# Introduction to Polynomials

## Math Background

**Previously, you**

- Identified the components in an algebraic expression
- Factored quadratic expressions using special patterns, grouping method and the ac method
- Worked with properties of integers
- Identified the zeros or roots of a quadratic function
- Recognized special patterns – perfect squares and the difference of squares

**In this unit you will**

- Identify polynomials and perform operations with polynomials
- Use the Binomial Theorem to expand binomials raised to positive integer powers
- Factor and divide polynomials
- Use the Factor Theorem and Remainder Theorem to factor polynomials completely

**You can use the skills in this unit to**

- Analyze the effect of a coefficient and a constant on a variable.
- Identify sums, differences, and products of polynomials as polynomials.
- Use polynomial identities to describe numerical relationships.
- To prove polynomial identities using properties of operations.
- Factor sum and difference of two cubes.
- Apply Pascal's Triangle to the expansion of binomials.
- Use the Binomial Theorem to calculate a specific term in a polynomial or to expand binomials.
- Factor a polynomial using synthetic or long division.

**Vocabulary**

- **Algebraic Expression** – It is a combination of numbers, variables and operation signs to represent a certain quantity.
- **Binomial** – A polynomial representing the addition or subtraction of exactly two distinct terms.
- **Binomial Theorem –** A rule for writing out the expansion of a binomial raised to any positive integer power.
- **Closed** – Describing a set for which a given operation (such as addition or multiplication) gives a result that is also a member of the same set.
- **Combinations** – The number of possible ways of selecting m objects out of n objects when you don't care about the order in which the m objects are arranged.
- **Factor Theorem** – It states that if $f(a)=0$ in which $f(x)$ represents a polynomial in *x*, then $(x-a)$ is one of the factors of $f(x)$.
- **Monomial** – An algebraic expression consisting of only one term. It is either a constant, a variable, or a product of a constant and one or more variables.

- **Pascal's Triangle** – A triangular array of numbers in which each row starts and ends with 1 and each number in between is the sum of the pair of numbers above it. The number at the apex is 1. The numbers at the nth row of Pascal's triangle are the same as the coefficients of $x$ and $y$ in the expansion of $(x+y)^{n-1}$.
- **Polynomial** – An algebraic expression that consists of two or more terms.
- **Polynomial Identity** – They are patterns involving sums and differences of like powers.
- **Remainder Theorem** – The theorem stating that if a polynomial in $x$, $f(x)$, is divided by $(x-a)$, where $a$ is any real or complex number, then the remainder is $f(a)$.
- **Root** – The value(s) of a variable that makes the equation true.
- **Synthetic Division** – A method of performing polynomial long division, with less writing and fewer calculations.
- **Synthetic Substitution** – The process of using synthetic division to evaluate $f(c)$ for a polynomial $f(x)$ and a number $c$.

**Essential Questions**

- Just as operations performed in the subset of decimals or set of integers yield decimals and integers, respectively, do the operations of addition, subtraction, and multiplication across polynomials yield polynomials?
- Why do we rewrite polynomials expressions?

**Overall Big Ideas**

The system of polynomials is closed under the operations of addition, subtraction, and multiplication. Polynomial identities allow us to derive and generate numerical and polynomial relationships such as those seen in Pythagorean triples and the binomial theorem.

Polynomial expressions can sometimes be rewritten to find relevant characteristics for the situation including finding zeros or extrema.

## Skill

**To identify, classify, evaluate, add and subtract polynomials.**

**To multiply polynomials and use binomial expansions to expand a binomial expression raised to positive integer powers.**

**To factor polynomials.**

**To divide polynomials.**

**To apply the Factor Theorem and Remainder Theorem.**

**Related Standards**

**A.SSE.A.1a**

Interpret parts of an expression, such as terms, factors, and coefficients. *(Modeling Standard)

**A.APR.A.1-2**

Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials of degree 2 and higher.

**A.APR.C.4**

Prove polynomial identities and use them to describe numerical relationships. For example, the polynomial identity $\left(x^2+y^2\right)^2=\left(x^2-y^2\right)^2+\left(2xy\right)^2$ can be used to generate Pythagorean triples.

**A.APR.C.5**

Know and apply the Binomial Theorem for the expansion of $\left(x+y\right)^n$ in powers of *x* and *y* for a positive integer *n*, where *x* and *y* are any numbers, with coefficients determined for example by Pascal's Triangle.

**A.APR.B.2**

Know and apply the Remainder Theorem: For a polynomial $p(x)$ and a number *a*, the remainder on division by $x-a$ is $p(a)$, so $p(a)=0$ if and only if $(x-a)$ is a factor of $p(x)$.

**A.APR.B.3**

Identify zeros of polynomials when suitable factorizations are available, and use the zeros to construct a rough graph of the function defined by the polynomial.

**A.SSE.A.2-2**

Use the structure of an expression, including polynomial and rational expressions, to identify ways to rewrite it. For example, see $x^4-y^4$ as $\left(x^2\right)^2-\left(y^2\right)^2$, thus recognizing it as a difference of squares that can be factored as $\left(x^2-y^2\right)\left(x^2+y^2\right)$.

