7a Graph linear and quadratic functions and show intercepts, maxima, and minima. *(Modeling Standard)

LEARNING TARGETS:

- 11.1 To review number sets and introduce the existence of imaginary numbers.
- 11.2 To use the graph of a quadratic function to solve its related quadratic equation.
- 11.3 To solve quadratic equations by taking the square root.
- 11.4 To solve quadratic equations by factoring.
- 11.5 To solve quadratic equations by completing the square.
- 11.6 To solve quadratic equations by using the quadratic formula.
- 11.7 To choose the best algebraic method to solve a quadratic equation.
- 11.8 To model real world situations with quadratic equations.
- 11.9 To solve a simple system involving a linear and quadratic equation, both algebraically and graphically.

ESSENTIAL QUESTIONS:

- What makes an equation a quadratic equation?
- How can algebra be used to solve quadratic equations?
- How many solutions can quadratic equations have? Are all solutions always real numbers?
- How can I decide which method to use to solve quadratic equations?

BIG IDEAS:

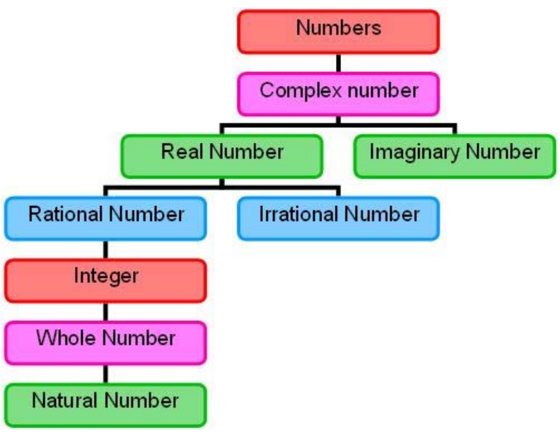
Solving quadratics can be done by different techniques (graphing, by inspection, factoring, completing the square, quadratic formula), each more efficient than the others based on the characteristics of the quadratic function. Some quadratics do not have real solutions. The discriminant gives us good information about the nature of the roots of a quadratic equation. Completing the square is a good method to use when a quadratic equation is difficult to factor or cannot be factored. It can also be used to derive the quadratic formula. The solutions to some real-world problems can be found my modeling them with quadratic equations and graphs. Systems of equations may be solved using graphs, tables, eliminations or substitution. The solution to a linear and quadratic system, if it exists, is an ordered pair or two ordered pairs.

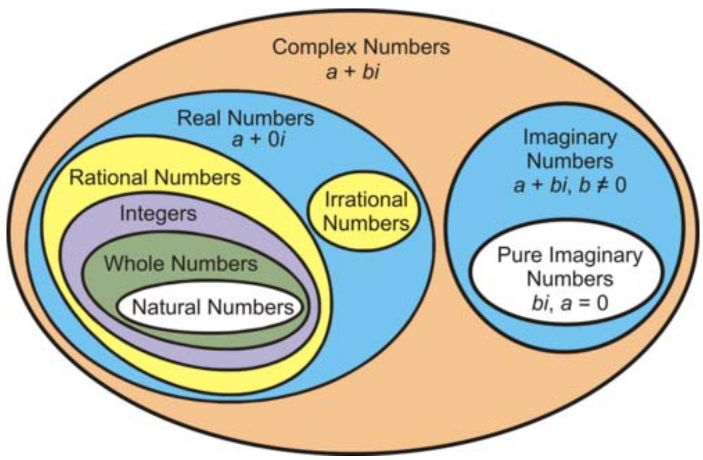
Notes, Examples and Exam Questions

Unit 11.1 To review number sets and introduce the existence of imaginary numbers.

Students in Algebra I should be introduced to the imaginary number set. They are not expected to write answers to a quadratic equation as a complex number. They will answer "no real solution", but the students should be aware that this is a possibility and that it will occur when the value of the discriminant is negative.

Review the real number system with the students and introduce the new set, imaginary numbers. This can be done using a tree diagram or a Venn diagram as seen below.





Up until this point in your math studies so far, you have worked with real numbers. How do we solve, $x^2 = -4$? This led mathematicians to define the imaginary numbers. The **imaginary unit, *i*,** is defined to be $i^2 = -1$, which means that $i = \sqrt{-1}$. With this definition, we can now solve quadratic equations that before we were not able to solve under the real number system.

