Unit 7 – Systems and Linear Programming

PREREQUISITE SKILLS:

- students should be able to solve linear equations
- students should be able to graph linear equations
- students should be able to create linear equations in slope-intercept and standard form
- students should be able to graph linear inequalities

VOCABULARY:

1. system of linear equations: a set of two equations with two variables
2. solution of a system of linear equations: an ordered pair (x,y) that satisfies both equations in the system
3. linear combination method: a method used to solve a system of linear equations where an expression is constructed from a set of terms by multiplying each term by a constant and adding the results, also known as the elimination method
4. elimination method: a method used to solve a system of linear equations where you add or subtract to linear equations to get an equation in one variable, also known as the linear combination method
5. substitution: a method for solving systems of linear equations by solving one of the equations for one of the variables and then plugging this back into the other equation, "substituting" for the chosen variable and solving for the other
6. intersection: the point where lines cross and share an element (ordered pair)
7. simultaneous equations: a set of two or more equations, each containing two or more variables whose values can simultaneously satisfy both or all the equations in the set
8. linear programming: also called linear optimization, it is a method to achieve the best outcome in a mathematical model whose requirements are represented by linear relationships
9. constraints: a condition of a linear programming problem that the solution must satisfy
10. feasible region: also known as a solution space, it is the set of all possible points of linear programming problem that satisfy the problem's constraints
11. objective function: a real-valued function in a linear programming problem whose value is to be either minimized or maximized over the set of feasible alternatives
12. corner-point principle: the principle says that the maximum (or minimum) solution occurs at a corner point (vertex) of the feasible region

SKILLS:

- Find a solution(s) that satisfies two linear equations or inequalities
- Graph linear equations on a coordinate plane
- Graph one or more linear inequalities on a coordinate plane

STANDARDS:

8.EE.C.8 Analyze and solve pairs of simultaneous linear equations.

A.REI.C.5 Prove that, given a system of two equations in two variables, replacing one equation by the sum of that equation and a multiple of the other produces a system with the same solutions.

A.REI.C.6 Solve systems of linear equations exactly and approximately (e.g., with graphs), focusing on pairs of linear equations in two variables.

A.REI.D.11-1 Explain why the x-coordinates of the points where the graphs of the equations y = f(x) and y = g(x) intersect are the solutions of the equation f(x) = g(x); find solutions to f(x) = g(x) approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where f(x) and/or g(x) are linear, quadratic, or exponential functions. *(Modeling Standard)

A.REI.D.12-1 Graph the solutions to a linear inequality in two variables as a half-plane (excluding the boundary in the case of a strict inequality), and graph the solution set to a system of linear inequalities in two variables as the intersection of the corresponding half-planes.

A.CED.A.3-1 Represent constraints by linear equations or inequalities, and by systems of linear equations and/or inequalities, and interpret solutions as viable or non-viable options in a modeling context. For example, represent inequalities describing nutritional and cost constraints on combinations of different foods. *(Modeling Standard)

LEARNING TARGETS:

- 7.1 To solve a linear system of equations by graphing, substitution, and elimination
- 7.2 To know that a system can have zero, one, or infinitely many solutions
- 7.3 To solve and graph systems of linear inequalities
- 7.4 To graph constraints and identify vertices of the feasible region
- 7.5 To use the vertices to maximize or minimize the objective function

BIG IDEAS:

We may use prior skills and concepts to "eliminate" variables and transform a system of two equations into a one variable equation. Each step produces a system which has the exact same solutions as all the others. Systems of linear equations may be solved using graphs, tables, linear combination (elimination), or substitution. The solution to a linear system, if it exists, is an ordered pair that is common to both equations. The solution for two functions being equal, are the points of intersection between the system of equations. Constraints are necessary to balance a mathematical model with real-world context. Variable quantities may be able to take on only certain values and expressing these restrictions, or constraints, algebraically is an important part of modeling with mathematics.

Notes, Examples and Exam Questions

Units 7.1, 7.2 To solve a linear system of equations by graphing, substitution, and elimination and to know that a system of two linear equations can have zero, one, or infinitely many solutions.

Checking if an ordered pair is a solution to a system of two linear equations:

Ex 1: Is the ordered pair $(3,-4)$ a solution of the system $\begin{matrix} 2x+y=2 \\ x-y=4 \end{matrix}$?

Step One: Substitute the ordered pair for (x,y) in both equations.

$2x+y=2 \Rightarrow 2(3)+(-4)=6-4=2 \therefore (3,-4)$ *is* a solution of the first equation.

$x-y=4 \Rightarrow (3)-(-4)=7\neq 4 \therefore (3,-4)$ *is NOT* a solution of the second equation.

Step Two: If the ordered pair is a solution to both equations, then it is a solution of the system.

So, NO, $(3,-4)$ *is not* a solution of the system.

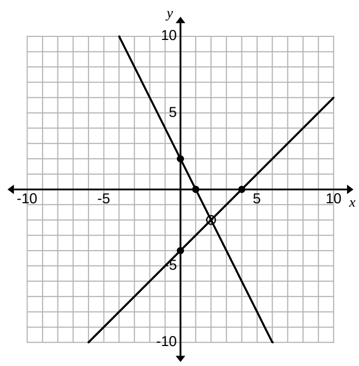
Solving a System of Two Linear Equations by Graphing:

Ex 2: Solve the system by graphing: $\begin{matrix} 2x+y=2 \\ x-y=4 \end{matrix}$

Step One: Graph both equations on the same coordinate plane.

To graph these lines, we will use intercepts.

$2x+y=2$	$x-y=4$
$2x+0=2$	$x-0=4$
$x=1$	$x=4$
$2(0)+y=2$	$0-y=4$
$y=2$	$y=-4$



Step Two: Find the coordinates of the point of intersection of the two lines.

The lines appear to intersect at the point $(2,-2)$.

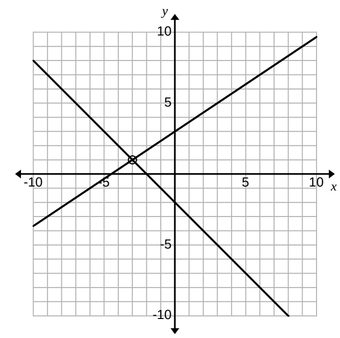
Step Three: Substitute the ordered pair found in Step Two into the original equations of the system to determine if this point is the solution of the system of equations.

$2x+y=2 \Rightarrow 2(2)+(-2)=2 \therefore (2,-2)$ is a solution of the first equation.

$x-y=4 \Rightarrow (2)-(-2)=4 \therefore (2,-2)$ is a solution of the second equation.

So, $\boxed{(2,-2)}$ is the solution of the system of equations.

Ex 3: Solve the system by graphing: $\begin{array}{l} x+y=-2 \\ 2x-3y=-9 \end{array}$



Step One: Graph both equations on the same coordinate plane.

To graph these lines, we will rewrite the equations in slope-intercept form.

$$\begin{array}{l} x+y=-2 \\ y=-x-2 \end{array} \qquad \begin{array}{l} 2x-3y=-9 \\ -3y=-2x-9 \\ y=\dfrac{2}{3}x+3 \end{array}$$

Step Two: Find the coordinates of the point of intersection of the two lines.

The lines appear to intersect at the point $(-3,1)$.

Step Three: Substitute the ordered pair found in Step Two into the original equations of the system to determine if this point is the solution of the system of equations.