**A.APR.D.6**

Rewrite simple rational expressions in different forms; write $\frac{a(x)}{b(x)}$ in the form $q(x)+\frac{r(x)}{b(x)}$, where $a(x), b(x), q(x),$ and $r(x)$ are polynomials with the degree of $r(x)$ less than the degree of $b(x)$, using inspection, long division, or, for the more complicated examples, a computer algebra system.

**Notes, Examples, and Exam Questions**

**UNIT 4.1 Identify, classify, evaluate, add and subtract polynomials**

**Polynomial Function:** a function of the form $f(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_2 x^2 + a_1 x + a_0$, $a_n \neq 0$

**Note:** Exponents are whole numbers and coefficients are real numbers.

Leading Coefficient: $a_n$  Constant Term: $a_0$  Degree: *n*, the largest exponent of *x*

**Standard Form of a Polynomial Function:** The terms are written in descending order of the exponents

**Classifying Polynomial Functions:**

This is kind of tricky- but $a_n$ is the name of the coefficient with the same degree. So, $a_n$ is the coefficient of the term that is the $n^{th}$ degree and $a_{n-1}$ is the coefficient of the term that is degree *n*−1.

I. Number of Terms

- 1 term = **monomial**
- 2 terms = **binomial**
- 3 terms = **trinomial**
- 4 or more terms = **polynomial**

II. Degree

| Degree | Name (Degree) | Standard Form | Example | Classification of Example |
|---|---|---|---|---|
| 0 | **Constant** | $f(x) = a$ | $y = -3$ | Constant Monomial |
| 1 | **Linear** | $f(x) = ax + b$ | $y = 2x$ | Linear Monomial |
| 2 | **Quadratic** | $f(x) = ax^2 + bx + c$ | $y = -x^2 + 5x + 7$ | Quadratic Trinomial |
| 3 | **Cubic** | $f(x) = ax^3 + bx^2 + cx + d$ | $y = 5x^3 - 2$ | Cubic Binomial |
| 4 | **Quartic** | $f(x) = ax^4 + bx^3 + cx^2 + dx + e$ | $y = x^4 - x^2 + 9x + 1$ | Quartic Polynomial |

**Identifying Polynomial Functions**

**Ex 1**: Is $f(x) = 5x^3 + x^{-2} - 8$ a polynomial function? If yes, write it in standard form.

No. In order to be a polynomial function, all exponents must be whole numbers.

**Ex 2**: Is $f(x) = \frac{5}{2}x - \pi x^4 + 8x^3$ a polynomial function? If yes, write it in standard form.

Yes. All exponents are whole numbers and all coefficients are real numbers.

Standard Form: $f(x) = -\pi x^4 + 8x^3 + \frac{5}{2}x$  Note: This is a quartic trinomial (degree = 4).

## Evaluating Polynomial Functions Using Direct Substitution

**Ex 3**: Find $f(2)$ if $f(x) = -3x^4 + x^3 - 5x^2 + 6x + 1$.

$$f(2) = -3(2)^4 + (2)^3 - 5(2)^2 + 6(2) + 1 = -3(16) + 8 - 5(4) + 12 + 1$$
$$= -48 + 8 - 20 + 12 + 1 = -40 - 20 + 12 + 1 = -60 + 12 + 1 = -48 + 1 = -47$$

$So: \boxed{f(2) = -47}$

## Adding Polynomials

**Ex 4**: Add the polynomials $(3x^3 - 5x^4 - 10x + 1) + (17x^4 - 3 + x^3)$.

Vertical Method: Write each polynomial in standard form and line up like terms. Then, add the like terms. Note the second polynomial does not have a linear term so "0x" is added to the line.

$$\begin{array}{l} -5x^4 + 3x^3 - 10x + 1 \\ \underline{17x^4 + x^3 \quad + 0x - 3} \\ 12x^4 + 4x^3 - 10x - 2 \end{array}$$

## Subtracting Polynomials

**Ex 5**: Subtract the polynomials $(9x^3 - 4 + x^2 + 8x) - (7x^3 - 3x + 7)$.

To subtract, we will rewrite the problem as an addition problem by adding the *opposite*.

$$(9x^3 - 4 + x^2 + 8x) + (-7x^3 + 3x - 7)$$

Horizontal Method: Combine each set of like terms.

$$(9x^3 - 4 + x^2 + 8x)(-7x^3 + 3x - 7)$$
$$= 2x^3 - 11 + 11x + x^2$$

Write the final answer in standard form. $\boxed{2x^3 + x^2 + 11x - 11}$

**Ex 6**: Subtract the polynomials $(5x^4 + 2x - 6x^2 + 12) - (5x^2 + 3x^3 - 8x + 7)$.

$$(5x^4 + 2x - 6x^2 + 12) + (-5x^2 - 3x^3 + 8x - 7)$$

Horizontal Method:

