UNIT 4 – FUNCTIONS AND FUNCTION NOTATION

PREREQUISITE SKILLS:

- students should understand how to evaluate variable expressions
- students should understand how to solve equations with one or two variables

VOCABULARY:

- relation: a set of ordered pairs
- function: a special relation that has a rule that establishes a mathematical relationship between two quantities, called the input and the output. For each input, there is exactly one output
- domain: the collection of all input values
- range: the collection of all output values
- independent variable: the variable in a function with a value that is subject to choice
- dependent variable: the variable in a relation with a value that depends on the value of the independent variable (input)
- function notation: a way to name a function that is defined by an equation. In function notation, the y in the equation is replaced with $f(x)$
- sequence: A list of numbers set apart by commas.
- arithmetic sequence: A sequence which has a constant difference between its terms.
- recursive formula: A formula that requires the computation of all previous terms in order to find the value of a numbered term.
- explicit formula: A formula that allows direct computation of any term for a sequence.

SKILLS:

- determine if a given relation is a function
- describe and model functions using an input-output table, mapping diagram, and writing a function rule with and without technology
- determine and differentiate between the domain and range of functions
- use equations of functions to make predictions or interpretations
- evaluate functions using function notation for given values of the variable
  translate among verbal descriptions, graphic, tabular, and algebraic representations of a function with and without technology

STANDARDS:

F.IF.A.1 Understand the concept of a function and use function notation.
Understand that a function from one set (called the domain) to another set (called the range) assigns to each element of the domain exactly one element of the range. If f is a function and x is an element of its domain, then $f(x)$ denotes the output of f corresponding to the input x. The graph of f is the graph of the equation $y = f(x)$.

F.IF.A.2 Use function notation, evaluate functions for inputs in their domains, and interpret statements that use function notation in terms of a context.

F.IF.A.3 Recognize that sequences are functions, sometimes defined recursively, whose domain is a subset of the integers. For example, the Fibonacci sequence is defined recursively by $f(0) = f(1) = 1$, $f(n+1) = f(n) + f(n-1)$ for $n \geq 1$.

F.LE.A.2 Construct linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two input-output pairs (include reading these from a table). *(Modeling Standard)

F.IF.B.4 Interpret functions that arise in applications in terms of the context.
For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship.

F.IF.B.5. Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes.

F.BF.A.1a Determine an explicit expression, a recursive process, or steps for calculation from a context. *(Modeling Standard)

F.BF.A.2 Write arithmetic and geometric sequences both recursively and with an explicit formula, use them to model situations, and translate between the two forms. *(Modeling Standard)

8.F.B.5 Describe qualitatively the functional relationship between two quantities by analyzing a graph (e.g., where the function is increasing or decreasing, linear or nonlinear). Sketch a graph that exhibits the qualitative features of a function that has been described verbally.

A.CED.A.2-1 Create linear, exponential, and quadratic equations in two or more variables to represent relationships between quantities and graph equations on coordinate axes with labels and scales. Limit exponentials to have integer inputs only. *(Modeling Standard)

LEARNING TARGETS:

- 4.1 To describe a relationship given a graph and to sketch a graph given a description.
- 4.2 To determine if a relation is a function.
- 4.3 To use functions to model real world situations.
- 4.4 To define and recognize an arithmetic sequence.
- 4.5 To graph an arithmetic sequence.
- 4.6 To write recursive and explicit formulas of an arithmetic sequence.
- 4.7 To use an arithmetic sequence to model a real world situation.

BIG IDEA:

Relationships among quantities can be represented using tables, graphs, verbal descriptions, equations and inequalities. Symbols are used to represent unknowns and variables. We can interpret and make critical predictions from functional relationships.

Notes, Examples and Exam Questions

<u>Relation</u>: a set of ordered pairs

<u>Domain</u>: the set of input values (x) in a relation; x is also called the "<u>independent</u>" variable.

<u>Range</u>: the set of output values (y) in a relation; y is also called the "<u>dependent</u>" variable.

Ex 1. State the domain and range using the relation: $\{(-1,2),(0,4),(0,-3),(1,-3)\}$.

To list the domain and range you list them from least to greatest in set notation.

Solution: Domain $\{-1,0,1\}$; Range $\{-3,2,4\}$

Ex 2. Use the following: Popcorn Prices: Small $3.^{\underline{00}}$ Medium $4.^{\underline{00}}$ Large $5.^{\underline{00}}$

How much would ten large popcorns cost?

$(1, 5.^{\underline{00}}), (2, 10.^{\underline{00}}), (3, 15.^{\underline{00}}), \ldots \quad (10, ?)$

The total cost depends on the number of popcorns you purchase, so the number of popcorns is the independent variable (input) and the cost is the dependent variable (output). We can write a rule for that to find the cost of any number of large popcorns purchased:

$$\underbrace{\text{Cost}}_{C}\quad \underbrace{\text{equals}}_{=}\quad \underbrace{\begin{matrix}\$5\\ \text{(per popcorn)}\end{matrix}}_{\$5}\quad \underbrace{\begin{matrix}\text{multiply}\\ \text{by}\end{matrix}}_{\times}\quad \underbrace{\begin{matrix}\text{\# of popcorns}\\ \text{purchased}\end{matrix}}_{n}$$

$$(10, ?) \qquad C = \$5 \times 10 = \$50$$

Three people in front of you in the line all buy some large popcorn: $(2, 10.^{\underline{00}})$, $(3, 15.^{\underline{00}})$ and $(1, 5.^{\underline{00}})$. You order 3 large popcorns and the popcorn guy says, "That will be $\$18.^{\underline{00}}$." Is everything functioning here? No, the rule was not followed for your order. The input of 3 large popcorns should have exactly 1 output, $15.

