Radicals & Functions

Math Background

Previously, you

- Worked with exponents and used exponent properties
- Evaluated expressions with exponents
- Identified the domain, range and x-intercepts of real-life functions
- Graphed quadratic, polynomial and rational functions
- Solved quadratic, polynomial and rational functions
- Transformed parent functions of quadratic and polynomial functions

In this unit you will

- Use the properties of exponents to convert between radical notation and terms with rational exponents
- Evaluate expressions with rational exponents
- Graph square root and cube root functions with and without technology
- Solve problems involving radical equations

You can use the skills in this unit to

- Use the structure of an expression to identify ways to rewrite it.
- Interpret the domain and its restrictions of a real-life function.
- Describe how a square root or cube root graph is related to its parent function.
- Identify extraneous solutions

Vocabulary

- **Base** – A number that is used as a repeated factor. In 4^3, the "4" is the base.
- **Cube Root Function** – A function whose value is the cube root of its argument. $f(x) = \sqrt[3]{x}$
- **End Behavior** – The end behavior of a function is the behavior of the graph of *f(x)* as *x* approaches positive infinity or negative infinity.
- **Extraneous Solutions** – A root of a transformed equation that is not a root of the original equation because it was excluded from the domain of the original equation.
- **Exponent** – A number used to indicate the number of times a term is used as a factor to multiply itself. The exponent is normally placed as a superscript after the term.
- **Power** – A short way of writing the same number multiplied by itself several times, written as a^n. It includes the base and the exponent.
- **Radical** – The taking of a root of a number. The symbol is $\sqrt{\ }$.
- **Radical Equation** – An equation containing radical expressions with variables in the radicands.
- **Radicand** – A number or expression inside the radical symbol.
- **Root Index** – A number written as a superscript to the left of the radical symbol giving the n^{th} root to be found. If there is no index, it is implied to find the square root. If the index is a "3", it tells us to find the cube root of the expression.
- **Square Root Function** – A function that maps the set of non-negative real numbers onto itself and when graphed is a half of a parabola with a vertical directrix.

Essential Questions

- How can numbers with rational exponents be written in other notations?
- What do the key features of the graphs of square root and cube roots tell you about the function?
- How do I solve a radical equation? How are extraneous solutions generated from a radical equation?

Overall Big Ideas

Using the properties of exponents, we can rewrite radical expressions to exponential form and vice versa.

The graph shows the solutions for the functions illustrating domain and range.

We solve radical equations by transforming the equation into a simpler form to solve. However, this can produce solutions that do not exist in the original domain.

Skill

To simplify and perform operations with radical expressions by applying properties of radicals.

To use properties of rational exponents to simplify and evaluate expressions.

To solve equations containing radicals or rational exponents.

To derive and verify inverse functions both algebraically and graphically.

To graph square root and cube root equations using "parent" functions, inverse functions and transformations.

Related Standards

N.RN.A.1
Explain how the definition of the meaning of rational exponents follows from extending the properties of integer exponents to those values, allowing for a notation for radicals in terms of rational exponents. For example, we define 5^(1/3) to be the cube root of 5 because we want (5^(1/3))^3 = 5^((1/3)3) to hold, so (5^(1/3))^3 must equal 5.

N.RN.A.2
Rewrite expressions involving radicals and rational exponents using the properties of exponents.

A.REI.A.2
Solve simple rational and radical equations in one variable, and give examples showing how extraneous solutions may arise.

F.IF.A.1

Understand that a function from one set (called the domain) to another set (called the range) assigns to each element of the domain exactly one element of the range. If f is a function and x is an element of its domain, then f(x) denotes the output of f corresponding to the input x. The graph of f is the graph of the equation y = f(x).

F.IF.A.2

Use function notation, evaluate functions for inputs in their domains, and interpret statements that use function notation in terms of a context.

F.BF.B.4b

Verify by composition that one function is the inverse of another.

F.BF.B.4c

Read values of an inverse function from a graph or a table, given that the function has an inverse.

F.BF.B.4a-2

Solve an equation of the form f(x) = c for a simple radical, rational, power, or exponential function f that has an inverse and write an expression for the inverse. For example, f(x) = 2x3 or f(x) = (x + 1)/(x – 1) for x ≠ 1.

G.CO.A.2

Represent transformations in the plane using, e.g., transparencies and geometry software; describe transformations as functions that take points in the plane as inputs and give other points as outputs. Compare transformations that preserve distance and angle to those that do not (e.g., translation versus horizontal stretch).