Ex 1: Solve $x^2 = -64$

$\sqrt{x^2} = \sqrt{-64}$ Take the square root of both sides

$x = \pm\sqrt{-64}$

$x = \pm\sqrt{-1} \cdot \sqrt{64}$

$x = \pm 8i$

There are no real solutions, but there are two imaginary solutions, $8i$ and $-8i$.

Ex 2: Solve $x^2 = \sqrt{-18}$

$\sqrt{x^2} = \sqrt{-18}$ Take the square root of both sides

$x = \pm\sqrt{-18}$

$x = \pm\sqrt{-1} \cdot \sqrt{9} \cdot \sqrt{2}$

$x = \pm 3i\sqrt{2}$

There are no real solutions, but there are two imaginary solutions, $\boxed{3i\sqrt{2} \text{ and } -3i\sqrt{2}}$.

Sample Exam Questions

1. How many real solutions does the equation $x^2 + 4 = 0$ **have?**

A. 0

B. 1

C. 2 ANS: A

2. The equation $x^2 = a$ **has no real solutions. What must be true?**

A. $a < 0$

B. $a = 0$

C. $a > 0$ ANS: A

<u>Unit 11.2 To use the graph of a quadratic function to solve its related quadratic equation.</u>

<u>Quadratic Equation</u>: an equation that can be written in the *standard form:* $ax^2 + bx + c = 0$, $a \neq 0$

<u>Solving a Quadratic Equation by Graphing</u>

Step One: Write the equation in the form $ax^2 + bx + c = 0$.

Step Two: Graph the function $y = ax^2 + bx + c$.

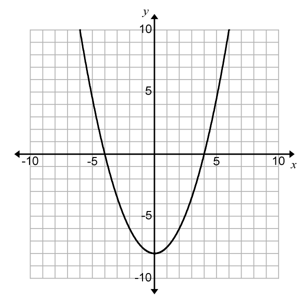
Step Three: Find the zero(s) or root(s) of the function. These are the solution(s) to the equation.

Note: The words *x*-intercept, zero, root, and solution can be used interchangeably for the above value.

Ex 3: Solve the equation $\frac{1}{2}x^2 = 8$ by graphing.

Step One: Write the equation in the form $ax^2 + bx + c = 0$. $\frac{1}{2}x^2 - 8 = 0$

Step Two: Graph the function $y = ax^2 + bx + c$. $y = \frac{1}{2}x^2 - 8$

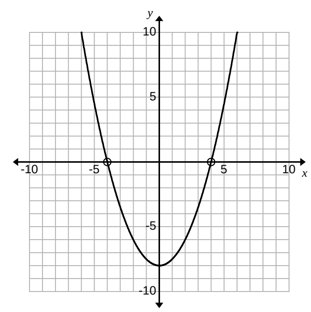


Vertex: $x = -\frac{b}{2a} = -\frac{0}{2\left(\frac{1}{2}\right)} = 0$ $\quad (0, -8)$

$y = \frac{1}{2}(0)^2 - 8 = -8$

Step Three: Find the x-intercept(s) of the function.

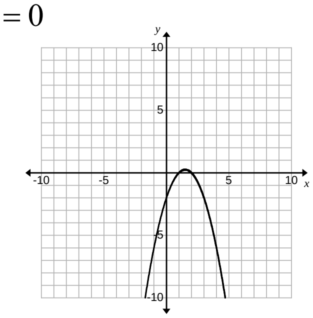
The zeros are at −4 and 4, so the solutions are $\boxed{x = -4, 4}$.



Ex 4: Solve the equation $-x^2 + 3x = 2$ by graphing.

Step One: Write the equation in the form $ax^2 + bx + c = 0$. $-x^2 + 3x - 2 = 0$

Step Two: Graph the function $y = ax^2 + bx + c$. $y = -x^2 + 3x - 2$



Vertex:

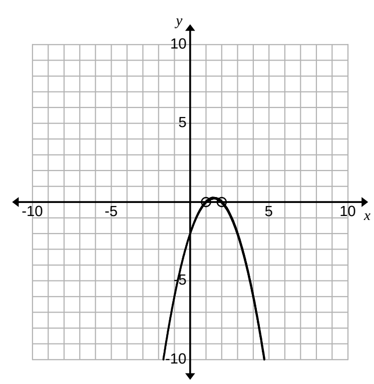
$x = -\frac{b}{2a} = -\frac{3}{2(-1)} = \frac{3}{2}$

$y = -\left(\frac{3}{2}\right)^2 + 3\left(\frac{3}{2}\right) - 2 = -\frac{9}{4} + \frac{9}{2} - 2 = -\frac{9}{4} + \frac{18}{4} - \frac{8}{4} = \frac{1}{4}$

$\left(\frac{3}{2}, \frac{1}{4}\right)$

Step Three: Find the x-intercept(s) of the function.