$x+y=-2 \Rightarrow (-3)+(1)=-2 \therefore (-3,1)$ is a solution of the first equation.

$2x-3y=-9 \Rightarrow 2(-3)-3(1)=-9 \therefore (-3,1)$ is a solution of the second equation.

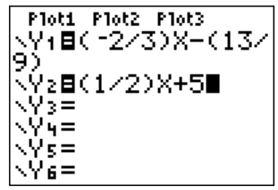
So, $\boxed{(-3,1)}$ is the solution of the system of equations.

Solving a Linear System on the Graphing Calculator:

Ex 4: Solve the linear system $\begin{aligned} 6x+9y&=-13 \\ -x+2y&=10 \end{aligned}$ on the graphing calculator.

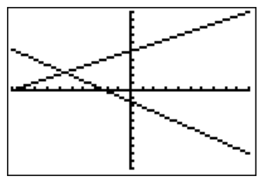
Step One: Solve both equations for *y*. (Rewrite in slope-intercept form.)

$$\begin{aligned} 6x+9y&=-13 \\ 9y&=-6x-13 \\ y&=-\frac{2}{3}x-\frac{13}{9} \end{aligned} \qquad \begin{aligned} -x+2y&=10 \\ 2y&=x+10 \\ y&=\frac{1}{2}x+5 \end{aligned}$$



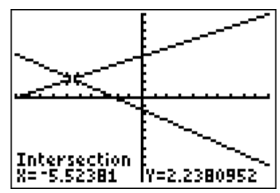
Step Two: Type the two equations into Y1 and Y2.

Step Three: Graph the equations.



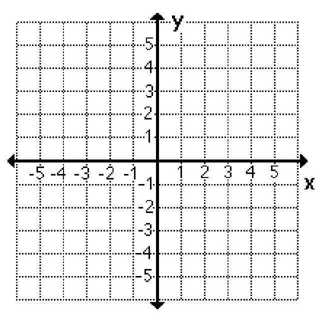
Step Four: Use the *Intersect* command to find the point of intersection.

Note: You may enter in a value for the "guess" or arrow over to the point.



Approximate Solution: $(-5.524, 2.238)$

You Try: Solve the system by graphing. Check your answer on the graphing calculator. $\begin{aligned} x-2y&=6 \\ y&=-3x+4 \end{aligned}$



QOD: Explain what the graph of a system of two equations with no solution would look like.

Solving a System of Linear Equations by Substitution:

Ex 5: Solve the system $\begin{matrix} 3x-5y=27 \\ x+4y=-8 \end{matrix}$ by substitution.

Step One: Solve one of the equations for one of the variables (if necessary). Note: You may choose which variable to solve for.

$$x+4y=-8$$
$$x=-4y-8$$

Step Two: Substitute the expression from Step One into the *other* equation of the system and solve for the other variable.

$$3(-4y-8)-5y=27$$
$$-12y-24-5y=27$$
$$-17y=51$$
$$y=-3$$

Step Three: Substitute the value from Step Two into the equation from Step One and solve for the remaining variable.

$$x=-4(-3)-8$$
$$x=4$$

Step Four: Write your answer as an ordered pair and check in both of the original equations.

Solution: $(4,-3)$

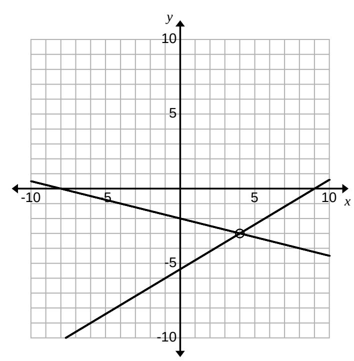
Step Five: Check your solution by substituting the ordered pair into both equations.

$$3x-5y=27 \Rightarrow 3(4)-5(-3)=12+15=27$$
$$x+4y=-8 \Rightarrow (4)+4(-3)=4-12=-8$$

You can also check the answer by graphing on the graphing calculator.

Write each equation in slope-intercept form:

$3x-5y=27$	$x+4y=-8$
$-5y=-3x+27$	$4y=-x-8$
$y=\frac{3}{5}x-\frac{27}{5}$	$y=-\frac{1}{4}x-2$



Graph and find the point of intersection:

Ex 6: Solve the system $\begin{array}{l} 2x+6y=15 \\ 2y=x \end{array}$ by substitution.

Use the steps listed above.

Step One: This step is complete. The second equation is solved for *x*.

Step Two: $\begin{array}{l} 2(2y)+6y=15 \\ 4y+6y=15 \end{array} \Rightarrow \begin{array}{l} 10y=15 \\ y=\dfrac{3}{2} \end{array}$

Step Three: $x=2y \Rightarrow x=2\left(\dfrac{3}{2}\right)=3$

Step Four: $2x+6y=15 \Rightarrow 2(3)+6\left(\dfrac{3}{2}\right)=6+9=15$

$2y=x \Rightarrow 2\left(\dfrac{3}{2}\right)=3$

Solution: $\boxed{\left(3,\dfrac{3}{2}\right)}$

<u>You Try:</u> Solve the system $\begin{array}{l} 2x+y=6 \\ 3x-2y=2 \end{array}$ by substitution. Check your answer by substituting the ordered pair into both of the original equations.

<u>QOD</u>: The first step of solving a system of equations by substitution is to solve one equation for one of the variables. Describe how you would choose which variable you would solve for.

Sample Exam Questions

1. What is the *x*-coordinate of the point of intersection for the two lines below?

$$\begin{array}{l} \mathbf{y=x-2} \\ \mathbf{4x+4y=8} \end{array}$$

A. –2

B. 0

C. $\dfrac{5}{4}$

D. 2

Ans: D

2. **A system of equations is shown below.**

$$\begin{cases} 3x - 2y = 8 \\ -x + 3y = -5 \end{cases}$$

What is the solution of the system of equations?

A $(4,2)$

B $(2,-1)$

C $\left(8,-\frac{5}{2}\right)$

D $\left(\frac{14}{3},3\right)$

Ans: B

3. The table shows points on two linear functions, *f and g*.

x	–2	–1	0	1	2	3
$f(x)$	–0.4	0.1	0.6	1.1	1.6	2.1
$g(x)$	–7.0	–4.6	–2.2	0.2	2.6	5.0

What is the approximate x-value of the intersection of $y = f(x)$ and $y = g(x)$?

(A) $x \approx -1.2$

(B) $x \approx 0.6$

(C) $x \approx 0.9$

(D) $x \approx 1.5$

Ans: D

4. What is the x-coordinate of the point of intersection of these two lines?

$$\begin{cases} y = -2x - 5 \\ 4x + y = 1 \end{cases}$$

(A) –11

(B) 1

(C) 3

(D) The lines do not intersect.

Ans: C

5. Use this system of equations.

$$\begin{cases} -4x+2y=8 \\ 3x+10y=6 \end{cases}$$

If the second equation is rewritten as

3*x* + 5(*m*) = 6,

which expression is equivalent to *m*?

(A) $-3x + 6$

(B) $2x + 4$

(C) $4x + 8$

Ans: C

Solving a System of Equations by Linear Combinations (Elimination):

Some systems of equations do not have an equation that can be solved "nicely" for one of the variables. If this occurs, we can solve the system using a new method. This method has many names – linear combinations, elimination method, addition method…

Ex 7: Solve the system $\begin{matrix} 5x-3y=1 \\ 4x-6y=-10 \end{matrix}$ by linear combinations.

