



Pre-Algebra Notes – Unit One: Rational Numbers and Decimal Expansions

Rational Numbers

Rational numbers are numbers that can be written as a quotient of two integers.

Since decimals are special fractions, all the rules we learn for fractions will work for decimals. The only difference is the denominators for decimals are powers of 10; i.e., 10^1 , 10^2 , 10^3 , 10^4 , etc.... Students normally think of powers of 10 in standard form: 10, 100, 1000, 10,000, etc.

In a decimal, the numerator is the number to the right of the decimal point. The denominator is not written, but is implied by the number of digits to the right of the decimal point. The number of digits to the right of the decimal point is the same as the number of zeros in the power of 10: 10, 100, 1000, 10,000...

Therefore, one place is tenths, two places are hundredths, and three places are thousandths.

Examples:

1) 0.56	2 places	→	$\frac{56}{100}$
2) 0.532	3 places	→	$\frac{532}{1000}$
3) 3.2	1 place	→	$3\frac{2}{10}$

The correct way to say a decimal numeral is to:

- 1) Forget the decimal point (if it is less than one).
- 2) Say the number.
- 3) Then say its denominator and add the suffix “*ths*”.

Examples:

1)	0.53	Fifty-three hundredths
2)	0.702	Seven hundred two thousandths
3)	0.2	Two tenths
4)	5.63	Five and sixty-three hundredths

When there are numbers on both sides of the decimal point, the decimal point is read as “and”. You say the number on the left side of the decimal point, and then the decimal point is read as “and”. You then say the number on the right side with its denominator.

- Examples:*
- 1) Write 15.203 in word form.
Fifteen and two hundred three thousandths
 - 2) Write 7.0483 in word form.
Seven and four hundred eighty-three ten-thousandths
 - 3) Write 247.45 in word form.
Two hundred forty-seven and forty-five hundredths

Converting Fractions to Decimals: Terminating and Repeating Decimals

NVACS 8.NS.A.1: Understand informally that every number has a decimal expansion; show that the decimal expansion of a rational number repeats eventually or terminates.

A *rational number*, a number that can be written in the form of $\frac{a}{b}$ (quotient of two integers), will either be a terminating or repeating decimal. A *terminating decimal* has a finite number of decimal places; you will obtain a remainder of zero. A *repeating decimal* has a digit or a block of digits that repeat without end.

One way to convert fractions to decimals is by making equivalent fractions.

Example: Convert $\frac{1}{2}$ to a decimal.

Since a decimal is a fraction whose denominator is a power of 10, look for a power of 10 that 2 will divide into evenly.

$$\frac{1}{2} = \frac{5}{10}$$

Since the denominator is 10, we need only one digit to the right of the decimal point, and the answer is 0.5.



Example: Convert $\frac{3}{4}$ to a decimal.

Again, since a decimal is a fraction whose denominator is a power of 10, we look for powers of 10 that the denominator will divide into evenly. 4 won't go into 10, but 4 will go into 100 evenly.

$$\frac{3}{4} = \frac{75}{100}$$

Since the denominator is 100, we need two digits to the right of the decimal point, and the answer is 0.75.

There are denominators that will never divide into any power of 10 evenly. Since that happens, we look for an alternative way of converting fractions to decimals. Could you recognize numbers that are not factors of powers of ten? Using your Rules of Divisibility, factors of powers of ten can only have prime factors of 2 or 5. That would mean 12, whose prime factors are 2 and 3, would not be a factor of a power of ten. That means that 12 will never divide into a power of 10 evenly. For example, a fraction such as $\frac{5}{12}$ will not terminate – it will be a repeating decimal.

Not all fractions can be written with a power of 10 as the denominator. We need to look at another way to convert a fraction to a decimal: *divide the numerator by the denominator*.

Example: Convert $\frac{3}{8}$ to a decimal.

This could be done by equivalent fractions since the only prime factor of 8

is 2. $\frac{3}{8} \rightarrow \frac{3}{8} \cdot \frac{125}{125} = \frac{375}{1000}$

However, it could also be done by division. $\frac{0.375}{8 \overline{)3.000}}$

Doing this division problem, we get 0.375 as the equivalent decimal.

Example: Convert $\frac{5}{12}$ to a decimal.