Function: a special type of relation in which each input has exactly one output.
Functions can be represented in several different ways; ordered pairs, table of values, mapping diagrams, graphs and in function notation.

Ordered Pairs: given a relation, it is a function if each input is paired with exactly 1 output (check to see if x repeats).

Ex 3. Is the relation a function? If so, state the domain and range.

a. $\{(3,5),(-4,6),(-2,4),(3,2)\}$

No, the input 3 has 2 output values.

b. $\{(-2,6),(0,10),(1,12),(3,16)\}$

Yes, each input has exactly 1 output.
Domain: {-2, 0, 1, 3}
Range: {6, 10, 12, 16}

Table of Values: given a table of values of a relation, it is a function if each input is paired with exactly 1 output (check to see if *x* repeats).

Ex 4. Use the following input-output table. Is the relation a function? If so, state the domain and range.

a.

Input	Output
3	0
6	4
9	0
12	–4

Yes, each input has exactly 1 output.
Domain: {3, 6, 9, 12}; Range: {-4, 0, 4}

b.

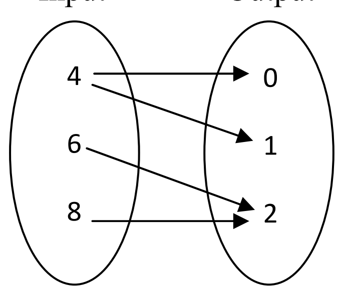
Input	Output
2	2
8	6
2	1
10	-6

No, the input 2 has 2 output values.

Mapping Diagrams: given a mapping diagram of a relation, we can tell if it is a function if each input is paired with exactly 1 output.

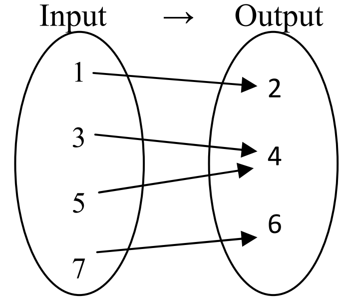
Ex 5. Is the relation a function? If so, state the domain & range.

a) Input → Output



No, the input 4 has two output values.

b) Input → Output



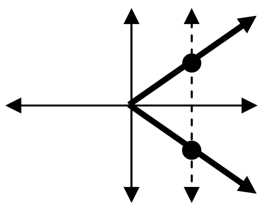
Yes, every input has exactly one output.
Domain: $\{1,3,5,7\}$; Range: $\{2,4,6\}$

Graphs of Functions: Given the graph, we can use the "vertical line test" to determine if a relation is a function.

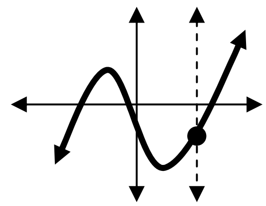
Vertical Line Test: a graph is a function if all vertical lines intersect the graph no more than once. If you can draw a vertical line between any two points on the graph, then it flunks the vertical line test.

The two points would have the same x value, but different y values; which means that there is more than one output (y) for that particular input (x).

Ex 6. Which of the graphs is a function?



Not a function – fails the vertical line test



Function – passes the vertical test

Function Rule: A function can be represented by an equation that describes the mathematical relationship that exists between the independent (x) and dependent (y) variables.

Ex 7. The equation $y = 2x + 1$, tells us that the output value is equal to 1 more than twice the input value.

We can use the function rule to pair x values with y values and create ordered pairs. Let's input the value 3 into the function rule for x and determine what output (y) value it is paired with.

$$y = 2x + 1$$
$$y = 2(3) + 1$$
$$y = 6 + 1$$
$$y = 7$$

So, the ordered pair (3, 7) is a solution to the function rule.

Function Notation: function notation is a way to name a function that is defined by an equation. For an equation in x and y, the symbol $f(x)$ replaces y and is read as "the value of the function at x" or simply "f of x". Remember, that the x value is the independent variable and the y value is dependent on what the value of x is. So, y is a function of x; $y = f(x)$ or in other words, y and the function of x are interchangeable.

Ex 8. Let's look at the equation $y = 2x + 1$. To write it using function notation we replace the y with $f(x)$ since they are interchangeable. So, the equation $y = 2x + 1$ becomes $f(x) = 2x + 1$.

Why use function notation? It helps us to relate the function rule to its graph. Each solution (ordered pair) to the function rule represents a point that falls on the graph of the function. When using function notation we can see the ordered pair.

Let's use the function rule expressed in function notation to find the value of the function when the input (x) is 3.

$f(x) = 2x + 1$
$f(3) = 2(3) + 1$
$f(3) = 6 + 1$
$f(3) = 7$

Notice that throughout the process you can see what the input value is. In the final result, you can see the ordered pair. Following the function rule; when x has a value of 3, y has a value of 7.

Does function notation always have to be expressed as $f(x)$? No. You can use any letter to represent a function. For example; $g(x)$, $h(x)$ or $k(x)$. When comparing multiple functions or their graphs you need some way to distinguish between them.

Ex 9. $f(x) = -2x + 3$ $\quad g(x) = x - 4$ $\quad h(x) = 5$ $\quad k(x) = x^2 + 1$

Evaluate the following expressions given the function rules above.