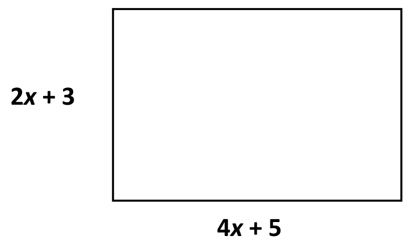
$$(5x^4 + 2x - 6x^2 + 12) + (-5x^2 - 3x^3 + 8x - 7) =$$
$$\boxed{5x^4 - 3x^3 - 11x^2 + 10x + 5}$$

Vertical Method:

$$\begin{array}{l} 5x^4 \qquad\quad - 6x^2 + 2x + 12 \\ \underline{\quad - 3x^3 - 5x^2 + 8x - 7} \\ 5x^4 - 3x^3 - 11x^2 + 10x + 5 \end{array}$$

## SAMPLE EXAM QUESTIONS

**1.** Which expression represents the perimeter of the rectangle?

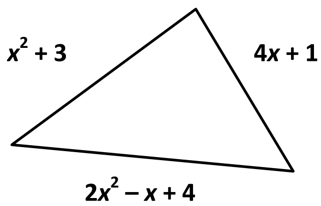



(A) 6x + 8 (C) 6x + 16

(B) 12x + 8 (D) 12x + 16

Ans: D

**2.** Which expression represents the perimeter of the triangle shown below?




(A) $x^2 - 5x$ (C) $x^2 - 3x + 3$

(B) $3x^2 - 5x + 6$ (D) $3x^2 + 3x + 8$

Ans: D

**3.** The function $g(x)$ is the amount of money Shawn has in the bank at the beginning of the month. The function $f(x)$ is the amount of money withdrawn from the account during the month. Which expression represents the amount of money left at the end of the month?

$$f(x) = x^2 - 3x + 12$$
$$g(x) = 6x^2 - 2x + 20$$

(A) $5x^2 - 5x + 8$ (C) $5x^2 + x + 8$

(B) $-5x^2 - x - 8$ (D) $-5x^2 - 5x + 8$

Ans: C

**4.** Subtract the following polynomials:

$$\left(4y^2+7y-5\right)-\left(2y^2-5y+3\right)$$

(A) $2y^2+2y-2$ (C) $2y^2+12y-8$

(B) $6y^2+2y-2$ (D) $6y^2+12y-8$

Ans: C

**UNIT 4.2 Multiply Polynomials and Binomial Expansion**

**Multiplying Polynomials**

**Ex 7**: Find the product $\left(x^3-x+2\right)\left(x^2+3x-4\right)$.

Horizontal Method: Use the distributive property by distributing each term of the first polynomial.

$$x^3\left(x^2+3x-4\right)-x\left(x^2+3x-4\right)+2\left(x^2+3x-4\right)$$
$$=x^5+3x^4-4x^3-x^3-3x^2+4x+2x^2+6x-8$$
$$=x^5+3x^4-4x^3-x^3-3x^2+2x^2+4x+6x-8$$

Combine like terms and write the answer in standard form.

$$\boxed{x^5+3x^4-5x^3-x^2+10x-8}$$

**Ex 8**: Multiply the polynomials $\left(3x-1\right)\left(2x^3+x^2-4x+7\right)$.

Vertical Method: Use long multiplication.

$$\begin{array}{rrrrrr} & 2x^3 & +x^2 & -4x & +7 & \\ \times & & & 3x & -1 & \\ \hline 6x^4 & +3x^3 & -12x^2 & +21x & & \\ + & -2x^3 & -x^2 & +4x & -7 & \\ \hline 6x^4 & +x^3 & -13x^2 & +25x & -7 & \end{array}$$

$$\boxed{6x^4+x^3-13x^2+25x-7}$$

**Ex 9**: Multiply the polynomials $\left(x-5\right)\left(2x+1\right)\left(4+x\right)$.

Multiply the polynomials two at a time. Because they are binomials, we can use FOIL to multiply the first two.

$$\left(2x^2+x-10x-5\right)\left(4+x\right)=\left(2x^2-9x-5\right)\left(4+x\right)$$

Use the distributive property. $\left(2x^2-9x-5\right)(4)+\left(2x^2-9x-5\right)(x)$

$=8x^2-36x-20+2x^3-9x^2-5x$

Combine like terms and write in standard form. $\boxed{2x^3-x^2-41x-20}$

**<u>Review:</u> Special Products** (Allow students to come up with these on their own.) Memorize these!

<u>Sum and Difference Product</u> $(a+b)(a-b)=a^2-b^2$

<u>Square of a Binomial</u>

$(a+b)^2=a^2+2ab+b^2$

$(a-b)^2=a^2-2ab+b^2$

<u>Cube of a Binomial</u>

$(a+b)^3=a^3+3a^2b+3ab^2+b^3$

$(a-b)^3=a^3-3a^2b+3ab^2-b^3$

**Ex 10**: Simplify the expression $(3y-2)^3$.

Using the cube of a binomial: $(3y)^3-3(3y)^2(2)+3(3y)(2)^2-(2)^3$

$=\boxed{27y^3-54y^2+36y-8}$

**<u>Application Problems</u>**

**Ex 11**: Find a polynomial expression for the volume of a rectangular prism with sides

$(x-3),(x+4)$, and $(x-2)$.

