Section 4.2: One – to – One and Inverse Functions

Activity 1: Inverse Functions Reversing the Process

You might have experienced converting between degrees Fahrenheit and degrees Celsius when measuring a temperature. The standard formula for determining temperature in degrees Fahrenheit, when given the temperature in degrees Celsius, is $F = \frac{9}{5}C + 32$. We can use this formula to define a function named g, namely $F = g(C) = \frac{9}{5}C + 32$, where C is the number of degrees Celsius and $g(C)$ is a number of degrees Fahrenheit. The function g defines a process for converting degrees Celsius to degrees Fahrenheit.

a. What is the value of $g(100)$? What does it represent?

b. Solve the equation g(C) = 112 and describe the meaning of your answer.

c. What happens if you want to input degree Fahrenheit and output degree Celsius? Reverse the process of the formula $F = \frac{9}{5}C + 32$ by solving for C

Two ***functions*** are said to be inverses of each other if they are the reverse process of each other. Notice, in the activity, the formula found in *c* was the **reverse process of *g*.** Instead of inputting in Celsius and outputting Fahrenheit, the new function inputs Fahrenheit and outputs degrees Celsius. We can notate this by using inverse function notation g^{-1} (note: $g^{-1} \neq \frac{1}{g}$).

Use inverse notation to rewrite the formula found in *c****

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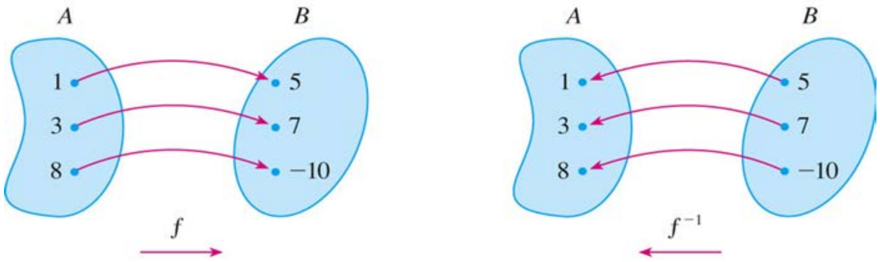
More formally, if a function $y = f(x)$ assigns values of the input quantity *x* to values of the output quantity *y*, the inverse of *f*, denoted $x = f^{-1}(y)$ is the function which assigns output values of *f* to corresponding input values.



Example 1: The functions f and g are defined in the table below. Based on this table, evaluate the following:

x	-3	-2	-1	0	1	2	3
$f(x)$	9	2	6	-4	-5	-8	-9
$g(x)$	3	0	1	2	-3	-1	-5

$a.$ $g(3)$ b) $f(2)$ c) $f^{-1}(2)$ d) $g^{-1}(3)$ e) $f^{-1}(f(2))$

Example 2: Use the graph below to determine the following:

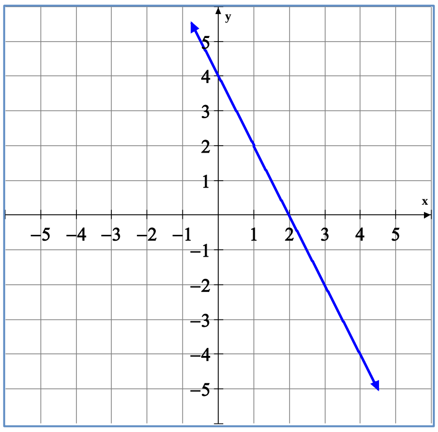
a. $f(2)$

b. $f^{-1}(2)$

c. $f^{-1}(4)$

d. $f\ (4)$

e. $f^{-1}(0)$

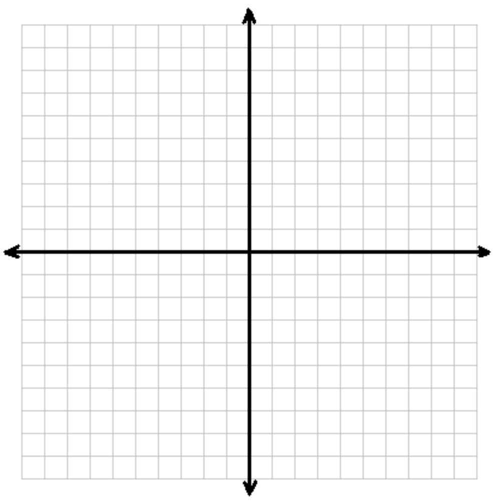


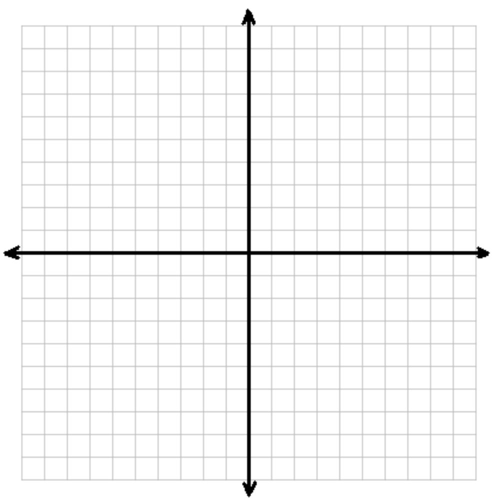
Because most graphing calculators only accept inputs of x, it has become common practice to interchange the variables x and y in order to obtain a function that is easier to graph