This could not be done by equivalent fractions since one of the factors of 12 is 3. We can still convert it to a decimal by division.

$$\begin{array}{r} 0.41666\dots \\ 12 \overline{)5.00000} \end{array}$$

Six is repeating, so we can write it as $0.41\overline{6}$.



The *vinculum* is written over the digit or digits that repeat.



Example: Convert $\frac{4}{11}$ to a decimal.

This would be done by division.

$$\frac{4}{11} \rightarrow 11 \overline{)4.0000} \begin{array}{l} 0.3636\dots \\ \text{or } 0.\overline{36} \end{array}$$

Converting Decimals to Fractions

NVACS 8.NS.1: Convert a decimal expansion which repeats eventually into a rational number.

To convert a decimal to a fraction:

- 1) Determine the denominator by counting the number of digits to the right of the decimal point.
- 2) The numerator is the number to the right of the decimal point.
- 3) Simplify, if possible.

Examples: 1) Convert 0.52 to a fraction.

$$\begin{aligned} 0.52 &= \frac{52}{100} \\ &= \frac{13}{25} \end{aligned}$$

2) Convert 0.613 to a fraction.

$$0.613 = \frac{613}{1000}$$

3) Convert 8.32 to a mixed number and improper fraction.

$$\begin{aligned} 8.32 &= 8\frac{32}{100} \\ &= 8\frac{8}{25} \text{ or } \frac{208}{25} \end{aligned}$$

But what if we have a repeating decimal?

While the decimals 0.3 and $0.\overline{3}$ look alike at first glance, they are different. They do not have the same value. We know 0.3 is three tenths, $\frac{3}{10}$. How can we say or write $0.\overline{3}$ as a fraction?

As we often do in math, we take something we don't recognize and make it look like a problem we have done before. To do this, we eliminate the repeating part – the vinculum (line over the 3).

Example: Convert $0.\overline{3}$ to a fraction.

$$0.\overline{3} = 0.333333\dots$$

Let's let $x = 0.333333\dots$

Notice, and this is important, that only *one number is repeating*. If I multiply both sides of the equation above by 10 (*one zero*), then subtract the two equations, the repeating part disappears.

$$\begin{array}{r} 10x = 3.3333\dots \\ - x = 0.3333\dots \\ \hline 9x = 3.0000 \\ \frac{9x}{9} = \frac{3.0000}{9} \\ x = \frac{1}{3} \end{array}$$

$\frac{1}{3}$ is the equivalent fraction for $0.\overline{3}$

Example: Convert $0.\overline{345}$ to a fraction.

The difficulty with this problem is the decimal is repeating. So we eliminate the repeating part by letting $x = 0.\overline{345}$.

$$0.\overline{345} = 0.345345345\dots$$

Note, *three digits are repeating*. By multiplying both sides of the equation by 1000 (*three zeros*), the repeating parts line up. When we subtract, the repeating part disappears.

$$\begin{array}{r} 1000x = 345.345345345\dots \\ - x = 0.345345345\dots \\ \hline \end{array}$$

$$\frac{999x}{999} = \frac{345}{999}$$

$$x = \frac{345}{999} \text{ or } \frac{115}{333}$$



Example: Convert $0.\overline{13}$ to a fraction.

Note, *one digit is repeating, but one is not.* By multiplying both sides of the equation by 10, the repeating parts line up. When we subtract, the repeating part disappears.

$$10x = 1.33333\dots$$

$$\begin{array}{r} - x = .13333\dots \\ \hline \end{array}$$

$$9x = 1.2$$

$$\frac{9x}{9} = \frac{1.2}{9}$$

$$x = \frac{1.2}{9} \text{ or } \frac{12}{90} \left(\text{which simplifies to } \frac{2}{15} \right)$$

Ready for a “short cut”? Let’s look at some patterns for repeating decimals.

$$\frac{1}{9} = 0.111\dots \text{ or } 0.\overline{1}$$

$$\frac{1}{11} = 0.0909\dots \text{ or } 0.\overline{09}$$

$$\frac{2}{9} = 0.222\dots \text{ or } 0.\overline{2}$$

$$\frac{2}{11} = 0.1818\dots \text{ or } 0.\overline{18}$$

$$\frac{3}{9} = 0.333\dots \text{ or } ?$$

$$\frac{3}{11} = 0.2727\dots \text{ or } ?$$

$$\frac{4}{9} = ?$$

$$\frac{4}{11} = ?$$

It is easy to generate the missing decimals when you see the pattern!