$g(6)$	$f(-2)$	$h(13)$	$k(0)$
$g(x) = x - 4$	$f(x) = -2x + 3$	$h(x) = 5$	$k(x) = x^2 + 1$
$g(6) = (6) - 4$	$f(-2) = -2(-2) + 3$	$h(13) = 5$	$k(0) = (0)^2 + 1$
$g(6) = 2$	$f(-2) = 4 + 3$		$k(0) = 0 + 1$
	$f(-2) = 7$		$k(0) = 1$

$f(x) - h(x)$	$h(x) \bullet g(x)$	$k(h(x))$	$f(g(x))$
$(-2x + 3) - (5)$	$(5) \bullet (x - 4)$	$k(5)$	$f(x - 4)$
$-2x - 2$	$5x - 20$	$(5)^2 + 1$	$-2(x - 4) + 3$
		$25 + 1$	$-2x + 8 + 3$
		26	$-2x + 11$

Graphing Solutions Given a Domain.

Ex 10: Graph the function for the given domain $-x+2y=6;$ domain: $\{-4,-2,0,2\}$.

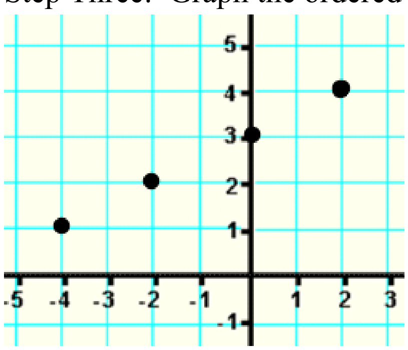
Step One: Solve for *y* and write in function notation.

$$\begin{aligned} -x+2y &= 6 \\ +x \quad & \quad +x \\ \frac{2y}{2} &= \frac{x+6}{2} \\ y &= \frac{x}{2}+\frac{6}{2} \\ f(x) &= \frac{1}{2}x+3 \end{aligned}$$

Step Two: Substitute the given values of the domain into the function rule.

$$\begin{aligned} f(-4) &= \frac{1}{2}(-4)+3 \\ f(-4) &= -2+3 \\ f(-4) &= 1 \\ &(-4,1) \end{aligned} \qquad \begin{aligned} f(-2) &= \frac{1}{2}(-2)+3 \\ f(-2) &= -1+3 \\ f(-2) &= 2 \\ &(-2,2) \end{aligned} \qquad \begin{aligned} f(0) &= \frac{1}{2}(0)+3 \\ f(0) &= 0+3 \\ f(0) &= 3 \\ &(0,3) \end{aligned} \qquad \begin{aligned} f(2) &= \frac{1}{2}(2)+3 \\ f(2) &= 1+3 \\ f(2) &= 4 \\ &(2,4) \end{aligned}$$

Step Three: Graph the ordered pair.



On a graph, the domain represents all possible *x* values. The domain is limited to −4, −2, 0 and 2.

Domain : {−4, −2, 0, 2}

On a graph, the range represents all possible y values. Since the domain is limited, the range will also be limited.

Range: {1, 2, 3, 4}

Graphing a Function Given a Domain of All Real Numbers: Create a table of values. Choose values for the independent variable and evaluate the function rule at these values. Plot the points to graph the function.

Ex 11: Graph the function $f(x) = -2x + 1$

Step One: Use the function rule to generate several ordered pairs by choosing values for *x*. (It is best to choose two negative values, zero, and two positive values.)

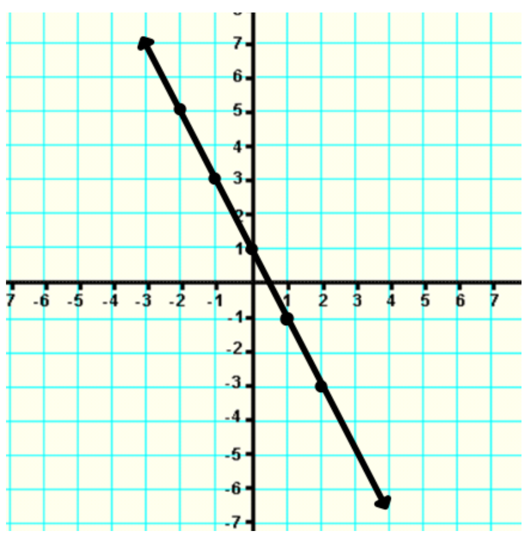
x	-2	-1	0	1	2
f(x)					

$\Rightarrow$

x	-2	-1	0	1	2
f(x)	5	3	1	-1	-3

Step Two: Plot enough points on the coordinate plane to see a pattern for the graph.

Step Three: Since the domain is all real numbers, the domain is infinite and there will be infinite ordered pairs that fall on the graph of the function. To indicate this, use arrowheads on both "ends" of the line to represent that the pattern continues infinitely in both directions.



On a graph, the domain represents all possible *x* values (from how far left to how far right the graph goes). Look at the end behavior of the line (arrows). This line will go forever to the left and forever to the right. So,

Domain : $-\infty \le x \le \infty$ or $\mathbb{R}$ (all real numbers)

On a graph, the range represents all possible *y* values (from how far down to how far up the graph goes). Look at the end behavior of the line (arrows). This line goes forever down and forever up. So,

Range: $-\infty \le x \le \infty$ or $\mathbb{R}$ (all real numbers)

Ex 12: Graph the function $g(x) = |x| + 2$

Step One: Use the function rule to generate several ordered pairs by choosing values for *x*. (It is best to choose two negative values, zero, and two positive values.)