Volume of a Rectangular Prism = Length x Width x Height

FOIL: $(x-3)(x+4)(x-2)$

$=\left(x^2+x-12\right)(x-2)$

Vertical Method:

$$\begin{array}{r} x^2+x-12 \\ \times \quad x-2 \\ \hline x^3+x^2-12x \\ +\; -2x^2-2x+24 \\ \hline \boxed{x^3-x^2-14x+24} \end{array}$$

**Ex 12**: From 1985 through 1996, the number of flu shots given in one city can be modeled by

$A=-11.33t^4-8.325t^3+2194t^2-4190t+7592$ for adults and by

$C=-6.87t^4+106t^3-251t^2+135t+540$ for children, where $t$ is the number of years since 1985.

Write a model for the total number $F$ of flu shots given in these years.

To find the total flu shots, we need to add the polynomials.

Vertical Method:

$$\begin{array}{r} -11.33t^4 - 8.325t^3 + 2194t^2 - 4190t + 7592 \\ -6.87t^4 \quad + 106t^3 \quad - 251t^2 \quad + 135t \quad + 540 \\ \hline -18.2t^4 + 97.675t^3 + 1943t^2 - 4055t + 8132 \end{array}$$

Solution: $\boxed{F = -18.2t^4 + 97.675t^3 + 1943t^2 - 4055t + 8132}$

**QOD:** What is the advantage of the vertical method when adding, subtracting, or multiplying polynomials?

## BINOMIAL EXPANSION:

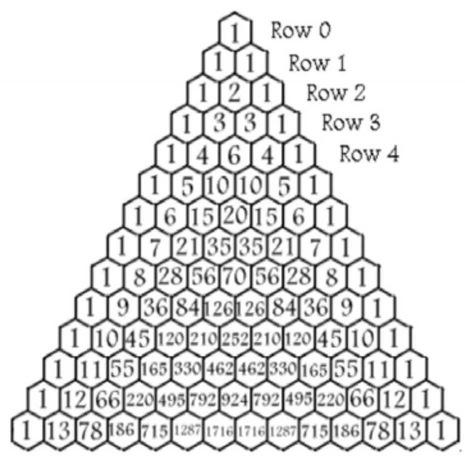
**Exploration:** Expand the following. Notice the pattern????

$(x+1)^0 = 1$ $\qquad$ $(x+1)^2 = 1x^2 + 2x + 1$

$(x+1)^1 = 1x + 1$ $\qquad$ $(x+1)^3 = 1x^3 + 3x^2 + 3x + 1$

Pascal's Triangle

1 Row 0
1 1 Row 1
1 2 1 Row 2
1 3 3 1 Row 3
1 4 6 4 1 Row 4
1 5 10 10 5 1 Row 5




This pattern can be continued…

The values in the triangle correspond to the coefficients of the binomial expansion of $(x+1)^n$.

They also correspond to ${}_nC_r$:

$${}_0C_0 = \frac{0!}{0!(0-0)!} = \frac{1}{1} = 1$$

$${}_1C_0 = \frac{1!}{0!(1-0)!} = \frac{1}{1} = 1 \qquad {}_1C_1 = \frac{1!}{1!(1-1)!} = \frac{1}{1} = 1$$

$${}_2C_0 = \frac{2!}{0!(2-0)!} = 1 \qquad {}_2C_1 = \frac{2!}{1!(2-1)!} = 2 \qquad {}_2C_2 = \frac{2!}{2!(2-2)!} = 1 \qquad \ldots$$

**Notation**: Combination (Binomial Coefficient)

$${}_nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

$$(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r$$

**The Binomial Theorem**

**Expanding a Binomial**

**Ex 13:** Expand $(x-3)^4$.

Since n = 4, we use row 4 from Pascal's Triangle for the coefficients: 1 4 6 4 1

Exponent of first term ($x$) starts at 4 and goes down by 1 each time; exponent of second term (−3) starts at 0 and goes up by 1 each time.

$$(x-3)^4 = 1(x^4)(-3)^0 + 4(x^3)(-3)^1 + 6(x^2)(-3)^2 + 4(x^1)(-3)^3 + 1(x^0)(-3)^4 = \boxed{x^4 - 12x^3 + 54x^2 - 108x + 81}$$

NOTE: When expanding a binomial: The exponent of the first term starts at *n* and goes **down** by 1 each time. The exponent of the second term starts at 0 and goes **up** by 1 each time. The **sum** of the two exponents of each term should equal *n*. If the second term is negative, the terms of the expansion alternate signs, like in the example above. We can also use the binomial theorem to find the expansion. Using the formula, we get:

$$(x-3)^4 = \binom{4}{0}x^{4-0}(-3)^0 + \binom{4}{1}x^{4-1}(-3)^1 + \binom{4}{2}x^{4-2}(-3)^2 + \binom{4}{3}x^{4-3}(-3)^3 + \binom{4}{4}x^{4-4}(-3)^4$$

$$= 1x^4 + 4x^3(-3) + 6x^2(9) + 4x(-27) + 1(81) = \boxed{x^4 - 12x^3 + 54x^2 - 108x + 81}$$