Let’s continue to look at a few more repeating decimals, converting back into fractional form.

▼ Because we are concentrating on the pattern, we will choose NOT to simplify fractions where applicable. This would be a step to add later.

$$0.\overline{5} = \frac{5}{9}$$

$$0.\overline{13} = \frac{13}{99}$$

$$0.\overline{123} = \frac{123}{999}$$

$$0.\overline{6} = \frac{6}{9}$$

$$0.\overline{25} = \frac{25}{99}$$

$$0.\overline{154} = \frac{154}{999}$$

$$0.\overline{7} = \frac{?}{9}$$

$$0.\overline{37} = \frac{?}{99}$$

$$0.\overline{421} = \frac{?}{999}$$

$$0.\overline{8} = \frac{?}{?}$$

$$0.\overline{56} = \frac{?}{?}$$

$$0.\overline{563} = \frac{?}{?}$$

The numerator of the fraction is the same numeral as the numeral under the vinculum. We can also quickly determine the denominator: it is 9^{th} s for *one* place under the vinculum, 99^{th} s for *two* places under the vinculum, 999^{th} s for *three* places under the vinculum, and so on.

But what if the decimal is of a form where not all the numerals are under the vinculum? Let's look at a few.

$$0.2\overline{3} = \frac{21}{90}$$

The numerator is generated by subtracting the number *not* under the vinculum from the entire number (including the digits under the vinculum).

$$0.37\overline{59} = \frac{3722}{9900}$$

$$0.3\overline{5} = \frac{32}{90}$$

$$0.42\overline{76} = \frac{4234}{9900}$$

$$0.4\overline{27} = \frac{423}{990}$$

$$0.201\overline{5} = \frac{1814}{9000}$$

$$0.3\overline{25} = \frac{322}{990}$$

We still determine the number of nines in the denominator by looking at the number of digits under the vinculum. The number of digits *not* under the vinculum gives us the number of zeroes.

$$0.60\overline{24} = \frac{5964}{9900}$$

$$0.4\overline{276} = \frac{4272}{9990}$$

$$0.814\overline{37} = \frac{80623}{99000}$$

$$0.23\overline{5} = \frac{212}{900}$$

$$0.5534\overline{1} = \frac{49807}{90000}$$

▼ Note that again we chose not to simplify fractions where applicable as we want to concentrate on the pattern.

Does $\overline{.9} = 1$????

Do you believe it? Let's look at some reasons why it's true. Using the method we just looked at:



$$\begin{array}{r}
 10x = 9.9999999\dots \\
 - \quad x = .9999999\dots \\
 \hline
 9x = 9
 \end{array}$$

Surely if $9x = 9$, then $x = 1$. But since x also equals $.9999999\dots$ we get that $.9999999\dots = 1$.

But this is unconvincing to many people. So here's another argument. Most people who have trouble with this fact oddly *don't* have trouble with the fact that $1/3 = .3333333\dots$. Well, consider the following addition of equations then:

$$\begin{array}{r}
 \frac{1}{3} = .3333333\dots \\
 + \frac{2}{3} = .6666666\dots \\
 \hline
 \frac{3}{3} = .9999999\dots
 \end{array}$$

This seems simplistic, but it's very, very convincing, isn't it? Or try it with some other denominator:

$$\begin{array}{r}
 \frac{1}{11} = .09090909\dots \\
 + \frac{10}{11} = .90909090\dots \\
 \hline
 \frac{11}{11} = .99999999\dots
 \end{array}$$

Which works out very nicely. Or even:

$$\begin{array}{r}
 \frac{2}{7} = .285714285714\dots \\
 + \frac{5}{7} = .714285714285\dots \\
 \hline
 \frac{7}{7} = .999999999999\dots
 \end{array}$$

It will work for any two fractions that have a repeating decimal representation and that add up to 1. The problem, though, is BELIEVING it is true.

So, you might think of $0.9999\dots$ as another name for 1, just as $0.333\dots$ is another name for $1/3$.