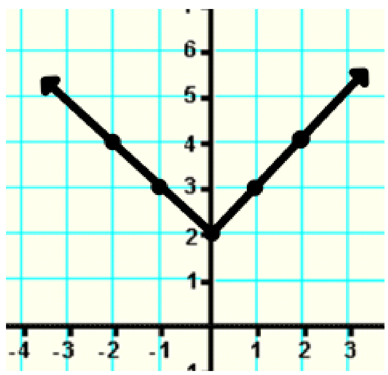
x	-2	-1	0	1	2
f(x)					

$\Rightarrow$

x	-2	-1	0	1	2
f(x)	4	3	2	3	4

Step Two: Plot enough points on the coordinate plane to see a pattern for the graph.

Step Three: Since the domain is all real numbers, the domain is infinite and there will be infinite ordered pairs that fall on the graph of the function. To indicate this, use arrowheads on both “ends” of the line to represent that the pattern continues infinitely in both directions.



On a graph, the domain represents all possible *x* values (from how far left to how far right the graph goes). Look at the end behavior of the line (arrows). This graph will go forever to the left and forever to the right. So,

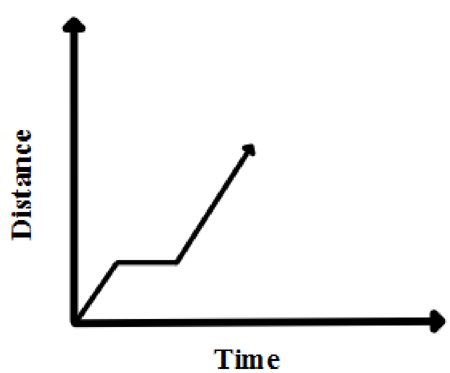
Domain : $-\infty \le x \le \infty$ or $\mathbb{R}$ (all real numbers)

On a graph, the range represents all possible *y* values (from how far down to how far up the graph goes). This graph has a low point or minimum at (0, 2). Looking at the end behavior, this graph goes forever upward in both directions; so,

Range: $2 \le x \le \infty$

Analyzing Graphs:

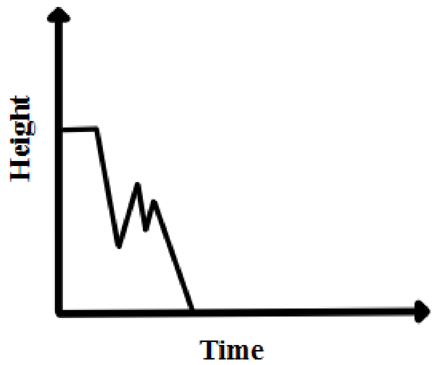
Ex 13: The graph represents the distance Sue has peddled on her bicycle.



To analyze the graph, read the labels on the *x*- and *y*-axes. Notice, that as the value of *x* increases, time is passing and as the value of *y* increases, the distance she has gone is increasing.

-The graph starts at the origin (0, 0). This means that 0 time has passed and Sue has gone 0 distance.
-Then, Sue’s distance increases at a steady rate (positive line) for a period of time.
-Then, Sue maintains the same distance for a period of time (flat line) which means she stopped for a while.
-Finally, Sue continues on her way at the same speed she was peddling before she stopped (positive line with same slope as before).

Ex 14: The graph represents a bungee jumper's height from the ground during a jump.



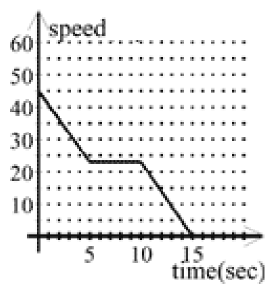
To analyze the graph, read the labels on the *x*- and *y*-axes. Notice, that as the value of *x* increases, time is passing and as the value of *y* increases, the height the jumper is from the ground increases.

-The graph starts at the highest point of the graph. The jumper maintains that height (flat line) for a period of time. Maybe the jumper is standing on the platform scared to jump and building up courage.
-Then, the jumper jumps and moves downward at a steady speed (negative line) for a period of time.
-Then, the jumper springs back up (positive line) for a period of time before falling again.
-The jumper springs up/down again, but the second time is a smaller bounce (lines don't go up/down as far as before).
-Finally, the jumper falls to the ground at the same speed as before(negative line).

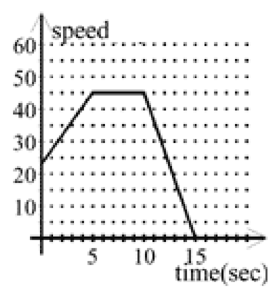
Ex 15: Which graph below would match the situation described?

A car travelling at 23 mi/h accelerates to 45 mi/h in 5 seconds. It maintains that speed for the next 5 seconds, and then slows to a stop during the next 5 seconds.

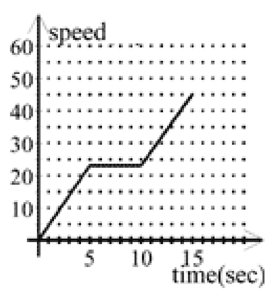
a.



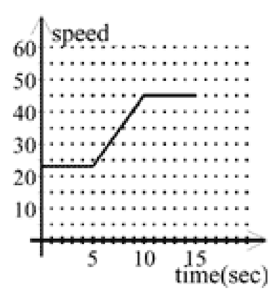
c.



b.



d.