Example 3: Find the inverse of each function as though you were going to enter it into a graphing calculator. Then graph f and f^{-1} on the same set of axis. Then find the domain and range of f and f^{-1}.

a. $y = f(x) = 1 - 3x$

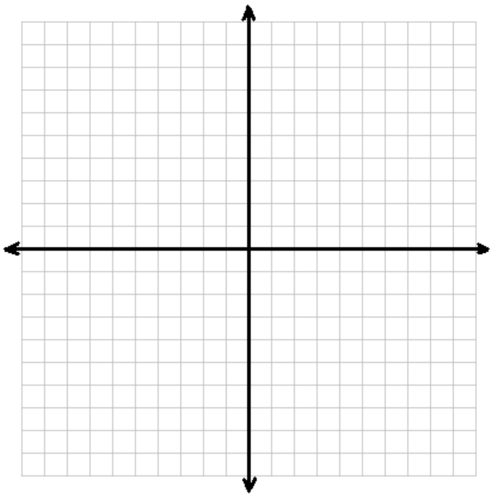
b. $f(x) = \frac{2x-5}{6}$

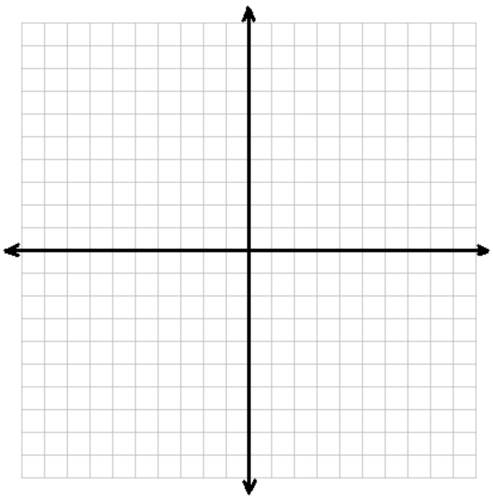




c. $f(x) = x^3$

d. $f(x) = x^2 + 9, x \geq 0$ $(why\ does\ x \geq 0)$

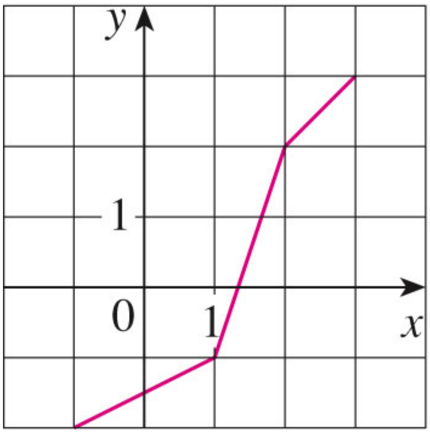


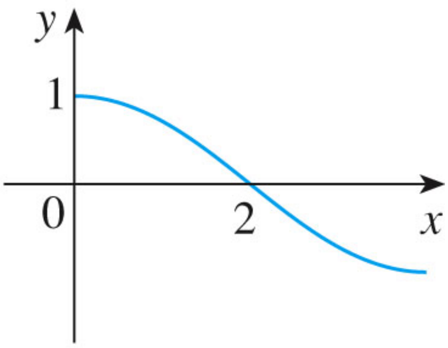


Fact: The graph of the inverse of a function is the same as the graph of the original function but reflected about the line $y = x$.

Draw the line $y = x$ onto each of the graphs in example 3. Does this fact hold true?

Example 4: Use the given graph of f to sketch the graph of f^{-1}.





Fact: If $f\left(g(x)\right) = x$ and $g\left(f(x)\right) = x$ then the functions f and g are inverses of each other.

Example 5: Find the inverse function of $g(x) = -\frac{2x}{x-1}$ and call it *f(x)*. Use the fact above to then verify your solution.

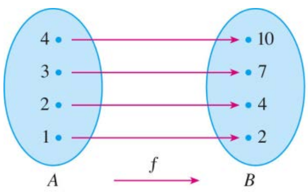
Definition: Functions are defined as **one-to-one functions** if each output corresponds to *exactly one* input.

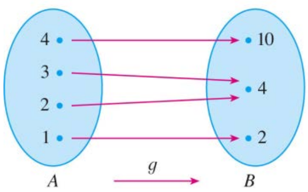
Why are all inverse functions always one-to-one functions?

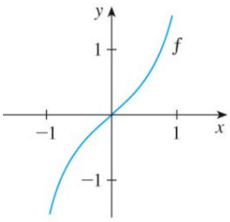
Example 6: Determine whether each function is one – to – one.