A.CED.A.2-2

Create equations in two or more variables to represent relationships between quantities and graph equations on coordinate axes with labels and scales. Use all types of equations. *(Modeling Standard)

F.IF.C.7b-2

Graph square root and cube root functions. *(Modeling Standard)

F.BF.B.3-2

Identify the effect on the graph of replacing *f*(*x*) by *f*(*x*) + k, *k f*(*x*), *f*(k*x*), and *f*(*x* + k) for specific values of k (both positive and negative); find the value of *k* given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. Include recognizing even and odd functions from their graphs and algebraic expressions for them. Include simple radical, rational, and exponential functions, note the effect of multiple transformations on a single graph, and emphasize common effects of transformations across function types.

Notes, Examples, and Exam Questions

REVIEW Radicals:

n^{th} Root: If $b^n = a$, then b is the n^{th} root of a. This is written $\sqrt[n]{a} = b$. n is called the **index** of the radical. a is called the **radicand**.

Note: The square root has an implied index of 2.

Ex: $\sqrt{9} = \boxed{3}$, because $3^2 = 9$ **Ex**: $\sqrt[3]{8} = \boxed{2}$, because $2^3 = 8$

Multiple Roots

Ex: Find the real *n*th root(s) of *a* if *a* = 16 and *n* = 4.

$\pm\sqrt[4]{16} = \boxed{\pm 2}$, because $(-2)^4 = 16$ and $2^4 = 16$.

Units 4.1 and 4.2 To simplify and perform operations with radical expressions by applying properties of radicals. To use properties of rational exponents to simplify and evaluate expressions.

This unit should be review. In Algebra II, Unit 5, students used properties of rational exponents to rewrite radicals, evaluated expressions with rational exponents, solved radical equations and inequalities, and graphed and transformed radical functions. Learning target 4.4 is a review of Algebra II, Unit 1.2 and 1.4.

Roots as Rational Exponents: The n^{th} root, $\sqrt[n]{a}$, can be written as an exponent $a^{\frac{1}{n}}$.

$a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m$ - Notice the placement of the *m* and *n*. The root index is the denominator and the exponent is the numerator.

Ex 1: Evaluate $256^{\frac{1}{4}}$.

$256^{\frac{1}{4}} = \boxed{4}$, because $4^4 = 256$. Note: This could also be written as $\sqrt[4]{256} = \boxed{4}$.

Ex 2: Evaluate $\sqrt[3]{-27}$.

$\sqrt[3]{-27} = (-27)^{\frac{1}{3}} = \boxed{-3}$, because $(-3)^3 = -27$.

Ex 3: Evaluate $25^{\frac{3}{2}}$.

It often helps to find the root first (the denominator) and then do the multiplication (the numerator).

$$25^{\frac{3}{2}} = \left(\sqrt{25}\right)^3 = (5)^3 = \boxed{125}$$

Ex 4: Evaluate $(243)^{-\frac{2}{5}}$.

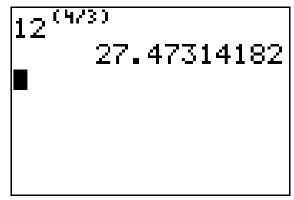
$$(243)^{-\frac{2}{5}} = \frac{1}{(243)^{\frac{2}{5}}} = \frac{1}{\left(\sqrt[5]{243}\right)^2} = \frac{1}{(3)^2} = \boxed{\frac{1}{9}}$$

Finding Roots on the Calculator

Ex 5: Use the graphing calculator to approximate $\left(\sqrt[3]{12}\right)^4$.

We will rewrite $\left(\sqrt[3]{12}\right)^4$ with a rational exponent: $(12)^{\frac{4}{3}}$. Type this into your calculator using the carrot key [^] for the exponent. Be sure to put the exponent in parentheses.

[1] [2] [^] [(] [4] [÷] [3] [)] [ENTER]



Question: What expression is your calculator evaluating if you do not use parentheses?

Ex 6: Use the properties of exponents to simplify the expression $\left(7^5 \cdot 3^5\right)^{-\frac{1}{5}}$.