ANS: C

EXAM QUESTIONS

1. Explain using the definition of a function why the vertical line tests determine whether a graph is a function.

   ANS: A function is a special relation where for each input there is exactly one output. If a vertical line can connect two points on a graph, then that graph contains two ordered pair that share the same x value, but have different y values. Therefore, for that particular input value, there is more than 1 output value.

2. Which of the following tables represent functions?

I.

Input	Output
0	1
0	2
2	3
3	4

III.

Input	Output
−3	2
−2	−2
2	−3
3	3

II.

Input	Output
0	0
1	0
2	1
3	1

IV.

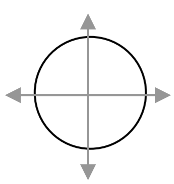
Input	Output
2	−11
0	8
2	4
4	7

   A. I and IV only
   B. II and III only
   C. I, II and III only
   D. II, III and IV only

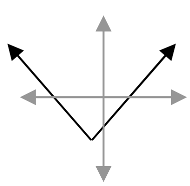
ANS: B

3. Determine which of the following are functions:

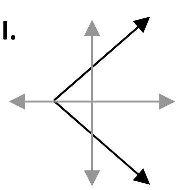
I.



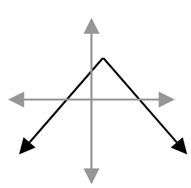
II.



III.



IV.



   A. I and III only
   B. II and IV only
   C. II, III, and IV only
   D. I, II, III and IV

ANS: B

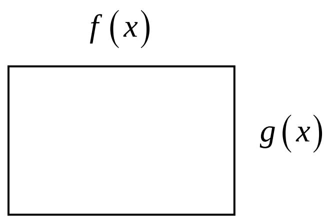
4. What is the range of the following relation? $\{(3,5),(-2,8),(5,1),(-3,-1)\}$

A. $\{-1, 1, 5, 8\}$ B. $\{-3, -2, 3, 5\}$

C. $\{-1 \le x \le 8\}$ D. $\{-3 \le x \le 5\}$

ANS: A

5. Use the diagram below when $f(x) = 5$ and $g(x) = 2x + 3$.



a. Write algebraic expressions for the area and the perimeter.

ANS:

area	perimeter
$l \bullet w$	$2l + 2w$
$f(x) \bullet g(x)$	$2 \bullet f(x) + 2 \bullet g(x)$
$5(2x + 3)$	$2(5) + 2(2x + 3)$
$10x + 15$	$10 + 4x + 6$
	$4x + 16$

b. If the perimeter is 24 inches, what is the value of x?

ANS: If the perimeter is $4x + 16$, we can write an equation using the expression we found in part a.

$P = 4x + 16$, now substitute 24 for the perimeter and solve for x.

$24 = 4x + 16$

$-16 \qquad -16$

$\frac{8}{4} = \frac{4x}{4}$

$x = 2$

6. Compare and contrast a relation and a function.

ANS: both relations and functions are sets of ordered pair. In a function, each input is paired with exactly 1 output. That is not necessarily true in a relation.

7. Which input-output table represents the function $f(x) = 2x - 3$?

A.

Input	Output
2	1
3	2
6	9
8	13

B.

Input	Output
2	1
3	2
6	5
8	13

C.

Input	Output
2	1
3	3
6	5
8	13

D.

Input	Output
2	1
3	3
6	9
8	13

ANS: D

8. Let a function be defined as $f(x) = -4x^2 + x - 3$. What is $f(1)$?
   A. –18
   B. –6
   C. 0
   D. 14

ANS: B

9. Kathy has two sets of numbers, *A* and *B*. The sets are defined as follows:

   $A = \{1, 2, 3\}$ $B = \{10, 20, 30\}$

   Kathy created four relations using elements from Set *A* for the domains and elements from Set *B* for the ranges. Which of Kathy's relations is NOT a function?
   A. {(1, 10), (1, 20), (1, 30)}
   B. {(1, 10), (2, 10), (3, 10)}
   C. {(1, 10), (2, 20), (3, 30)}
   D. {(1, 10), (2, 30), (3, 20)}

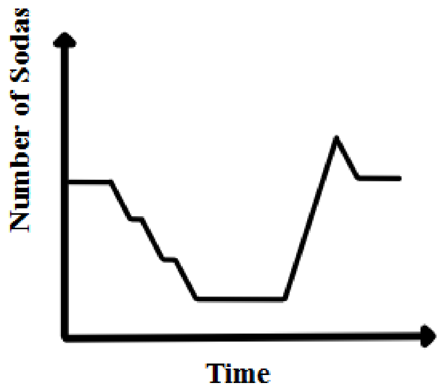
ANS: A

10. Translate the following statements into coordinate points.

    $f(-3) = 1$ $g(2) = 4$ $g(0) = 7$ $k(5) = 8$

    ANS: (-3, 1) (2, 4) (0, 7) (5, 8)

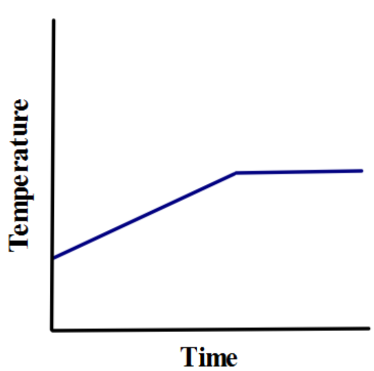
11. The graph represents soda bottles in according to the graph.