Step One: Write the two equations in standard form.

$$\begin{aligned} 5x-3y&=1 \\ 4x-6y&=-10 \end{aligned}$$

Step Two: Multiply one or both of the equations by a constant to obtain coefficients that are opposites for one of the variables.

We can multiply the first equation by -2 to obtain a *y*-coefficient of 6 in the first equation (the opposite of -6)

$$\begin{aligned} 5x-3y=1 &\Rightarrow -10x+6y=-2 \\ 4x-6y=-10 &\Rightarrow 4x-6y=-10 \end{aligned}$$

Step Three: Add the two equations from Step Two. One of the variable terms should be eliminated. Solve for the remaining variable.

$$-6x + 0y = -12$$
$$-6x = -12$$
$$x = 2$$

Step Four: Substitute the value from Step Three into either one of the original equations to solve for the other variable.

$$5(2) - 3y = 1$$
$$10 - 3y = 1$$
$$-3y = -9$$
$$y = 3$$

Step Five: Write your answer as an ordered pair and check in the original system.

Solution: $\boxed{(2,3)}$

$$5x - 3y = 1 \Rightarrow 5(2) - 3(3) = 10 - 9 = 1$$
$$4x - 6y = -10 \Rightarrow 4(2) - 6(3) = 8 - 18 = -10$$

Ex 8: Solve the system $\begin{matrix} 5x + 3y = 9 \\ -2x - 5y = 23 \end{matrix}$ by linear combinations. Use the steps listed above.

Step One: This step is complete. The equations are written in standard form.

Step Two: Eliminate the *x* term by multiplying the first equation by 2 and the second equation by 5. (Note: You could have also eliminated the *y* term by multiplying the first equation by 5 and the second equation by 3.)

$$\begin{matrix} 5x + 3y = 9 & \Rightarrow \times 2 \Rightarrow & 10x + 6y = 18 \\ -2x - 5y = 23 & \Rightarrow \times 5 \Rightarrow & -10x - 25y = 115 \end{matrix}$$

Step Three: $\begin{matrix} -19y = 133 \\ y = -7 \end{matrix}$

Step Four: $\begin{matrix} 5x + 3y = 9 \\ 5x + 3(-7) = 9 \end{matrix} \Rightarrow \begin{matrix} 5x - 21 = 9 \\ 5x = 30 \\ x = 6 \end{matrix}$

Step Five:
$$5x+3y=9 \Rightarrow 5(6)+3(-7)=30-21=9$$
$$-2x-5y=23 \Rightarrow -2(6)-5(-7)=-12+35=23$$

Solution: $\boxed{(6,-7)}$

Ex 9: Solve the system $\begin{matrix} 5b-20=-4a \\ b=-\frac{5}{4}a+4 \end{matrix}$ by linear combinations. Use the steps listed above.

Step One:
$$\begin{matrix} 5b-20=-4a \\ 4a+5b=20 \end{matrix} \qquad \begin{matrix} b=-\frac{5}{4}a+4 \\ 4b=-5a+16 \\ 5a+4b=16 \end{matrix}$$

Standard Form: $\begin{matrix} 4a+5b=20 \\ 5a+4b=16 \end{matrix}$

Step Two: Eliminate the a term by multiplying the first equation by −5 and the second equation by 4. (Note: You could have also eliminated the b term by multiplying the first equation by −4 and the second equation by 5.)

$$4a+5b=20 \Rightarrow \times -5 \Rightarrow -20a-25b=-100$$
$$5a+4b=16 \Rightarrow \times 4 \quad \Rightarrow \quad 20a+16b=64$$

Step Three:
$$-9b=-36$$
$$b=4$$

Step Four:
$$\begin{matrix} 5b-20=-4a \\ 5(4)-20=-4a \end{matrix} \quad \Rightarrow \quad \begin{matrix} 0=-4a \\ a=0 \end{matrix}$$

Step Five:
$$5b-20=-4a \Rightarrow 5(4)-20=-4(0) \Rightarrow 0=0$$
$$b=-\frac{5}{4}a+4 \Rightarrow 4=-\frac{5}{4}(0)+4 \Rightarrow 4=4$$

Solution: $\boxed{(0,4)}$

<u>You Try:</u> Solve the system $\begin{matrix} 15x+2y=31 \\ 4x+6y=11 \end{matrix}$ by linear combinations.

<u>QOD</u>: When solving a system of linear equations by linear combinations, why is it important to write the equations in standard form?

Sample Exam Questions

1. What is the *y*-coordinate of the point of intersection for the two lines below?

$$\begin{aligned} x - y &= 9 \\ x + 6y &= -12 \end{aligned}$$

A. –6
B. –3
C. 3
D. 6

Ans: B

2. A system of equations is shown below.

$$\begin{cases} 2x - 4y = -5 \\ 3x + 5y = 9 \end{cases}$$

What is the solution of the system of equations?

A $(2, 2.25)$
B $(0.5, 1.5)$
C $(-0.5, 1)$
D $(-2, 3)$

Ans: B

3. Use the system of equations.

$$\begin{cases} x + 8y = 3 \\ 2x - 2y = -7 \end{cases}$$

Which step(s) would create equations so that the coefficients of one of the variables are opposites?

(A) Multiply the first equation by 7. Multiply the second equation by 3.

(B) Multiply the first equation by –2. Multiply the second equation by 4.

(C) Multiply the first equation by 2.

(D) Multiply the second equation by 4.

Ans: D

Application Problems

When solving an application problem, it is helpful to have a problem solving plan. We will use the following plan to solve the application problems that follow.
Problem-Solving Plan:

Step One: Write a verbal model.

Step Two: Assign labels.

Step Three: Write an algebraic model.

Step Four: Solve the algebraic model using one of the methods for solving a system of equations.

Step Five: Answer the question asked and label the answer appropriately.

Application Problems with Systems of Equations

Ex 10: A sporting goods store receives a shipment of 124 golf bags. The shipment includes two types of bags, full-size and collapsible. The full-size bags cost \$38.50 each. The collapsible bags cost \$22.50 each. The bill for the shipment is \$3430. How many of each type of golf bag are in the shipment?

Step One: # of Full-Size Bags + # of Collapsible Bags = Total # of Golf Bags in the Shipment

Cost of Full-Size Bags · # of Full-Size Bags + Cost of Collapsible Bags · # of Collapsible Bags = Cost of Shipment

Step Two: # of Full-Size Bags = F # of Collapsible Bags = C Total # of Bags = 124
Cost of Full-Size Bags = 38.50   Cost of Collapsible Bags = 22.50
Cost of Shipment = 3430

Step Three:
$$F + C = 124$$
$$38.5F + 22.5C = 3430$$

Step Four: We will use substitution.

$$F = 124 - C$$
$$38.5(124 - C) + 22.5C = 3430$$
$$4774 - 38.5C + 22.5C = 3430$$
$$-16C = -1344$$
$$C = 84$$

$$F = 124 - C$$
$$F = 124 - 84$$
$$F = 40$$

Step Five: There are 40 full-size bags and 84 collapsible bags in the shipment.

Ex 11: Sheila and a friend share the driving on a 280 mile trip. Sheila's average speed is 58 miles per hour. Her friend's average speed is 53 miles per hour. Sheila drives one hour longer than her friend. How many hours did each of them drive?

Step One: We will use the equation $d = rt$.