Power of a Product: $\left(7^5 \cdot 3^5\right)^{-\frac{1}{5}} = \left[(7 \cdot 3)^5\right]^{-\frac{1}{5}}$

Power of a Power: $\left[(7 \cdot 3)^5\right]^{-\frac{1}{5}} = (7 \cdot 3)^{-1} = 21^{-1} = \boxed{\frac{1}{21}}$

****Note: The product and quotient properties for exponents can be extended to radicals, as we now know that a radical is simply a rational exponent.**

Product Property: $\sqrt[n]{a \cdot b} = \sqrt[n]{a}\sqrt[n]{b}$

Quotient Property: $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$

Note that while we can "break up" products and quotients under a radical, we can't do the same thing for sums or differences. In other words, $\sqrt[n]{a+b} \neq \sqrt[n]{a} + \sqrt[n]{b}$ AND $\sqrt[n]{a-b} \neq \sqrt[n]{a} - \sqrt[n]{b}$

Ex 7: Use the properties of exponents to simplify the expression $\left(\frac{\sqrt[4]{30}}{\sqrt[4]{6}}\right)^2$.

Power of a Quotient: $\left(\frac{\sqrt[4]{30}}{\sqrt[4]{6}}\right)^2 = \left(\sqrt[4]{\frac{30}{6}}\right)^2 = \left(\sqrt[4]{5}\right)^2$

Rewrite with Rational Exponents: $\left(\sqrt[4]{5}\right)^2 = \left(5^{\frac{1}{4}}\right)^2$

Power of a Power: $\left(5^{\frac{1}{4}}\right)^2 = 5^{\frac{2}{4}} = \boxed{5^{\frac{1}{2}}}$

<u>Simplest Form of a Radical:</u> A radical is said to be in simplified radical form (or just simplified form) if each of the following are true.

1. All exponents in the radicand must be less than the index.
2. Any exponents in the radicand have no factors in common with the index.
3. No fractions appear under a radical.
4. No radicals appear in the denominator of a fraction.

Ex 8: Simplify the expression $(320)^{\frac{1}{3}}$.

Step One: Factor out the perfect cube (3rd root). $\sqrt[3]{64\cdot 5}$

Step Two: Rewrite using the product property. $\sqrt[3]{64}\cdot\sqrt[3]{5}$

Step Three: Simplify by taking the cube root of the perfect cube. $\boxed{4\sqrt[3]{5} = 4(5)^{\frac{1}{3}}}$

Ex 9: Simplify the expression $\left(\frac{8}{3}\right)^{\frac{1}{4}}$.

Step One: Rewrite in radical form. $\frac{\sqrt[4]{8}}{\sqrt[4]{3}}$

Step Two: Multiply the numerator and denominator by a root that will make the denominator's radicand a perfect 4th root. (We can multiply 3 by 27 for a product of 81, which is a perfect 4th root.

$$\frac{\sqrt[4]{2}}{\sqrt[4]{3}}\cdot\frac{\sqrt[4]{27}}{\sqrt[4]{27}} = \frac{\sqrt[4]{2\cdot 27}}{\sqrt[4]{3\cdot 27}} = \frac{\sqrt[4]{54}}{\sqrt[4]{81}}$$

Step Three: Simplify by taking the 4th root of the denominator.

$$\frac{\sqrt[4]{54}}{3} = \frac{1}{3}(54)^{\frac{1}{4}}$$

Ex 10: Simplify the expression $\dfrac{\sqrt{x^2-7x+12}}{\sqrt{x^2+2x-15}}$.

Step One: Factor the numerator and denominator. $\dfrac{\sqrt{(x-3)(x-4)}}{\sqrt{(x-3)(x+5)}}$

Step Two: Rewrite using the product property. $\dfrac{\sqrt{(x-3)}\sqrt{(x-4)}}{\sqrt{(x-3)}\sqrt{(x+5)}}$

Step Three: Cancel the common factor. $\dfrac{\sqrt{(x-4)}}{\sqrt{(x+5)}}$

Step Four: Rationalize the denominator. $\dfrac{\sqrt{(x-4)}}{\sqrt{(x+5)}} \cdot \dfrac{\sqrt{(x+5)}}{\sqrt{(x+5)}} = \boxed{\dfrac{\sqrt{x^2+x-20}}{x+5}}$

<u>Simplifying Variable Expressions</u>: Note: For the following exercises, we must assume all variables are positive.

Ex 11: Simplify the expression $\sqrt{xy} \cdot \sqrt{x^3y^5}$.

Step One: Rewrite using rational exponents. $(xy)^{\frac{1}{2}}(x)^{\frac{3}{2}}(y)^{\frac{5}{2}}$

Step Two: Use the product of powers property. $\left(x^{\frac{1}{2}+\frac{3}{2}}\right)\left(y^{\frac{1}{2}+\frac{5}{2}}\right)$

Step Three: Simplify. $\left(x^{\frac{4}{2}}\right)\left(y^{\frac{6}{2}}\right) = \boxed{x^2y^3}$

Ex 12: Simplify the expression $\sqrt[3]{\dfrac{8x^6}{y^9}}$.