SOLUTION: To analyze the graph, read the labels on the *x*- and *y*-axes. Notice, that as the value of *x* increases, time is passing and as the value of *y* increases, the number of sodas in the machine increases.

- The graph starts at a higher point of the graph and maintains that height (flat line) for a period of time. The machine has sodas left over from yesterday.
- Then, the level of sodas decrease at a steady pace for a period of time, possibly during first lunch.
- Then, the level remains the same for a short period of time, possibly passing period between first and second lunches.
- Then, the level of sodas decrease at a steady pace again for a period of time, possibly during second lunch.
- Then, the level remains the same for a short period of time, possibly passing period between second and third lunches.
- Then, the level of sodas decrease again at a steady pace for a period time, possibly during third lunch.
- Then, the level of sodas remain the same for a period of time.
- Then, the level of sodas steadily increase rapidly. Possibly the attendant came to refill the machine.
- Then, the level of sodas remain the same until after school when it decreases again for a period of time. Possibly a few students and teachers purchased sodas on their way out.

12. Describe the graph. Can you determine what the temperature will be after 3 hours? Explain…



Solution:

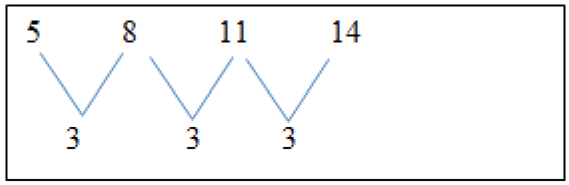
The temperature starts rising at a constant rate and then remains constant again.

No, the temperature cannot be determined after 3 hours because there are no intervals labeled on the graph.

Some of the patterns we study in secondary math are arithmetic sequences. Each number in the sequence is called a ***term***. Each successive term of an arithmetic sequence is separated by a ***common difference***. For example, if you started with the first term of 5 then added 3, you get 8 for the second term. Then add 8 + 3 to get the next term, and so on. The arithmetic sequence 5, 8, 11, 14… is generated, and the common difference is 3.

In these problems we notice that the first difference is constant, and in this case is equal to 3. (***First difference*** refers to the differences we find the first time we subtract previous terms of the sequence, $a_n - a_{n-1}$.

Note: 3 is a rate of change.



Look at it another way. We could also generate the same sequence by using the expression, 3*x* + 2, and substituting 1, 2, 3, … for *x*.

Now we get to the interesting part. If we did not know the linear expression that generated this sequence, could we find it? Since the first differences between the terms remain constant, this is a linear or arithmetic sequence that could generally be written as $y = ax + b$. Let's substitute 1, 2, 3, … for *x* to generate the general-case sequence.

$$a_1 = 3(1) + 2 = 5 \qquad a_2 = 3(2) + 2 = 8 \qquad a_3 = 3(3) + 2 = 11$$

Note: The first difference is constant (does not change) and is equal to the coefficient of the variable.

Aha! Does it make sense that the rate of change is the coefficient of *x* in the linear equation *ax* + *b?* What does the *a* represent in a linear equation?

If *a* = 3, that means that the first term, *3(1)* + *b* = 5. Solving for *b,* 3 + b = 5, so b = 2.

Solution: The sequence was generated by the expression 3*x* + *2.*

Do you see the pattern? Can you describe it and write it?

Here is the pattern: Find the common difference – that is the rate of change. When you plug in any of the other terms, for example the first term, you see what you get by multiplying, then adjust by adding or subtracting to make it work for the value you need.

<u>Sequence</u>: an ordered list of numbers

<u>Finding the *n*th Term of a Sequence</u>: look for a pattern within the sequence and write the pattern in terms of *n*, which is the term number

Ex 16: Use the sequence 3, 6, 9, 12, …. Write the next three terms and write an expression for the *n*th term.

Next three terms: We are adding 3 to each term to obtain the next term, so they are $\boxed{15, 18, 21}$.

*n*th term: When $n = 1$, the term is 3; when $n = 2$, the term is 6; when $n = 3$, the term is 9;… To obtain each term, we multiply the term number by 3. So the expression for the *n*th term is $\boxed{3n}$.

Ex 17: Use the sequence $12, 8, 4, 0, -4, ...$ Write the next three terms and write an expression for the *n*th term.

Next three terms: We are subtracting 4 to each term to obtain the next term, so they are $\boxed{-8, -12, -16}$.

*n*th term: When $n = 1$, the term is 12; when $n = 2$, the term is 8; when $n = 3$, the term is 4;… To obtain each term, we multiply the term number by −4 and add 16. So the expression for the *n*th term is $\boxed{-4n + 16}$.

Challenge Ex: Use the sequence 4, 8, 16, …. Write the next three terms and write an expression for the *n*th term.

Next three terms: We are multiplying each term by 2 to obtain the next term, so they are $\boxed{32, 64, 128}$.

*n*th term: When $n = 1$, the term is 4; when $n = 2$, the term is 8; when $n = 3$, the term is 16;… To obtain each term, we raise 2 to the $n + 1$ power. So the expression for the *n*th term is $\boxed{2^{n+1}}$.

<u>You Try</u>: Use the sequence $5, 3, 1, -1, ...$ Write the next three terms and write an expression for the *n*th term.

<u>QOD</u>: When a sequence is created by adding or subtracting a number to obtain the next term, where is this number in the expression for the *n*th term?