Sheila's Distance + Friend's Distance = 280
Sheila's Distance = Sheila's Average Rate · Sheila's Time
Friend's Distance = Friend's Average Rate · Friend's Time
Sheila's Time = 1 + Friend's Time

Step Two: Sheila's Distance = d_S Friend's Distance = d_F
Sheila's Avg Speed = 58 Friend's Avg Speed = 53
Sheila's Time = t_S Friend's Time = t_F

Step Three: $d_S + d_F = 280$ $\quad d_S = 58t_S$, $d_F = 53t_F$ $\quad t_S = 1 + t_F$

Step Four: We will use substitution.

$$\begin{aligned} 58t_S + 53t_F &= 280 \\ 58(1+t_F) + 53t_F &= 280 \end{aligned} \Rightarrow \begin{aligned} 58 + 58t_F + 53t_F &= 280 \\ 111t_F &= 222 \\ t_F &= 2 \end{aligned} \qquad \begin{aligned} t_S &= 1 + t_F \\ t_S &= 1 + 2 = 3 \end{aligned}$$

Step Five: Sheila drove 3 hours, and her friend drove 2 hours.

Ex 12: A chemist needs to make 200L of a 62% solution by mixing together an 80% solution with a 30% solution. How much of each solution should she use?

Step One: Amount of 80% Sol. + Amount of 30% Sol. = Total Amount of 62% Sol.

% of Sol. · Amount of 80% Sol. + % of Sol. · Amount of 30% Sol. = % of Sol. · Amount of 62% Sol.

Step Two: Amount of 80% Sol. = *E* Amount of 30% Sol. = *T* Amount of 62% Sol. = 200
% of Sol. ⇒ We will write all percents as decimals: 62% = 0.62, 80% = 0.8, 30% = 0.3

Step Three:

$$\begin{aligned} E + T &= 200 \\ 0.8E + 0.3T &= 0.62(200) \end{aligned} \Rightarrow \begin{aligned} E + T &= 200 \\ 0.8E + 0.3T &= 124 \end{aligned}$$

Step Four: We will use linear combinations. (Multiply the first equation by -0.3 to eliminate the *T* term.)

$$\begin{aligned} -0.3E - 0.3T &= -60 \\ 0.8E + 0.3T &= 124 \end{aligned} \Rightarrow \begin{aligned} 0.5E &= 64 \\ E &= 128 \end{aligned} \qquad \begin{aligned} E + T &= 200 \\ 128 + T &= 200 \\ T &= 72 \end{aligned}$$

Step Five: The chemist should use 128L of the 80% solution and 72L of the 30% solution.

Ex 13: Shayna is 3 times as old as Tara. In 4 years, the sum of their ages will be 56. How old are Shayna and Tara?

Step One: Shayna's Age = 3 · Tara's Age

(Shayna's Age + 4) + (Tara's Age + 4) = 56

Step Two: Shayna's Age = *s* Tara's Age = *t*

Step Three: $s = 3t$

$$(s+4)+(t+4)=56$$
$$s+t+8=56$$
$$s+t=48$$

Step Four: We will use substitution.

$$(3t)+t=48 \qquad s=3t$$
$$4t=48 \qquad s=3(12)$$
$$t=12 \qquad s=36$$

Step Five: Shayna is 36 years old and Tara is 12 years old.

Graphing Calculator Note: You can solve any of these application problems on the graphing calculator by graphing and finding the intersection.

<u>You Try:</u>

1. The sum of two numbers is 82. One number is 12 less than 3 times the other. Find the numbers.

2. A health store wants to make trail mix with raisins and granola. The owner mixes granola, which costs \$4 per pound, and raisins, which cost \$2 per pound, together to make 25 lbs of trail mix. How many pounds of raisins should he include if he wants the mixture to cost him a total of \$80?

<u>QOD:</u> Write a unique application problem that involves systems of equations.

Sample Exam Questions

1. **Jason has 20 coins in nickels and quarters. He has a total of \$3.40 in coins. How many quarters does Jason have?**

A. 10
B. 12
C. 13
D. 18

Ans: B

2. A movie theater sells regular tickets (*r*) for \$12 each. Students receive discounted tickets (*d*) for \$9 each. One evening the theater sold 834 tickets and collected \$9,258 in revenue. Which system of linear equations can be used to determine the number of student tickets sold?

A. $\begin{cases} r + d = 9258 \\ 12r + 9d = 834 \end{cases}$

B. $\begin{cases} r + d = 9258 \\ 9r + 12d = 834 \end{cases}$

C. $\begin{cases} r + d = 834 \\ 12r + 9d = 9258 \end{cases}$

D. $\begin{cases} r + d = 834 \\ 9r + 12d = 9258 \end{cases}$

Ans: C

3. Michael has 34 coins in nickels and dimes. The total value of the coins is \$2.45. If Michael has *d* dimes and *n* nickels, which system of equations can be used to find the number of each coin?

(A) $\begin{cases} d + n = 15 \\ 5d + 10n = 245 \end{cases}$

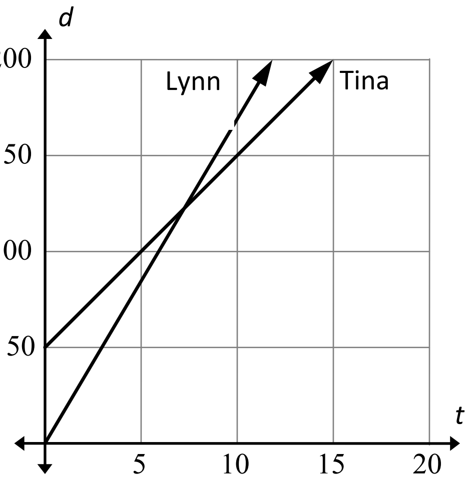
(B) $\begin{cases} d + n = 15 \\ 10d + 5n = 245 \end{cases}$

(C) $\begin{cases} d + n = 34 \\ 5d + 10n = 245 \end{cases}$

(D) $\begin{cases} d + n = 34 \\ 10d + 5n = 245 \end{cases}$

Ans: D

4. Lynn and Tina are planning a foot race. Lynn can run 16.9 feet per second and Tina can run 10 feet per second. Lynn gives Tina a 50-foot head start. The diagram below shows distance-time graphs for Lynn and Tina.



After about how much time will Lynn pass Tina?

(A) 5 seconds

(B) 7 seconds

(C) 10 seconds

(D) 12 seconds

Ans: B

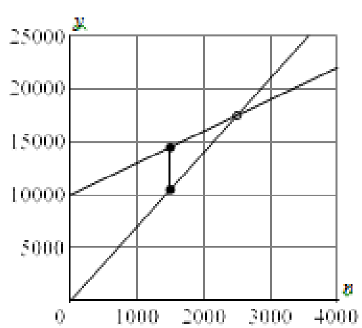
5. A toy company is manufacturing a new doll. The cost of producing the doll is $10,000 to start plus $3 per doll. The company will sell the doll for $7 each.

(a) Write functions $C(n)$ and $I(n)$ to represent the cost of producing the dolls and income from selling the dolls, respectively.

$$C(n) = \$10{,}000 + \frac{\$3}{\text{doll}}n$$

$$I(n) = \frac{\$7}{\text{doll}}n$$

(b) Graph the functions.



(c) How many dolls must be produced for the company to break even, i.e. $C(n) = I(n)$?

$$\$10{,}000 + \frac{\$3}{\text{doll}}n = \frac{\$7}{\text{doll}}n$$
$$\$10{,}000 = \frac{\$4}{\text{doll}}n$$
$$\frac{\$10{,}000}{\frac{\$4}{\text{doll}}} = n$$
$$2500 \text{ dolls} = n$$

(d) Compute $I(1500) - C(1500)$. What does this mean for the company?