Step One: Rewrite using rational exponents. $\dfrac{8^{\frac{1}{3}}\left(x^{\frac{6}{3}}\right)}{y^{\frac{9}{3}}}$

Step Two: Simplify. $\boxed{\dfrac{2x^2}{y^3}}$

Ex 13: Simplify the expression $\sqrt[3]{x}\cdot\sqrt[4]{x}$.

Step One: Rewrite using rational exponents. $(x)^{\frac{1}{3}}\cdot(x)^{\frac{1}{4}}$

Step Two: Use the product of powers property. $\left(x^{\frac{1}{3}+\frac{1}{4}}\right)=\left(x^{\frac{4}{12}+\frac{3}{12}}\right)$

Step Three: Simplify. $\left(x^{\frac{7}{12}}\right)=\boxed{\sqrt[12]{x^7}}$

Ex 14: Simplify the expression $\sqrt[4]{x^5y^8}$.

Step One: Rewrite the radicand extracting perfect 4th roots. $\sqrt[4]{x^4\cdot x\cdot y^4\cdot y^4}$

Note: Powers of 4 are perfect 4th roots $\Rightarrow\ \sqrt[4]{x^4}=\left(x^4\right)^{\frac{1}{4}}=x$.

Step Two: Take the 4th root of any perfect roots. $x\cdot y\cdot y\sqrt[4]{x}$

Step Three: Simplify. $\boxed{xy^2\sqrt[4]{x}}$

Ex 15: Simplify the expression $\dfrac{2\sqrt{x}\cdot\sqrt{x^3}}{\sqrt{18x^{10}}}$.

Step One: Multiplication (numerator). $\dfrac{2\sqrt{x^4}}{\sqrt{18x^{10}}}$

Step Two: Rewrite the radicands extracting perfect square roots. $\dfrac{2\sqrt{x^2\cdot x^2}}{\sqrt{9\cdot 2}\sqrt{x^2\cdot x^2\cdot x^2\cdot x^2\cdot x^2}}$

Step Three: Take the square root of any perfect roots. $\dfrac{2\cdot x\cdot x}{3\sqrt{2}\cdot x\cdot x\cdot x\cdot x\cdot x}$

Step Four: Multiply and simplify. $\dfrac{2x^2}{3\sqrt{2}\cdot x^5}=\dfrac{2}{3\sqrt{2}\cdot x^3}=\dfrac{2}{3x^3\sqrt{2}}$

Step Five: Rationalize the denominator. $\dfrac{2}{3x^3\sqrt{2}}\cdot\dfrac{\sqrt{2}}{\sqrt{2}}=\dfrac{2\sqrt{2}}{3x^3\cdot 2}=\boxed{\dfrac{\sqrt{2}}{3x^3}}$

Like Terms: Llike terms have identical variable and radical factors. To combine like terms means to add the coefficients while leaving the variable and radical factors unchanged.

Note: Only **like radicals** can be added or subtracted.

Ex 16: Simplify the following expression by adding/subtracting like radicals. $\sqrt[3]{16}+\sqrt[3]{5}-7\sqrt[3]{2}$

Step One: Simplify each radical by extracting any perfect cube roots.

$$\sqrt[3]{8}\cdot\sqrt[3]{2}+\sqrt[3]{5}-7\sqrt[3]{2}$$
$$=2\sqrt[3]{2}+\sqrt[3]{5}-7\sqrt[3]{2}$$

Step Two: Add/subtract the like radicals (cube roots of 2).

$$=(2-7)\sqrt[3]{2}+\sqrt[3]{5}$$
$$=\boxed{-5\sqrt[3]{2}+\sqrt[3]{5}}$$

Ex 17: Perform the indicated operation. Assume all variables are positive. $10\sqrt[4]{5s^7}-s\sqrt[4]{80s^3}$

Step One: Simplify each radical by extracting any perfect fourth roots.

$$10\sqrt[4]{5s^7}-s\sqrt[4]{80s^3}$$
$$=10\sqrt[4]{5s^4\cdot s^3}-s\sqrt[4]{16\cdot 5s^3}$$
$$=10s\sqrt[4]{5s^3}-2s\sqrt[4]{5s^3}$$

Step Two: Add/subtract the like radicals.