**Ex 14:** Expand $(3c+2d)^5$.

Since n = 5, we use row 5 from Pascal's Triangle for the coefficients: 1 5 10 10 5 1

Exponent of first term (3c) starts at 5 and goes down by 1 each time; exponent of second term (2d) starts at 0 and goes up by 1 each time.

$$(3c+2d)^5 = 1(3c)^5(2d)^0 + 5(3c)^4(2d)^1 + 10(3c)^3(2d)^2 + 10(3c)^2(2d)^3 + 5(3c)^1(2d)^4 + 1(3c)^0(2d)^5$$

$$= (243c^5)(1) + 5(81c^4)(2d) + 10(27c^3)(4d^2) + 10(9c^2)(4d^3) + 5(3c)(16d^4) + (1)(32d^5)$$

$$= \boxed{243c^5 + 810c^4d + 1080c^3d^2 + 360c^2d^3 + 240cd^4 + 32d^5}$$

**A common mistake made by students is that the variable of each term is raised to appropriate powers but not the coefficient. For instance, one may think that the first term in the expansion in the example above is $3c^5$ instead of $(3c)^5 = 243c^5$. Also, students may forget to use the numbers from Pascal's Triangle in an expansion or will not multiply all coefficients together and simplify for each term.

**Finding a Particular Term of a Binomial Expansion:** Recall that the coefficients from Pascal's Triangle can be found individually by evaluating ${}_nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$, or by using technology.

**Ex 15:** Find the fifth term of the binomial expansion of $(2x+1)^{10}$.

The first term of the expansion has $r=0$, so the fifth term has $r=4$ in the combination formula.

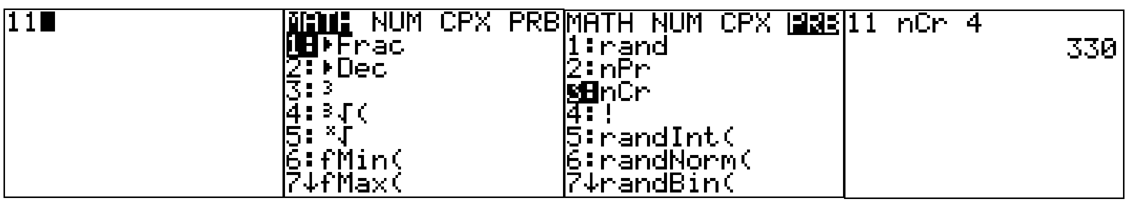
$$\text{Fifth term: } \binom{10}{4}(2x)^6(1)^4 = \frac{10!}{4!(10-4)!}\left(64x^6\right) = \frac{10\cdot 9\cdot 8\cdot 7}{4\cdot 3\cdot 2}\left(64x^6\right) = \boxed{13440x^6}$$

**Ex 16:** Find the coefficient of the term with $a^7$ in the expansion of $(4a-3b)^{11}$.

Sum of exponents = 11, so the term has $(-3b)^4$; therefore, $r=4$.

$$\binom{11}{4}(4a)^7(-3b)^4 = 330\left(16384a^7\right)\left(81b^4\right) = \boxed{437{,}944{,}320a^7b^4}$$

To evaluate $\binom{11}{4}$ on the TI-84, use the following steps: Type 11 into the calculator on the home screen, press MATH, arrow over to PRB, find ${}_nC_r$ ,hit enter and then type in 4.