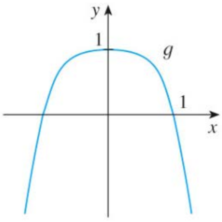
x	1	2	3	4
f(*x*)	1.5	2.1	1.6	2.0

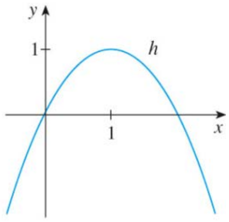
x	1	2	3	4
f(*x*)	1.5	2.0	1.6	2.0











The Horizontal Line Test
If a function is one – to – one then no horizontal line intersects the graph of the function more than once.

Example 7: A function *f* defined as $m = f(d) = 28d$ relates the number of miles of driving distance, *m*, to the number of inches on a map *d*. We say that the number of miles driven, *m*, is a function of the number of inches on a map *d*. The function *f* accepts the values of inches on a map as input and produces values of the number of miles driven as output.

a. Define a function f^{-1} that expresses the *number of inches on a map as a function of the number of miles* of driving distance.

b. Verify that these functions are inverses of each other by showing that

$$f\left(f^{-1}(m)\right) = m \quad and \quad f^{-1}(f(d)) = d$$

c. Find the value of $f^{-1}(280)$. Describe what this means in the context of the problem.

Example 8: The head circumference *C* of a child is related to the height H of the child (both in inches) through the function: $H(C) = 2.15C - 10.53$

a. Express the head circumference *C as a function of height H*.

b. Verify that *C=C(H)* is the inverse of *H=H(C)* by showing that *H(C(H))=H* and *C(H(C))=C.*

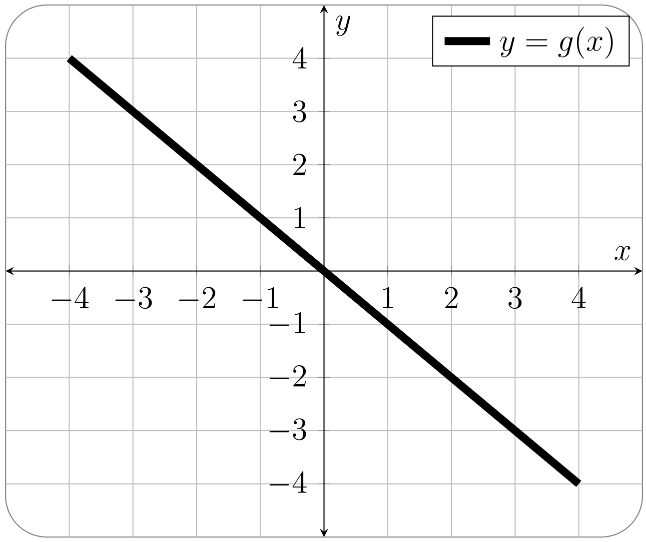
c. Predict the head circumference of a child who is 26 inches tall.

1. Use the following table and graph to evaluate the following:

   (a) $f(0)$ (c) $f^{-1}(\pi)$ (e) $f^{-1}(-13)$ (g) $f^{-1}(100)$

   (b) $g(2)$ (d) $g^{-1}(-4)$ (f) $g^{-1}(-3)$ (h) $g^{-1}(4)$

x	$f(x)$
0	4
10	π
20	17
25	-100
37	-13



2. Determine if the following functions are one-to-one by sketching their graphs, using your calculator if you need to. Hence determine if each has an inverse, but do not find the inverse. *Hint: you are testing to see if these functions pass/fail the horizontal line test.*

   (a) $f(x) = 3x - 5$ (c) $p(t) = 7t^2 + 2$ (e) $h(x) = |x - 8| + 4$

   (b) $g(x) = (x + 2)^2 - 1$ (d) $s(t) = \sqrt{2t - 1} + 15$ (f) $m(x) = 5\sqrt[3]{x + 2} - 6$

3. For each of the following, use composition to determine if g is the inverse of f by calculating $(f \circ g)(x)$

   (a) $f(x) = 3x + 5$; $g(x) = \frac{1}{3}x - \frac{1}{5}$ (c) $f(x) = (x-3)^3 + 4$; $g(x) = \sqrt[3]{x - 4} + 3$

   (b) $f(x) = 2x - 4$; $g(x) = \frac{1}{2x - 4}$ (d) $f(x) = 6x + 2$; $g(x) = \frac{x - 2}{6}$

4. The following functions are one-to-one. Find $f^{-1}(x)$, and then verify your result by finding $(f \circ f^{-1})(x)$.

   (a) $f(x) = 5x^3 + 2$ (c) $f(x) = 4 - \frac{1}{x}$

   (b) $f(x) = 2 + \sqrt{x - 3}$ (d) $f(x) = \frac{4}{3}x + 7$

5. Suppose at a particular college, the model

$$c(h) = 26h + 12,$$

   represents the cost, c, for students who are taking h hours of classes.

   (a) What would be a reasonable domain for this function?

   (b) Evaluate and interpret $c(10)$.

   (c) Is $c(h)$ one to one? If so, find the inverse function.

   (d) Use the inverse function to determine how much many hours of classes a student can enrole in with $272.