The zeros are at 1 and 2, so the solutions are $\boxed{x = 1, 2}$.

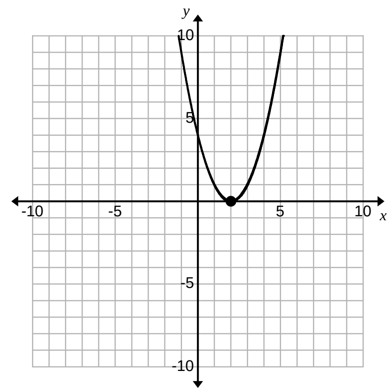


Ex 5: Solve the equation $x^2+4=4x$ by graphing.

Step One: Write the equation in the form $ax^2+bx+c=0$. $x^2-4x+4=0$

Step Two: Graph the function $y=ax^2+bx+c$.

$y=x^2-4x+4$



Vertex: $x=-\frac{b}{2a}=-\frac{(-4)}{2(1)}=\frac{4}{2}=2$ $(2,0)$

$y=(2)^2-4(2)+4=4-8+4=0$

Step Three: Find the *x*-intercept(s) of the function.

The root is at 2, so the solution is $\boxed{x=2}$.

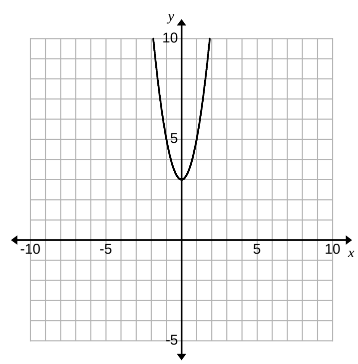
Check: $x^2+4=4x \Rightarrow (2)^2+4=4(2) \Rightarrow 4+4=8$ ☺

Ex 6: Solve the equation $2x^2+3=0$ graphically.

Step One: Write the equation in the form $ax^2+bx+c=0$. $2x^2+3=0$

Step Two: Graph the function $y=ax^2+bx+c$.

$y=2x^2+3$



Vertex: $x=-\frac{b}{2a}=-\frac{0}{2(3)}=\frac{0}{6}=0$ $(0,3)$

$y=2(0)^2+3=3$

Step Three: Find the *x*-intercept(s) of the function.

There is no zero, so this equation has $\boxed{\text{no real solution}}$.

Using a Graphing Calculator to Solve Quadratic Equations

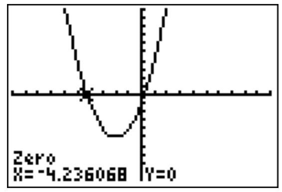
Ex 7: Approximate the solution(s) of $x^2=1-4x$ using a graphing calculator.

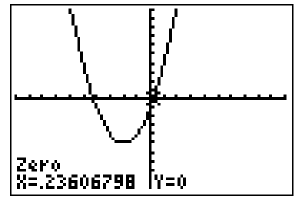
Step One: Write the equation in standard form. $x^2+4x-1=0$

Step Two: Graph the function $y = ax^2 + bx + c$. $y = x^2 + 4x - 1$

Step Three: Find the zero(s) of the function.

$x \approx -4.236, 0.236$





You Try: Solve the quadratic equation $6 = \frac{2}{3}x^2$ graphically. Then, check your answer algebraically.

QOD: How can you tell from the graph of a quadratic function if the equation has one, two, or no solution?

SAMPLE EXAM QUESTIONS

1. The graph of $y = -x^2 + x + 12$ has how many *x*-intercepts?

A. 12 **C.** 2

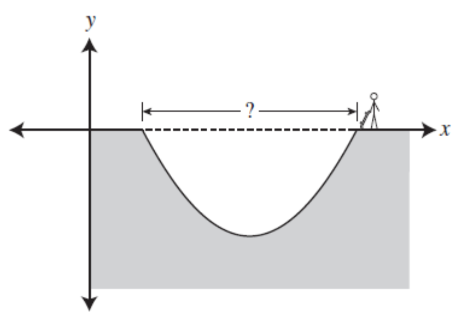
B. 1 **D.** 0

ANS: C

2. Fiona is designing a skateboard park. One skating area in the park will be shaped like the parabola that is described by the equation below.

$$y = \frac{1}{6}\left(x^2 - 18x + 45\right)$$

A sketch of Fiona's design for the skating area is shown below.