**QOD:** Is the sum of EVERY row of Pascal's triangle an even number? Explain.

## SAMPLE EXAM QUESTIONS

**1. Which polynomial represents the product of $(x+2)(x^3-8)$?**

**A.** $x^4+2x^3-8x-16$

**B.** $x^4+2x^3+8x+16$

**C.** $x^4-16$

**D.** $x^4+16$

Ans: A

**2. Suppose $xy=9$ and $(x+y)^2=21$. What is $x^2+y^2$?**

**A.** 3 **B.** 12 **C.** 36 **D.** 81

Ans: A

**3. What is the product of the polynomials?** $(3x-2)(5x^2-4x-6)$

**A.** $15x^3-2x^2-10x+12$

**B.** $15x^3-2x^2-26x+12$

**C.** $15x^3-22x^2-10x+12$

**D.** $15x^3-22x^2-26x+12$

Ans: C

**4. What is the 4th term of the expanded binomial** $(2x-1)^6$ **?**

**A.** $240x^3$ **B.** $60x^3$

**C.** $-240x^3$ **D.** $-160x^3$

Ans: D

**UNIT 4.3 Factor Polynomials**

**Review: Factoring Patterns**

**Factoring a General Trinomial**

**Ex 17**: Factor the trinomial $2x^2-5x-12$.

*ac* Method: $ac=-24$ $\quad -8\cdot 3=-24$, $-8+3=-5$

Split the middle term: $2x^2-8x+3x-12$

Factor by grouping: $2x(x-4)+3(x-4)$

$=\boxed{(2x+4)(x-4)}$

**Factoring a Perfect Square Trinomial**

**Ex 18**: Factor the trinomial $x^2+6x+9$.

Use $a^2+2ab+b^2=(a+b)^2$. $\quad (x)^2+2(3)(x)+(3)^2=\boxed{(x+3)^2}$

**Difference of Two Squares**

**Ex 19**: Factor $36x^2-49y^2$.

Use $a^2-b^2=(a+b)(a-b)$. $\quad (6x)^2-(7y)^2=\boxed{(6x+7y)(6x-7y)}$

**Common Monomial Factor**

**Ex 20**: Factor the trinomial completely. $2x^7 - 32x^3$

Factor the GCF and the binomial square .

$2x^3\left(x^4 - 16\right)$

$2x^3\left(x^2 + 4\right)\left(x^2 - 4\right)$

Since this is not completely factored, use $a^2 - b^2 = (a+b)(a-b)$.

$= \boxed{2x^3\left(x^2+4\right)(x+2)(x-2)}$

**Sum and Difference of Two Cubes**

$$a^3 + b^3 = (a+b)\left(a^2 - ab + b^2\right)$$
$$a^3 - b^3 = (a-b)\left(a^2 + ab + b^2\right)$$

**Ex 21**: Factor the binomial $8x^3 + 1$.

Use $a^3 + b^3 = (a+b)\left(a^2 - ab + b^2\right)$.

$(2x)^3 + (1)^3 = (2x+1)\left((2x)^2 - (2x)(1) + (1)^3\right)$

$= \boxed{(2x+1)\left(4x^2 - 2x + 1\right)}$

**Ex 22**: Factor the binomial $y^6 - 27x^3$.

Use $a^3 - b^3 = (a-b)\left(a^2 + ab + b^2\right)$.

$\left(y^2\right)^3 - (3x)^3 = \left(y^2 - 3x\right)\left(\left(y^2\right)^2 + \left(y^2\right)(3x) + (3x)^2\right)$

$= \boxed{\left(y^2 - 3x\right)\left(y^4 + 3xy^2 + 9x^2\right)}$

**Factoring by Grouping**

**Ex 23**: Factor the polynomial $x^3 - 2x^2 - 9x + 18$.

Group each pair of terms and factor the GCF. $x^2(x-2) - 9(x-2)$

Factor the common binomial factor. $= (x-2)\left(x^2 - 9\right)$

Factor the remaining terms if possible. $= \boxed{(x-2)(x+3)(x-3)}$

**Ex 24**: Factor the polynomial $4x^6 - 20x^4 - 24x^2$.

Factor out the common binomial factor. $4x^2\left(x^4 - 5x^2 + 6\right)$

Factor the remaining trinomial. $= \boxed{4x^2\left(x^2 - 3\right)\left(x^2 - 2\right)}$

**QOD:** Give an example of a binomial that can be factored either as the difference of two squares or as the difference of two cubes. Show the complete factorization of your binomial.

## SAMPLE EXAM QUESTIONS

**1. What is the factored form of the polynomial $x^3 - 27$?**

**A.** $(x+3)\left(x^2 - 3x + 9\right)$ **B.** $(x+3)\left(x^2 - 3x - 9\right)$

**C.** $(x-3)\left(x^2 - 3x - 9\right)$ **D.** $(x-3)\left(x^2 + 3x + 9\right)$

Ans: D

**2. Which expression is equivalent to $xc + xb + yc + yb$?**

**A.** $(x+b)(y+c)$

**B.** $(x+c)(y+b)$

**C.** $(x+y)(b+c)$

Ans: C

**3. Which is equivalent to $\left(4x^2 - 9y^4\right)$**

**A.** $\left(2x - 3y^2\right)^2$

**B.** $\left(2x - 3y^2\right)\left(2x + 3y^2\right)$

**C.** $\left(2x + 3y^2\right)(2x - 3y)(2x + 3y)$

Ans: A

## UNIT 4.4 Dividing Polynomials

Review: Long Division $43,581 \div 23$

$$
\begin{array}{r}
1894 \\
23\overline{)43581} \\
\underline{23}\phantom{581} \\
205\phantom{81} \\
\underline{184}\phantom{81} \\
218\phantom{1} \\
\underline{207}\phantom{1} \\
111 \\
\underline{92} \\
19
\end{array}
$$

Remainder is 19, so the quotient is $\boxed{1894\frac{19}{23}}$

**Long Division of Polynomials** (same process!)

The closure property states that when you combine any two elements of the set, the result is also in that set. Polynomials are closed with respect to addition, subtraction and multiplication. When we add, subtract or multiply polynomials, our result is a polynomial. However, when we divide polynomials, we might not get a polynomial. Therefore, we say that division of polynomials is not closed.