<u>Arithmetic Sequence:</u> a sequence in which the difference between consecutive terms is constant

<u>Common Difference</u>: the difference, *d*, between consecutive terms in an arithmetic sequence

<u>Explicit Rule:</u> a rule for a sequence that gives a_n as a function of the term's position number, *n*, in the sequence

<u>Recursive Rule</u>: a rule for a sequence that gives the beginning term(s) of the sequence and then an equation that tells how a_n is related to one or more preceding terms

*****Make sure that students are familiar with both function formatting and subscript formatting for arithmetic sequences.** Either method might be found on the End-of-Course exam.

Ex 18: Determine if the sequence is arithmetic. If it is, state the common difference. $-8,-5,-2,1,4,\ldots$

Find the difference of consecutive terms:

$-5-(-8)=3$ $\qquad -2-(-5)=3$ $\qquad 1-(-2)=3$ $\qquad 4-1=3$

The difference is constant, so it **is** an arithmetic sequence with $\boxed{d=3}$.

<u>Rule for the *n*th Term of an Arithmetic Sequence</u>

Given an arithmetic sequence with common difference *d*, $\boxed{f(n)=f(1)+(n-1)d}$.

This is the explicit rule in subscript form. $\boxed{a_n=a_1+(n-1)d}$

Ex 19: Write a rule for the *n*th term of the sequence $12,7,2,-3,\ldots$. Then, find $f(23)$ or a_{23}.

Use the rule, with $f(1)$ or $a_1=12$ and $d=-5$.

$f(n)=12+(n-1)(-5)\Rightarrow \boxed{f(n)=17-5n}$

$a_n=12+(n-1)(-5)\Rightarrow \boxed{a_n=17-5n}$

$f(23)=17-5(23)=\boxed{-98}$

$a_{23}=17-5(23)=\boxed{-98}$

Ex 20: Write a rule for an arithmetic sequence with $f(8)$ or $a_8 = 50$ and a common difference of 2.

Use the rule, with $f(n) = 50\ (a_n = 50), n = 8,$ and $d = 2$.

$$50 = f(1) + (8-1)(2)$$
$$36 = f(1)$$

$$f(n) \text{ or } a_n = 36 + (n-1)(2) \Rightarrow \boxed{f(n) \text{ or } a_n = 34 + 2n}$$

<u>**Recursive Rule for Arithmetic Sequences:**</u> (Teachers – have students come up with this.)

$$f(n) = f(n-1) + d \text{ or } a_n = a_{n-1} + d$$

Ex 21: Write the explicit and recursive rules for the arithmetic sequence where $f(1)$ or $a_1 = 15$ and $d = 5$.

Explicit Rule:

$$f(n) = f(1) + (n-1)d = 15 + (n-1)(5) \Rightarrow \boxed{f(n) = 10 + 5n}$$
$$a_n = a_1 + (n-1)d = 15 + (n-1)(5) \Rightarrow \boxed{a_n = 10 + 5n}$$

Recursive Rule:

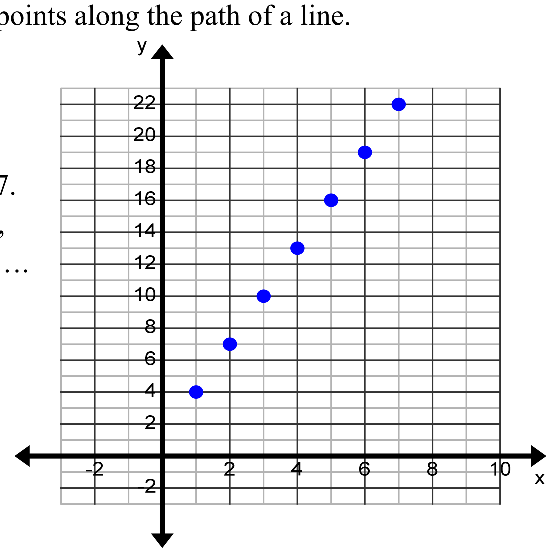
$$\boxed{\begin{array}{l} f(1) = 15,\ \ f(n) = f(n-1) + 5 \\ a_1 = 15,\ a_n = a_{n-1} + 5 \end{array}}$$

<u>Connection between arithmetic sequences and linear functions:</u>

Linear functions graph as lines and have a very special property: equal changes in the input give rise to equal changes in the output. Arithmetic sequences have this same property: equal changes in the input (e.g. moving from term to term) give rise to equal changes in the output (determined by the common difference). Thus, arithmetic sequences always graph as **points long a line**. Instead of a continuous linear function, an arithmetic sequence is a discrete function which is shown on the graph as discrete points along the path of a line.

Ex 22: Graph the arithmetic sequence 4, 7, 10, 13, …

When the input is 1 (for the first term in the sequence), the output is 4.
When the input is 2 (for the second term in the sequence), the output is 7.
When the input is 3 (for the third term in the sequence), the output is 10, and so on. That gives us the ordered pairs (1, 4), (2, 7), (3, 10), (4, 13),….



Ex 23: The table below shows typical costs for a construction company to rent a crane for one, two, three, or four months. Assuming that the arithmetic sequence continues, how much would it cost to rent a crane for one year?