What is the distance across the top of the skating area?

A. 12 units

B. 14 units

C. 15 units

D. 18 units

ANS: A

Unit 11.3 To solve quadratic equations by taking the square root.

Solving a Quadratic Equation by Finding Square Roots:

To use this method, the quadratic equation must be able to be written in the form
$ax^2 + c = 0, \; (b = 0) \quad or \;\; ax^2 = c$

Ex 8: Solve the equation $x^2 = 16$.

Step One: Find the square root of both sides.

$\sqrt{x^2} = \sqrt{16}$

$x = \pm 4$

Step Two: Solve for *x*. (Note that there are *two solutions*.)

$\boxed{x = -4, \; 4}$

This can also be written $\boxed{x = \pm 4}$

Ex 9: Solve the equation $x^2 + 4 = 4$.

Step One: Isolate the squared expression.

$x^2 + 4 = 4$

$x^2 = 0$

Step Two: Find the square root of both sides.

$\sqrt{x^2} = \sqrt{0}$

$x = \pm 0$

Step Three: Solve for *x*. (Note that there is *one solution*.)

$\boxed{x = 0}$

Ex 10: Solve the equation $3 + x^2 = 2$.

Step One: Isolate the squared expression.

$3 + x^2 = 2$

$x^2 = -1$

Step Two: Find the square root of both sides.

$\sqrt{x^2} = \sqrt{-1}$

$x = \pm\sqrt{-1}$ $\quad \sqrt{-1}$ is not a real number

$x = \pm i$

Step Three: Solve for *x*. (Note: The solution is imaginary.) This equation has no real solution.

Ex 11: Solve the equation $3x^2 - 108 = 0$.

Step One: Isolate the squared expression.

$$3x^2 = 108$$
$$x^2 = 36$$

Step Two: Find the square root of both sides.

$$\sqrt{x^2} = \sqrt{36}$$
$$x = \pm 6$$

Step Three: Solve for the variable.

$$\boxed{x = -6 \text{ or } x = 6}$$

Ex 12: Solve the equation $x^2 - 7 = 43$.

Step One: Isolate the squared expression.

$$x^2 - 7 = 43$$
$$x^2 = 50$$

Step Two: Find the square root of both sides.

$$\sqrt{x^2} = \sqrt{50}$$
$$x = \pm 5\sqrt{2}$$

Step Three: Solve for the variable.

$$\boxed{x = -5\sqrt{2} \text{ or } x = 5\sqrt{2}}$$

Ex 13: Solve the equation $2(2n-5)^2 = 162$.

Step One: Isolate the squared expression.

$$(2n-5)^2 = 81$$

Step Two: Find the square root of both sides.

$$\sqrt{(2n-5)^2} = \sqrt{81}$$
$$2n - 5 = \pm 9$$

Step Three: Solve for the variable.

$$2n - 5 = 9 \qquad 2n - 5 = -9$$
$$2n = 14 \qquad 2n = -4$$
$$\boxed{n = 7} \qquad \boxed{n = -2}$$

Ex 14: Solve the equation $\frac{1}{4}(a-8)^2 = 7$

Step One: Isolate the squared expression. $(a-8)^2 = 28$

Step Two: Find the square root of both sides.

$$\sqrt{(a-8)^2} = \sqrt{28}$$
$$a-8 = \pm 2\sqrt{7}$$

Step Three: Solve for the variable.

$$a-8 = 2\sqrt{7} \qquad a-8 = -2\sqrt{7}$$
$$a = 8+2\sqrt{7} \qquad a = 8-2\sqrt{7}$$
$$\boxed{a = 8 \pm 2\sqrt{7}}$$

Note: The $\pm$ ("plus or minus") symbol is used to write both solutions in a shorter way. In set notation, the solutions would be written $\left\{8-2\sqrt{7}, 8+2\sqrt{7}\right\}$.

Real-Life Application: Free Fall

On Earth, the equation for the height (*h*) of an object for *t* seconds after it is dropped can be modeled by the function $h = -16t^2 + h_0$, where h_0 is the initial height of the object.

Ex 15: A ball is dropped from a height of 81 ft. How long will it take for the ball to hit the ground?

Use the free-fall function. $h = -16t^2 + h_0$ $h_0 = 81, h = 0$

Initial height is 81 ft. The ball will hit the ground when its height is 0 ft.