$I(1500) - C(1500) = \$10{,}500 - \$14{,}500 = -\$3{,}000$, so if the company only produced and sold 1500 dolls, they would lose \$3,000.

Big Idea: identify the number of solutions to a system of equations.

Choosing an Appropriate Method: Substitution is the method of choice when one of the equations is easily solvable (or already solved) for one of the variables. If this is not the case, use elimination (linear combinations) to solve the system.

Ex 14: Which method would be BEST for solving the following system of equations?

a) $\begin{aligned} 3x - 7y &= 20 \\ -11x + 10y &= 5 \end{aligned}$ Elimination

b) $\begin{aligned} y &= x + 4 \\ y &= 2x + 5 \end{aligned}$ Substitution

Note: For extra practice, solve each system of equations using the method you chose.

Special Cases:

- If the graphs of the equations in a system are **parallel** (do not intersect), then the system has NO SOLUTION.
- If the graphs of the equations in a system are the same line (**coincident**), then the system has INFINITELY MANY SOLUTIONS.

⚠Note: If the graphs are not coincident and intersect, then the system has EXACTLY ONE SOLUTION.

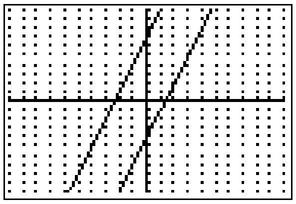
Ex 15: How many solutions does each system of equations have?

a) $\begin{aligned} -6x + 2y &= -8 \\ -3x + y &= 7 \end{aligned}$

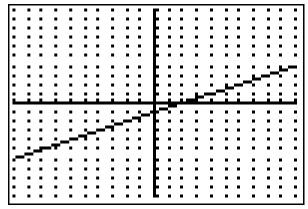
b) $\begin{aligned} -x + 2y &= -2 \\ 3x - 6y &= 6 \end{aligned}$

Step One: Graph both equations on the same coordinate plane.

a)



b)



b)

Step Two: Using the graph, determine the number of solutions.

a) The lines are parallel, therefore there is *no solution* to the system.

b) The lines are coincident (the same line), therefore there are *infinitely many solutions* to the system.

Special Cases: As we know from solving systems of equations by graphing, systems of equations can have exactly one solution, infinitely many solutions, or no solution.

System of Equations with Infinitely Many Solutions:

Ex 16: Solve the system $\begin{array}{l}-9x+6y=0\\-12x+8y=0\end{array}$ using the method of your choice.

Note: Because the equations are in standard form, and are not easily solvable for one of the variables, we will use linear combinations.

Step One: Done. The equations are in standard form.

Step Two: Multiply the first equation by 4 and the second equation by -3 to eliminate the y terms.

$$\begin{array}{l}-9x+6y=0 \Rightarrow \times 4 \quad \Rightarrow -36x+24y=0\\-12x+8y=0 \Rightarrow \times -3 \Rightarrow \quad 36x-24y=0\end{array}$$

Step Three: $\begin{array}{l}0x+0y=0\\0=0\end{array}$ Note: Both of the variables were eliminated!

If both of the variables are eliminated, and we end up with a **true** statement (i.e. $a=a$), then the equation has INFINITELY MANY SOLUTIONS.

Note: If we were to graph these two equations, the two lines would be coincident.

<u>Systems of Equations with No Solution:</u>

Ex 17: Solve the system $\begin{matrix} -2x+y=6 \\ 4x-2y=5 \end{matrix}$ using the method of your choice.

Note: Because the first equation is easily solvable for *y*, we will use substitution.

Step One: $-2x+y=6 \Rightarrow y=2x+6$

Step Two:
$$4x-2(2x+6)=5$$
$$4x-4x-12=5$$
$$-12=5$$
Note: The variable was eliminated!

If the variable is eliminated, and we end up with a **false** statement (i.e. $a \neq b$), then the equation has NO SOLUTION.

Note: If we were to graph these two equations, the two lines would be parallel.

<u>You Try:</u> Solve the system of equations using a method of your choice. State how many solutions the system has and describe the graphs of the lines.

1. $\begin{matrix} 3x+y=-1 \\ -9x-3y=3 \end{matrix}$
2. $\begin{matrix} y=-2x+4 \\ 4x-2y=0 \end{matrix}$
3. $\begin{matrix} x-5=2y \\ 4y-2x=2 \end{matrix}$

<u>QOD:</u> When choosing a method for solving a system of equations, when would you use linear combinations, and when would you use substitution?

Sample Exam Questions

1. How many solutions does the system have?

$$\begin{matrix} -x+4y=-20 \\ 3x-12y=4 \end{matrix}$$

A. no solution
B. one solution
C. two solutions
D. infinitely many solutions

Ans: A

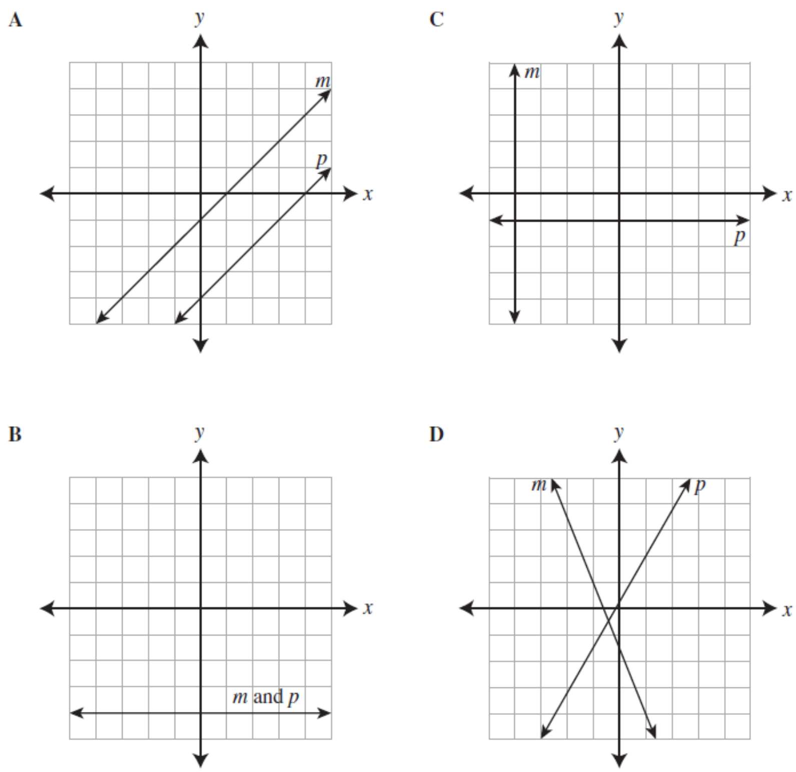
2. How many solutions does the system have?

$$-4x + 2y = -\frac{8}{3}$$
$$6x - 2y = 4$$

A. no solution
B. one solution
C. two solutions
D. infinitely many solutions

Ans: B

3. In which graph do line *m* and line *p* represent a system of equations that has NO solution?



Ans: A

4. Use the system of linear equations.

$$\begin{cases} kx - 3y = 8 \\ 2x - 6y = m \end{cases}$$

Which values of k and m make the lines parallel?