**Ex 25:** Find the quotient. $\left(x^4-8x^3+11x-6\right)\div\left(x+3\right)$

Note: Every term of the polynomial in the dividend must be represented. Since this polynomial is missing an $x^2$ term, we must include the term $0x^2$.

$$
\begin{array}{r}
x^3-11x^2+33x-88 \\
x+3\overline{)x^4-8x^3+0x^2+11x-6} \\
\underline{x^4+3x^3}\phantom{+0x^2+11x-6} \\
-11x^3+0x^2\phantom{+11x-6} \\
\underline{-11x^3-33x^2}\phantom{+11x-6} \\
33x^2+11x\phantom{-6} \\
\underline{33x^2+99x}\phantom{-6} \\
-88x-6 \\
\underline{-88x-264} \\
258
\end{array}
$$

We place the remainder, 258, over the divisor. Just like long division with natural numbers.

Solution: $\boxed{x^3-11x^2+33x-88+\frac{258}{x+3}}$

**Ex 26:** Find the quotient of $y^4 + 2y^2 - y + 5$ and $y^2 - y + 1$.

$$
\begin{array}{r l}
 & y^2 + y + 2 \\
y^2 - y + 1 \,\big)\!\! & \overline{y^4 + 0y^3 + 2y^2 - y + 5} \\
 & \underline{-y^4 + y^3 \quad - y^2} \text{ (add the opposite)} \\
 & \quad y^3 + \quad y^2 - y \text{ (bring down the next term)} \\
 & \quad \underline{-y^3 + y^2 - y} \\
 & \qquad 2y^2 - 2y + 5 \\
 & \qquad \underline{-2y^2 + 2y - 2} \\
 & \qquad\qquad 3 \text{ (remainder)}
\end{array}
$$

Remember to put a place for the missing term.

Solution: $y^2 + y + 2 + \dfrac{3}{y^2 - y + 1}$

**Dividing Polynomials Using Synthetic Division:** (Note: This procedure can only be used when the divisor is in the form $x - k$ (a linear binomial).)

**Ex 27**: Divide the polynomial $x^3 - 3x^2 - 7x + 6$ by $x + 3$.

With the polynomial in standard form, write the coefficients in a row. If a term is missing, make sure to put a zero in the row.

Put the *k* value (-3) to the upper left in the "box".

Bring down the first coefficient, then multiply by the *k* value. Add straight down the columns and repeat.

$$
\begin{array}{r|rrrr}
-3 & 1 & -3 & -7 & 6 \\
 & & -3 & 18 & -33 \\
\hline
 & 1 & -6 & 11 & \text{R} \; -27
\end{array}
$$

Note for graphing: This means that (-3,-27) is an ordered pair that is on the graph of the function.

The coefficients of the quotient and remainder appear in synthetic substitution.

Quotient: $1x^2 - 6x + 11 - \dfrac{27}{x+3}$

**Notice the quotient is now one degree lower than the dividend (the original polynomial).

**Synthetic Substitution:**

**Ex 28**: Find $f(2)$ if $f(x) = -3x^4 + x^3 - 5x^2 + 6x + 1$ using synthetic substitution.

Using the polynomial in standard form, write the coefficients in a row. Put the *x*-value to the upper left.

$$2\,|\; -3 \quad 1 \quad -5 \quad 6 \quad 1$$

Bring down the first coefficient, then multiply by the *x*-value.

multiply

$$\begin{array}{r|rrrrr} 2 & -3 & 1 & -5 & 6 & 1 \\ & \downarrow & -6 & & & \\ \hline & -3 & & & & \end{array}$$

Add straight down the columns, and repeat.

$$\begin{array}{r|rrrrr} 2 & -3 & 1 & -5 & 6 & 1 \\ & \downarrow & -6 & -10 & -30 & -48 \\ \hline & -3 & -5 & -15 & -24 & \boxed{-47} \end{array}$$

The number in the bottom right is the value of $f(2)$. *So* : $\boxed{f(2) = -47}$

**Exploration:** Use the polynomial function $f(x) = 3x^3 - 2x^2 + 2x - 5$. Use long division to divide $f(x)$ by $x - 2$. Then use synthetic substitution to find $f(2)$. What do you notice?

**SAMPLE EXAM QUESTIONS**

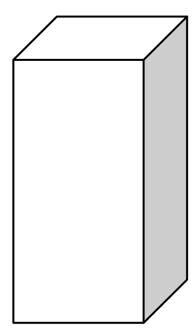
**1. Divide $(x^4 + 2x^3 - 7) \div (x^2 - 1)$ using long division.**

(A) $x^2 + 2x + 1 + \dfrac{6}{x^2 - 1}$ (B) $x^2 + 2x - 1 + \dfrac{2x - 6}{x^2 - 1}$

(C) $x^2 + 2x - 1 + \dfrac{2x + 6}{x^2 - 1}$ (D) $x^2 + 2x + 1 + \dfrac{2x - 6}{x^2 - 1}$

Ans: D

**2. The volume $V(x)$ and height ($h$) of the prism is given. Find a polynomial expression for the area of the base ($B$) in terms of $x$. (*Hint:* $V = Bh$)**