Solve for *t*.

$$0 = -16t^2 + 81$$
$$16t^2 = 81$$
$$\sqrt{t^2} = \sqrt{\frac{81}{16}}$$
$$t = \pm\frac{9}{4}$$
$$t = -\frac{9}{4}, \frac{9}{4}$$

Solution: Since time is positive, the only feasible answer is $\frac{9}{4} = \boxed{2.25 \text{ seconds}}$

You Try: Solve the equation $7 - 10x^2 = 1$.

QOD: Why do some quadratic equations have two, one, or no real solution?

SAMPLE EXAM QUESTIONS

1. If $p^2 = 25$ and $q^2 = 16$, which of these CANNOT equal $p + q$?

A. -1

B. 9

C. 41

ANS: C

For questions 2-3, use the equation $x^2 = (2x + p)^2$.

2. $x = 2x + p$

A. True

B. False

ANS: A

3. $x = -(2x + p)$

A. True

B. False

ANS: A

4. Solve the equation $\frac{u^2}{2} + P = h$ for *u*, where all variables are positive real numbers.

A. $u = \sqrt{2h - P}$

B. $u = \sqrt{\frac{h - P}{2}}$

C. $u = \sqrt{2(h - P)}$

D. $u = \sqrt{\frac{h}{2} - P}$

ANS: C

5. Solve the equation for *x*: $a(x-h)^2+k=p$

A. $x=h\pm\sqrt{\frac{p}{a}-k}$

B. $x=h\pm\sqrt{\frac{p}{a}}-\sqrt{k}$

C. $x=h\pm\frac{\sqrt{p-k}}{a}$

D. $x=h\pm\sqrt{\frac{p-k}{a}}$

ANS: D

6. What is the solution set of the equation $4(t-3)^2-1=8$**?**

A. $\left\{1\frac{1}{2}, 4\frac{1}{2}\right\}$

B. $\left\{\frac{3}{4}, 5\frac{1}{4}\right\}$

C. $\left\{3-\sqrt{3}, 3+\sqrt{3}\right\}$

D. $\left\{3-\sqrt{5}, 3+\sqrt{5}\right\}$

ANS: A

7. Solve each quadratic equation for *x*.

(1) $x^2-8=0$

ANS: $\boxed{x=\pm2\sqrt{2}}$

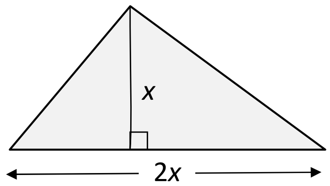
(2) $(x-2)^2-4=0$

ANS: $\boxed{0 \text{ or } 4}$

(3) $3(x+6)^2=15$

ANS: $\boxed{x=-6\pm\sqrt{5}}$

8. The area of the triangle below is 24 square units. What is the height of the triangle?



A. 6 units

B. 12 units

C. $\sqrt{12}$ units

D. $\sqrt{24}$ units

ANS: D

Unit 11.4 To solve quadratic equations by factoring.

Zero Product Property: If the product of two factors is 0, then one or both of the factors must equal 0.

Ex 16: Solve the equation $(x-3)(x+1)=0$ using the zero product property.

Since one or both of the factors must equal 0, we will solve the two linear equations $x-3=0$ and $x+1=0$.

$x-3=0$ $\quad$ $x+1=0$

$x=3$ $\quad$ $x=-1$ $\quad$ Solutions: $\boxed{x=-1,\ 3}$

Solving a Quadratic Equation by Factoring

Ex 17: Solve the equation $x^2-5x=-6$ by factoring.

Step One: Write the equation in standard form. $x^2-5x+6=0$

Step Two: Factor the quadratic. $(x-3)(x-2)=0$

Step Three: Set each factor equal to zero and solve. $x-3=0$, $x=3$ $\quad$ $x-2=0$, $x=2$ $\quad$ $\boxed{x=2,\ 3}$

Check this answer by graphing on the calculator.

Ex 18: Solve the equation $2x^2 - 4x - 8 = -x^2 + x$.

Step One: Write the equation in standard form. $3x^2 - 5x - 8 = 0$

Step Two: Factor the quadratic using the "*ac* method".

$a \cdot c = -24$ $\quad 3x^2 - 8x + 3x - 8 = 0$

$b = -5$ $\quad x(3x-8) + 1(3x-8) = 0$

-8 and 3 $\quad (3x-8)(x+1) = 0$

Step Three: Set each factor equal to zero and solve.