(A) $k = -2, m = -16$

(B) $k = 1, m = 10$

(C) $k = 1, m = 16$

(D) $k = 2, m = 8$

Ans: C

5. Use the system of equations.

$$\begin{cases} -8x - 4y = -64 \\ 2x + y = 16 \end{cases}$$

Which describes the solution set of the system?

(A) There is a single solution of (0, 16).

(B) There is a single solution of (8, 0).

(C) There are no solutions to the system.

(D) There are an infinite number of solutions to the system.

Ans: D

6. How many solutions does the system of equations have?

$$\begin{cases} -6x - 4y = -64 \\ 3x + 2y = 32 \end{cases}$$

(A) no solution

(B) one solution

(C) two solutions

(D) infinitely many solutions

Ans: D

Units 7.3 To solve and graph systems of linear inequalities.

VOCABULARY:

System of Linear Inequalities: a set of two or more linear inequalities

Solution of a System of Linear Inequalities: the ordered pairs that satisfy all of the linear inequalities in the system

Testing if an Ordered Pair is a Solution to a System of Linear Inequalities

Ex 18: Use the system of linear inequalities $\begin{matrix} 2x - y \ge 4 \\ y < 2x - 6 \end{matrix}$. Is the given point a solution?

a) $(0,0)$ b) $(7,5)$ c) $(0,-6)$

Test each point in both inequalities. It is a solution if and only if it satisfies **both** inequalities.

a) $2x - y \ge 4$; $2(0) - (0) \ge 4$; $0 \ge 4 \Rightarrow \text{false}$

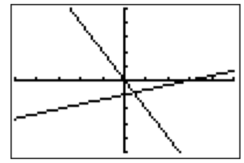
b) $2x - y \ge 4$; $2(7) - (5) \ge 4$; $9 \ge 4 \Rightarrow \text{true}$ — $y < 2x - 6$; $5 < 2(7) - 6$; $5 < 8 \Rightarrow \text{true}$

c) $2x - y \ge 4$; $2(0) - (-6) \ge 4$; $6 \ge 4 \Rightarrow \text{true}$ — $y < 2x - 6$; $-6 < 2(0) - 6$; $-6 < -6 \Rightarrow \text{false}$

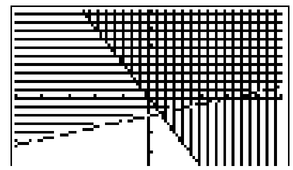
So $(0,0)$ is NOT a solution. So $(7,5)$ IS a solution. So $(0,-6)$ IS NOT a solution

Graphing a System of Linear Inequalities

Ex 19: Graph the system $\begin{array}{l} y-2\geq -2(x+1) \\ x-3y<3 \end{array}$.

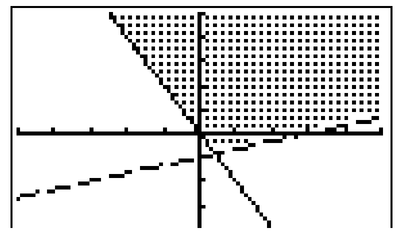


Step One: Graph each line on the same coordinate plane.



Step Two: Determine whether to use solid or dashed lines. (Recall: Use solid lines for $\leq$ and $\geq$, and use dashed lines for < and >.)

Step Three: Lightly shade the appropriate half-planes for each inequality.



Step Four: The solution to the system is the overlapping region formed by the shading in Step Three. Shade only this region.

Step Five: Choose a point in the shaded region and test it in the original system of inequalities.

Choose $(0,2)$.

$$\begin{array}{ll} y-2\geq -2(x+1) & x-3y<3 \\ (2)-2\geq -2(0+1) & (0)-3(2)<3 \\ 0\geq -2 \Rightarrow \text{true} & -6<3 \Rightarrow \text{true} \end{array}$$

Ex 20: Graph the system of linear inequalities. Label each vertex of the solution region. Describe the shape of the region.

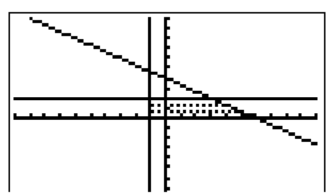
$x\geq -1$

$y\geq 0$

$y\leq 2$

$y\leq -\dfrac{2}{3}x+4$

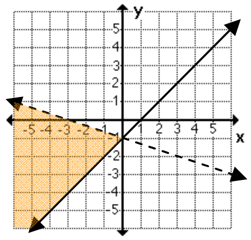
Graph:

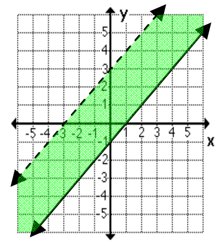


Vertices: $(-1,0),(-1,2),(3,2),(6,0)$

The shape of the region is a TRAPEZOID.

Ex 21: Write a system of inequalities for each graph.

a) 

b) 

Step One: Write the equations of the lines.

a) $y = -\frac{1}{3}x - 1$, $y = x - 1$ b) $y = x + 3$, $y = x - 1$

Step Two: Write the inequalities for each line.

a) $y < -\frac{1}{3}x - 1$, $y \geq x - 1$ b) $y < x + 3$, $y \geq x - 1$

<u>Application Problems Involving Systems of Inequalities</u>: Use the problem-solving plan.

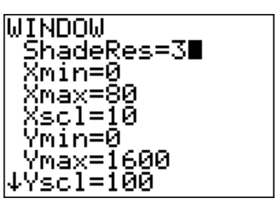
Ex 22: A contractor needs at least 500 bricks and 10 bags of sand. Bricks weigh 2 lb each and sand weighs 50 lb per bag. The maximum weight that can be delivered is 3000 lb. Write and graph a system of inequalities that represents the situation.

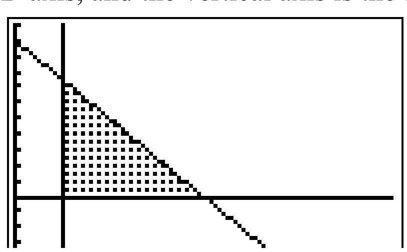
Step One: # of Bricks $\geq$ 500 # of Bags of Sand $\geq$ 10
Weight of Bricks · # of Bricks + Weight of Sand · # of Bags of Sand $\leq$ 3000

Step Two: # of Bricks = *B* # of Bags of Sand = *S* Weight of Bricks = 2 Weight of Sand = 50

Step Three: $B \geq 500$ $S \geq 10$ $2B + 50S \leq 3000$

Step Four: Graph (Note: The horizontal axis is the *B*-axis, and the vertical axis is the *S*-axis.)

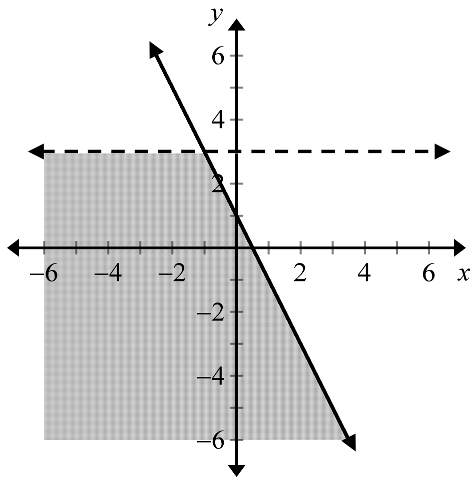




<u>QOD</u>: Describe how to determine if an ordered pair is a solution of a system of linear inequalities.