$h = x + 2$

$V(x) = 2x^3 + 5x^2 - 4$

(A) $2x^2 + x - 2$ (B) $2x^2 + 4x - 4$

(C) $x - 6$ (D) $x^2 - 5x + 3$

Ans: A

**3. Write an expression that represents the width of a rectangle with length $x+5$ and area $x^3+12x^2+47x+60$.**

(A) $x^3+7x^2+12x$

(B) $x^2+7x+12$

(C) $x^2+17x-38-\frac{50}{x+5}$

(D) $x^2+17x+132+\frac{720}{x+5}$

Ans: B

**Unit 4.5 Factor Theorem and Remainder Theorem**

**Remainder Theorem:** If a polynomial $f(x)$ is divided by $x-k$, then the remainder is $r=f(k)$.

**Ex 29:** Find the remainder without using division for $x^4-8x^3+11x-6$ divided by $x+3$.

$f(x)=x^4-8x^3+11x-6$ $\quad r=f(-3)=(-3)^4-8(-3)^3+11(-3)-6=\boxed{258}$

Note: Compare your answer to the long division problem **Ex 25** above.

**Ex 30:** Find $f(-3)$ if $f(x)=x^5-2x^3+7x^2-11$ using synthetic substitution.

This polynomial function is in standard form, however it is missing two terms. We can rewrite the function as $f(x)=x^5+0x^4-2x^3+7x^2+0x-11$ to fill in the missing terms.

| | | | | | | |
|---|---|---|---|---|---|---|
| $-3$ | 1 | 0 | $-2$ | 7 | 0 | $-11$ |
| | | $-3$ | 9 | $-21$ | 42 | $-126$ |
| | 1 | $-3$ | 7 | $-14$ | 42 | $\boxed{-137}$ |

$\boxed{f(-3)=-137}$

This also means that (-3,-137) is an ordered pair that would be a point on the graph.

**With the remainder theorem, we can find this value much quicker.

$$r=f(-3)=(-3)^5-2(-3)^3+7(-3)^2-11=\boxed{-137}$$

**Factor Theorem:** A polynomial $f(x)$ has a factor $x-k$ if and only if $f(k)=0$.

**Ex 31:** Determine if $x-3$ is a factor of $2x^3-3x^2-5x-12$ without dividing.

Show that $r=f(3)=0$: $2(3)^3-3(3)^2-5(3)-12=0$, so by the Factor Theorem, $x-3$ is a factor of $2x^3-3x^2-5x-12$.

**Ex 32**: Factor $f(x) = 3x^3 + 14x^2 - 28x - 24$ given that $f(-6) = 0$.

Because $f(-6) = 0$, we know that $x + 6$ is a factor of $f(x)$ by the Factor Theorem. We will use synthetic division to find the other factors.

$$\begin{array}{r|rrrr} -6 & 3 & 14 & -28 & -24 \\ & & -18 & 24 & 24 \\ \hline & 3 & -4 & -4 & \boxed{0} \end{array}$$

$f(x) = (x+6)(3x^2 - 4x - 4)$

$\boxed{f(x) = (x+6)(3x+2)(x-2)}$

Note for graphing: This means that (-6,0) is an ordered pair that is on the graph of the function. -6 is called a zero. It is also an x-intercept.

**Ex 33**: One zero of $f(x) = 2x^3 - 9x^2 - 32x - 21$ is $x = 7$. Completely factor the function.

Because $f(7) = 0$, we know that $x - 7$ is a factor of $f(x)$ by the Factor Theorem. We will use synthetic division to find the other factors.

$$\begin{array}{r|rrrr} 7 & 2 & -9 & -32 & -21 \\ & & 14 & 35 & 21 \\ \hline & 2 & 5 & 3 & \boxed{0} \end{array}$$

$f(x) = (x-7)(2x^2 + 5x + 3)$

$\boxed{f(x) = (x-7)(2x+3)(x+1)}$

**QOD:** If $f(x)$ is a polynomial that has $x + a$ as a factor, what do you know about the value of $f(-a)$?

## SAMPLE EXAM QUESTIONS

**1. Use the remainder theorem to find the remainder when f(x) is divided by x - c?**

$f(x) = x^4 + 8x^3 + 12x^2;\ x + 1$

**A.** 21 **B.** -21

**C.** 5 **D.** -5

Ans: C

**2. Use the factor theorem to determine whether x – c is a factor of f(x).**

$f(x) = 5x^4 + 19x^3 - 4x^2 + x - 4;\ x + 4$

**A.** Yes **B.** No

Ans: A

**3. Use the remainder theorem to find the remainder when f(x) is divided by x-k.**

$$f(x) = 2x^2 - 3x + 1;\ k + 2$$

**A.** 15 **B.** 9

**C.** 18 **D.** 3

Ans: D

**4. Use the factor theorem to determine whether the first polynomial is a factor of the second polynomial.**

$$x - 1;\ f(x) = x^3 - x^2 + x - 1$$

**A.** No solution **B.** no

**C.** yes **D.** 0

Ans: C