$3x - 8 = 0 \quad x + 1 = 0$

$x = \frac{8}{3} \quad x = -1$

The solutions can be written in set notation: $\left\{-1, \frac{8}{3}\right\}$

Ex 19: Solve the equation $25 = 30y - 9y^2$.

Step One: Write the equation in standard form. $9y^2 - 30y + 25 = 0$

Step Two: Factor the quadratic.

$(3y)^2 - 30y + (5)^2$

Note: $2(5)(3y) = 30y$

$(3y-5)^2 = 0$

Step Three: Set each factor equal to zero and solve.

$3y - 5 = 0$

$y = \frac{5}{3}$

The solution can be written in set notation: $\left\{\frac{5}{3}\right\}$

<u>Zero(s) of Quadratic Function</u>: the *x*-value(s) where the function intersects the *x*-axis

To find the zero(s), factor the quadratic and set each factor equal to 0.

Note: We can graph quadratic functions by plotting the zeros. The vertex is halfway between the zeros.

Ex 20: Find the zero(s) of the quadratic function $y = -x^2 - 2x + 3$ and graph the parabola.

Step One: Factor the quadratic polynomial.

$y = -(x^2 + 2x - 3)$

$y = -(x+3)(x-1)$

Step Two: Set each factor equal to 0 and solve.

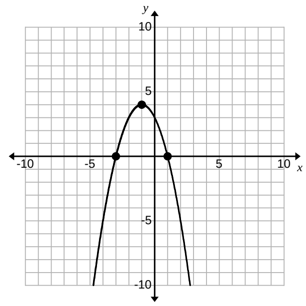
$$x+3=0 \quad x-1=0$$
$$x=-3 \qquad x=1$$

Step Three: Find the coordinates of the vertex.

$$x=\frac{-3+1}{2}=\frac{-2}{2}=-1$$

$$y=-(-1)^2-2(-1)+3=-1+2+3=4$$

Step Four: Plot the points and sketch the parabola.



<u>You Try</u>: Solve the quadratic equation $5t^2-25=4(t^2+6)$ by factoring.

<u>QOD</u>: What must be true about a quadratic equation before you can solve it using the zero product property?

SAMPLE EXAM QUESTIONS

1. **What is the solution set for the following equation?** $\mathbf{x^2-10x+9=0}$

A. {–9, –1}
B. {–9, 1}
C. {–1, 9}
D. {1, 9}

ANS: D

2. **Which of the following equations has roots of –7 and 4?**

A. $(x+7)(x-4)=0$
B. $(x-7)(x+4)=0$
C. $(x-7)(x-4)=0$
D. $(x+7)(x+4)=0$

ANS: A

For questions 3-5, consider the solutions to the equation $(x+5)(x-3)=0$.

3. $x^2-15=0$ **has the same solutions as the given equation.**

A. True

B. False

ANS: B

4. $x^2+2x-15=0$ **has the same solutions as the given equation.**

A. True

B. False

ANS: A

5. $(x+1)^2-14=0$ **has the same solutions as the given equation.**

A. True

B. False

ANS: B

6. The expression $4x^2+bx-3$ **is factorable into two binomials. Which could NOT equal** ***b*****?**

A. –7

B. –1

C. 1

D. 11

ANS: A

7. Which quadratic equation has solutions of $x = 2a$ **and** $x = -b$**?**

A. $x^2-2ab=0$

B. $x^2-x(b-2a)-2ab=0$

C. $x^2-x(b+2a)+2ab=0$

D. $x^2+x(b-2a)-2ab=0$

ANS: D

8. Which equation has roots of 4 and -6?

A. $(x-4)(x+6)=0$

B. $(x-4)(x-6)=0$

C. $(x+4)(x+6)=0$

D. $(x+4)(x-6)=0$

ANS: A

9. What value(s) of *x* make the equation $(x-m)(x-n)=0$ true? (*m* and *n* do not equal zero.)

A. $-m$ and $-n$

B. 0

C. m and n

D. mn

ANS: C

10. Solve the quadratic $4x^2=14x+8$.

A. $x=-2$ or $x=1$

B. $x=-\frac{1}{2}$ or $x=4$

C. $x=-\frac{1}{7}$ or $x=8$

D. $x=0$ or $x=-\frac{7}{4}$

ANS: B

11. When $2x^2+(4-p)x-2p=0$, $x=-2$ is a solution. Which is a factor of $2x^2+(4-p)x-2p$?

A. $2x-p$

B. $2x+p$

C. $4-p$

D. $x-2p$

ANS: A