Sample Exam Questions

1. Which system of inequalities is graphed below?



A. $\begin{cases} y < 3 \\ y < -2x + 1 \end{cases}$

B. $\begin{cases} y < 3 \\ y \le -2x + 1 \end{cases}$

C. $\begin{cases} y > 3 \\ y > -2x + 1 \end{cases}$

D. $\begin{cases} y > 3 \\ y \ge -2x + 1 \end{cases}$

Ans: B

2. Which point is in the solution set for the system of inequalities $x - y > 1$ and $y < 2x - 1$?

A. $(-2, -4)$

B. $(-1, 1)$

C. $(0, -2)$

D. $(2, 2)$

Ans: C

Units 7.4, 7.5 To graph constraints and identify vertices of the feasible region and To use the vertices to maximize or minimize the objective function.

Vocabulary:

Linear Programming: the process of optimizing a linear function (called the **objective function**) based upon a system of linear inequalities, which are called **constraints**.

Feasible Region: the graph of a system of constraints

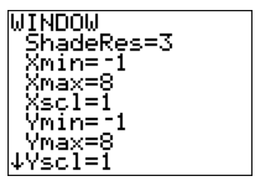
Optimization: the process of finding the minimum or maximum value of a quantity

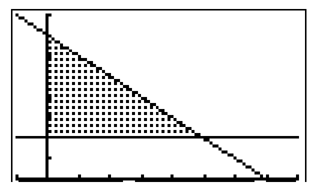
> Note: In a linear programming problem, the maximum or minimum always occurs at one of the **vertices (corner points)** of the feasible region.

Solving a Linear Programming Problem

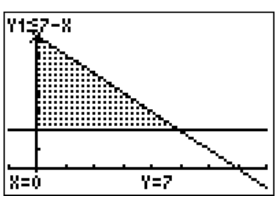
Ex 23: Find the minimum and maximum value of the function $P = 2x + 3y$ subject to the constraints $\begin{matrix} x \geq 0 \\ y \geq 2 \\ x + y \leq 7 \end{matrix}$.

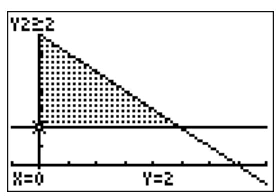
Step One: Graph the feasible region. (Use the constraints.)

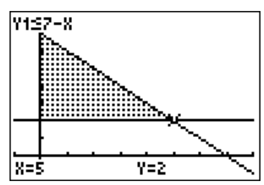




Step Two: Find the coordinates of the vertices of the shaded region (triangle).







The vertices are $(0,7), (0,2)$, and $(5,2)$.

Step Three: Evaluate the objective function for each of the vertices.

$(0,7): P = 2(0) + 3(7) = 21$

$(0,2): P = 2(0) + 3(2) = 6$

$(5,2): P = 2(5) + 3(2) = 16$

Solution: The maximum value of the function *P* is 21. It occurs when $x = 0$ and $y = 7$. The minimum value of the function *P* is 6. It occurs when $x = 0$ and $y = 2$.

Ex 24: Rei volunteers to bring origami swans and giraffes to sell at a charity crafts fair. It takes her three minutes to make a swan and six minutes to make a giraffe. She plans to sell the swans for \$4 each and the giraffes for \$6 each. If she only has 16 pieces of origami paper and can't spend more than one hour folding, how many of each animal should Rei make to **maximize** the charity's profit?

Step One: Write the objective function (what is being maximized)

Let $s =$ swans and let $g =$ giraffes

Objective function (profit): $P = 4s + 6g$

Step Two: Write the constraint inequalities from the information given in the problem.

Time to make the origami (in minutes): $3s + 6g \le 60$

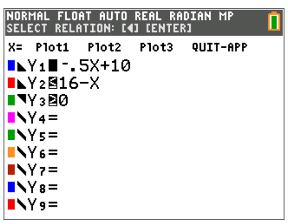
Only a limited amount of paper: $s + g \le 16$

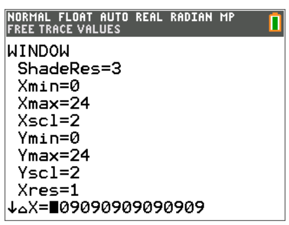
Number of swans: $s \ge 0$

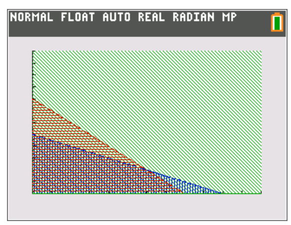
Number of giraffes: $g \ge 0$

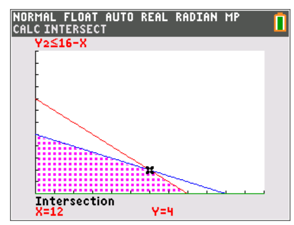
Step Three: Graph the constraint inequalities to find the feasible region.

Using a graphing calculator, solve the equations for g and find the shaded region.









Step Four: Identify the vertices (corner points) of the feasible region.

The vertices are (0, 0), (16, 0), (0, 10) and (12, 4).

Step Five: Evaluate the objective function for the vertices to find the maximum profit.

$$(0,0): P = 4(0) + 6(0) = 0$$
$$(16,0): P = 4(16) + 6(0) = 64$$
$$(0,10): P = 4(0) + 6(10) = 60$$
$$(12,4): P = 4(12) + 6(4) = 72$$

Therefore, the maximum profit is \$72. Rei should make 12 swans and 4 giraffes to maximize the charity's profit.

Ex 25: It takes a company 2 hours to manufacture a pair of skis and 1 hour to manufacture a snowboard. The finishing time for both skis and snowboards is 1 hour. The maximum time available for manufacturing is 40 hours and for finishing is 32 hours each week. The manufacturer must produce at least 8 snowboards every week. The profit for a pair of skis is \$70, and the profit for a snowboard is \$50. Write an objective function for profit and use linear programming to find the maximum profit. How many skis and snowboards should the company manufacture to **maximize the profit**?

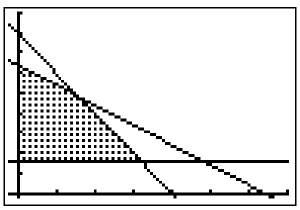
Step One: Write the objective function (what is being maximized/minimized).

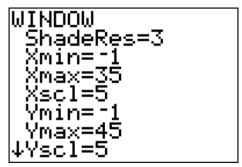
$P = 70S + 50B$ (S = # of skis, B = # of snowboards)

Step Two: Write and graph the constraints.

Manufacturing Time: $2S + B \leq 40$ Finishing Time: $S + B \leq 32$

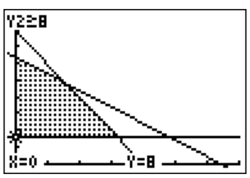
Number of Skis: $S \geq 0$ Number of Snowboards: $B \geq 8$

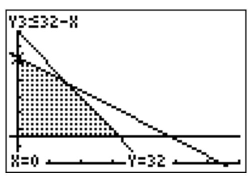


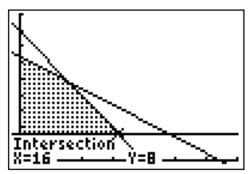


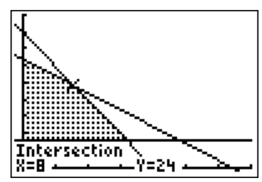
Note: The horizontal axis is the *S*-axis, and the vertical axis is the *B*-axis.

Step Three: Find the coordinates of the vertices of the feasible region.