Months	Cost
1	$75,000
2	$90,000
3	$105,000
4	$120,000

Find the common difference: 90,000 – 75,000 = 15,000, so d = 15,000
The first term of the sequence, *f(1)* is 75,000.
One year is equal to 12 months, so n = 12.

$$f(n) = 75,000 + (12-1)(15,000)$$
$$f(n) = 75,000 + 11(15,000)$$
$$f(n) = 75,000 + 165,000 \Rightarrow \boxed{f(n) = \$240,000}$$

Ex 24: A theater has 40 seats in the first row, 46 seats in the second row, 52 seats in the third row, and so on in the same increasing pattern. Find the number of seats in the last row, the 20th row.

Find the common difference: 52 – 46 = 46 – 40 = 6, so d = 6.
The first term of the sequence, $a_1 = 40$.
Find the number in row 20, n = 20.

$$a_n = 40 + (20-1)(6)$$
$$a_n = 40 + (19)(6)$$
$$a_n = 40 + 114 \Rightarrow \boxed{a_n = 154}$$

The twentieth row of the theater has 154 seats in the row.

SAMPLE EXAM QUESTIONS

1. A company is tracking the number of complaints received on its website. During the first 4 months, they record the following numbers of complaints: 20, 25, 30, and 35. Which is a possible explicit rule for the number of complaints they will receive in the *n*th month?

   a. $a_n = 20n + 5$ b. $a_n = 15 + 5n$

ANS: B

2. Find the 16th term in the following arithmetic sequence.–6, –13, –20, –27, –34, ...

   a. –105 c. –126
   b. –118 d. –111

ANS: D

3. Find the first 5 terms of the sequence with $a_1 = 6$ and $a_n = 2a_{n-1} - 1$ for $n \geq 2$.

   a. 1, 2, 3, 4, 5 c. 6, 12, 24, 48, 96
   b. 6, 7, 8, 9, 10 d. 6, 11, 21, 41, 81

ANS: D

4. Write a rule for the *n*th term of the arithmetic sequence. –10, –4, 2, 8, . . .

   a. $a_n = -16(6)^j$ c. $a_n = -16 + 6$
   b. $a_n = 6n - 16$ d. $a_n = 6n - 18$

ANS: B

5. Find a function that describes the arithmetic sequence 16, 17, 18, 19, ... Use *y* to identify each term in the sequence and *n* to identify each term's position.

   a. $y = 16n$ c. $y = n + 15$
   b. $y = 15n + 1$ d. $y = 17n - 1$

ANS: C

6. What is the first term of an arithmetic sequence with a common difference of 5 and a sixth term of 40?

ANS: 15

7. What is the first term of an arithmetic sequence with a common difference of –7 and a seventh term of 40?

ANS: 82

8. Write a rule for the *n*th term of the arithmetic sequence.16, 19, 22, . . .

ANS: $16 + (n-1)(3) = 3n + 13$

9. The first five terms of a sequence are given.

14 17 20 23 26

Which equation describes the n^{th} term of the sequence?

(A) $f(n) = 3 + 11n$ (B) $f(n) = 11 + 3n$

(C) $f(n) = 14 + 17n$ (D) $f(n) = 17 - 3n$

ANS: B

10. What are the first five terms of the sequence defined as

$\boldsymbol{a(1) = 3}$

$\boldsymbol{a(n+1) = a(n) - 4, \text{ for } n \geq 1?}$

(A) –3, –2, –1, 0, 1 (B) –1, –5, –9, –13, –17

(C) 3, –1, –5, –9, –13 (D) 3, –1, 0, 1, 2

ANS: C

11. A sequence *t* is defined as $t(n)=0.57-0.06n$, where $n\geq 1$. Which is an equivalent recursive definition for sequence *t*?

(A) $t(1)=0.57;\ t(n+1)=t(n)-0.06$, for $n\geq 1$

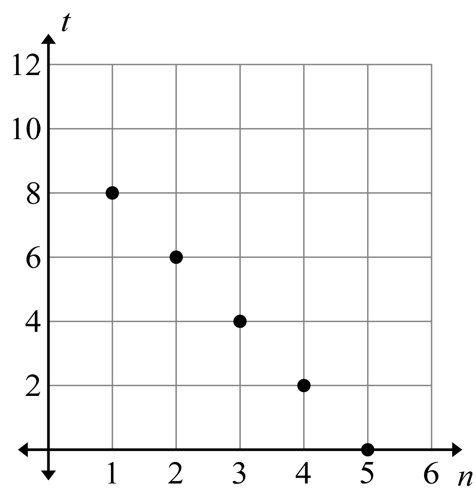
(B) $t(1)=0.51;\ t(n+1)=t(n)-0.06$, for $n\geq 1$

(C) $t(1)=0.57;\ t(n+1)=t(n)-0.51$, for $n\geq 1$

(D) $t(1)=0.51;\ t(n+1)=t(n)-0.51$, for $n\geq 1$

ANS: B

12. The graph shows the first five terms of an arithmetic sequence whose domain is the positive integers.



Which is a definition of the sequence?

(A) $t(n)=8-n$ (B) $t(n)=8-2n$

(C) $t(n)=10-n$ (D) $t(n)=10-2n$

ANS: D

13. Sam is beginning an exercise program that begins the first week with 30 minutes of daily exercise. Each week, daily exercise is increased by 5 minutes. Which function represents the number of minutes of daily exercise in week *n*?

(A) $f(1)=30;\ f(n)=30n$, for $n\geq 2$ (B) $f(1)=30;\ f(n)=5n+30$, for $n\geq 2$

(C) $f(1)=30;\ f(n)=f(n-1)+5$, for $n\geq 2$

(D) $f(1)=30;\ f(n)=5f(n-1)$, for $n\geq 2$

ANS: C