The vertices are $(0,8),(0,32),(16,8),$ and $(8,24)$.

Step Four: Evaluate the objective function for the vertices to find the maximum profit.

$(0,8): P = 70(0) + 50(8) = 400$

$(0,32): P = 70(0) + 50(32) = 1600$

$(16,8): P = 70(16) + 50(8) = 1520$

$(8,24): P = 70(8) + 50(24) = 1760$

Therefore, the maximum profit is \$1760. The company should manufacture 8 skis and 24 snowboards each week to maximize the profit.

You Try: A company makes tape players (for a profit of $28 each) and CD players (for a profit of $33 each). The company wants to produce at least 60 tape players and 100 CD players per day, but can't produce more than 200 total combined. Write an objective function for profit and use linear programming to find the maximum profit. How many tape players and CD players should the company manufacture to maximize the profit?

QOD: Are the vertices of a feasible region the only possible points that satisfy an objective function? Explain your answer.

Sample Exam Questions

1. In a community service program, students earn points for two tasks: painting over graffiti and picking up trash. The following constraints are imposed on the program.

1) A student may not serve more than 10 total hours per week.

2) A student must serve at least 1 hour per week at each task.

Let g = the number of hours a student spends in a week painting over graffiti.
Let t = the number of hours a student spends in a week picking up trash.

Which system represents the imposed constraints?

(A) $\begin{cases} g+t \le 10 \\ g \ge 1 \\ t \ge 1 \end{cases}$

(B) $\begin{cases} g+t \le 10 \\ g \ge 0 \\ t \ge 0 \end{cases}$

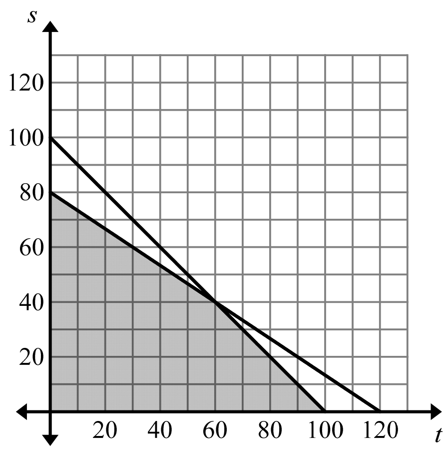
(C) $\begin{cases} g+t \le 8 \\ g \ge 1 \\ t \ge 1 \end{cases}$

(D) $\begin{cases} g+t \le 8 \\ g = t \end{cases}$

Ans: A

2. The volleyball team is having a fundraiser and can purchase t-shirts for $10 and sweatshirts for $15. The team has a budget of $1200. Due to shipping costs, no more than a total of 100 t-shirts and sweatshirts combined can be ordered. Let *t* represent the number of t-shirts sold and *s* represent the number of sweatshirts sold.

The constraints are illustrated in the graph.



The team makes a profit of $6 on each t-shirt and $10 on each sweatshirt. How many of each need to be sold to maximize profit?

The objective function for profit is $\boldsymbol{P = 6t + 10s}$.

(A) 60 t-shirts, 40 sweatshirts

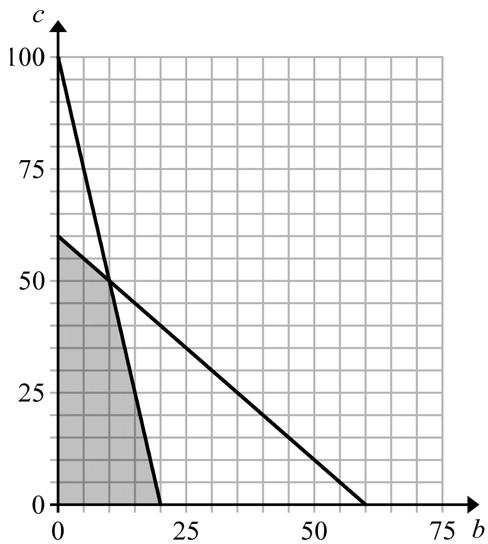
(B) 40 t-shirts, 60 sweatshirts

(C) 0 t-shirts, 80 sweatshirts

(D) 100 t-shirts, 0 sweatshirts

Ans: C

The area of a parking lot is 600 square meters. A car requires 6 square meters and a bus requires 30 square meters of space. The lot can handle a maximum of 60 vehicles. Let *b* represent the number of buses and *c* represent the number of cars. The diagram below represents the feasible region based on the constraints of the number of vehicles that can be parked in the lot.



3. To park in the lot, a bus costs \$8 and a car costs \$3. How many of each type of vehicle can be parked in the lot to maximize the amount of money collected?

A. 0 buses and 60 cars

B. 10 buses and 50 cars

C. 20 buses and 0 cars

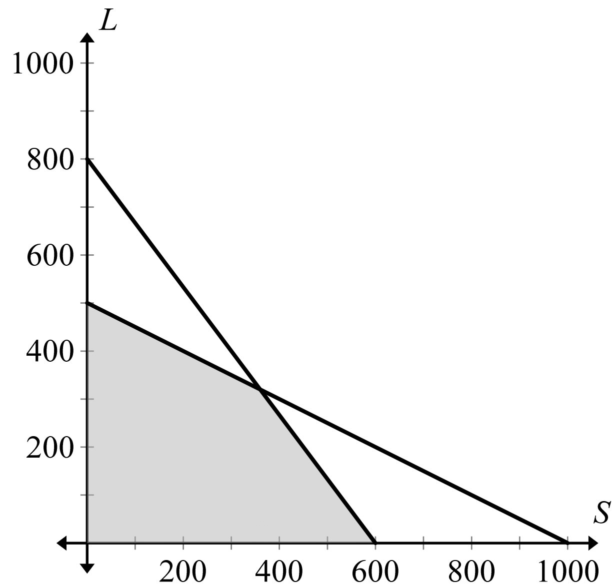
D. 30 buses and 30 cars

Ans: B

4. For a fundraiser, an art club is making paper frogs. Here are some conditions about the fundraiser.

- The club has 500 sheets of paper to make frogs.
- One sheet of paper will produce one large frog.
- One sheet of paper will produce two small frogs.
- The club can produce 15 small frogs per hour.
- The club can produce 20 large frogs per hour.
- The club has 40 hours to produce the frogs.

This graph shows how many of each size frog can be made under the conditions.



(a) Identify the vertices of the shaded region.

The vertices are (0, 0), (600, 0), (0, 500), and the intersection of $L = 500 - \frac{1}{2}S$ and $L = 800 - \frac{4}{3}S$ or (360, 320).

The club will sell large frogs for \$3 each and small frogs for \$2 each. Income is maximized using quantities from at least one of the vertices of the shaded region.

(b) What is the maximum income and how many of each size frog should be produced?

Income is maximized at \$1,680 when 360 small and 320 large frogs are produced.

Vertex	Small (x \$2)	Large (x \$3)	Income
(0, 0)	0	0	\$0
(600, 0)	600	0	\$1,200
(0, 500)	0	500	\$1,500
(360, 320)	360	320	\$1,680

(c) One boundary of the region is $\frac{1}{2}S + L = 500$. Explain what this equation means in context of the situation.

It takes $\frac{1}{2}$ sheet of paper to make a small frog and one sheet of paper to make a large frog. In this equation, $\frac{1}{2}S$ represents the number of sheets of paper used to make small frogs and L represents the number of sheets of paper needed to make large frogs, the total of which may not exceed the 500 sheets of paper the club has.