$$(10s-2s)\sqrt[4]{5s^3}$$
$$=\boxed{8s\sqrt[4]{5s^3}}$$

SAMPLE EXAM QUESTIONS

1. Write the radical expression $\left(\sqrt[3]{7}\right)^4$ in exponential notation.

A. 7^1 **B.** 7^{12}

C. $7^{\frac{3}{4}}$ **D.** $7^{\frac{4}{3}}$

Ans: D

2. If $f(x)=\sqrt{x^2-1}$ and $g(x)=\sqrt{x-1}$, which expression represents $\dfrac{f(x)}{g(x)}$ for $x>1$?

A. $\sqrt{x}$ **B.** $\sqrt{x-1}$

C. $\sqrt{x+1}$ **D.** $\dfrac{1}{\sqrt{x+1}}$

Ans: C

3. If $x > 1$ and $\frac{\sqrt{x}}{x^3} = x^m$, what is the value of *m*?

(A) $-\frac{7}{2}$

(B) -3

(C) $-\frac{5}{2}$

(D) -2

(E) $-\frac{3}{2}$

Ans: E

4. Assuming all variables are positive, what is the simplified form of $-z^2\sqrt{16z^3} + 3\sqrt{36z^7}$?

(A) $-z^3\sqrt{z}$

(B) $14z^3\sqrt{z}$

(C) $14z^4\sqrt{z}$

(D) $92z^3\sqrt{z}$

Ans: B

5. Simplify $\left(4\sqrt{3} - 6\sqrt{2}\right)^2$.

(A) $120 - 24\sqrt{6}$

(B) 120

(C) $120 - 48\sqrt{6}$

(D) $48 - 48\sqrt{6}$

Ans: C

6. Use the rules of exponents to simplify the expression. Write the answer with positive exponents. Assume that all variables represent positive real numbers. $\frac{x^{\frac{1}{2}}}{x^{\frac{3}{4}} \cdot x^{-2}}$.

(A) $x^{\frac{7}{4}}$

(B) $\frac{1}{x^{\frac{13}{4}}}$

(C) $\frac{1}{x^{\frac{7}{4}}}$

(D) $x^{\frac{13}{4}}$

Ans: A

Unit 4.3 To solve equations containing radicals or rational exponents.

Solving radical equations are very similar to solving other types of equations. The objective is to get the variable by itself. However, now there are radicals within the equations. Recall that the opposite of the square root of something is to square it and the opposite of the cube root of something is to cube it. Make sure to ALWAYS check your answers when solving radical equations. Sometimes you will solve an equation, get a solution, and then plug it back in and it will not work. These types of solutions are called **extraneous solutions** and are not actually considered solutions to the equation. **This unit should be treated as review of Algebra II, Unit 5.**

Steps for Solving a Radical Equation:

- Isolate the radical.
- Raise both sides to the *n*th power, where *n* is the index of the radical.
- Isolate the variable and solve the equation.
- Check your solution in the original equation. This is crucial, as you may obtain extraneous solutions – solutions that do not work in the original equation.

Ex 18: Solve the equation $12-\sqrt[4]{3x}=9$.

Step One, isolate the radical:

$$-\sqrt[4]{3x}=-3$$
$$\sqrt[4]{3x}=3$$

Step Two, raise both sides to the third power:

$$\left(\sqrt[4]{3x}\right)^4=(3)^4$$
$$3x=81$$

Step Three, solve:

$$x=27$$

Step Four, check the solution :

$$12-\sqrt[4]{3\cdot 27}=9$$
$$12-\sqrt[4]{81}=9$$
$$12-3=9$$

The solution works in the original equation, so $\boxed{x=27}$

Ex 19: Solve the equation $2\sqrt{6x-5}+20=6$.

Step One, isolate the radical:

$$2\sqrt{6x-5}=-14$$
$$\sqrt{6x-5}=-7$$

Step Two, square both sides:

$$\left(\sqrt{6x-5}\right)^2=(-7)^2$$
$$6x-5=49$$

Step Three, solve:

$$6x = 54$$
$$x = 9$$

Step Four, check the solution:

$$2\sqrt{6(9)-5} + 20 = 6$$
$$2\sqrt{49} + 20 = 6$$
$$14 + 20 \neq 6$$

The solution does not work in the original equation. Therefore, 9 is an extraneous solution, and this equation has NO SOLUTION.

Question: Could you have determined earlier in the process of solving that this equation had no solution? Explain.

Ex 20: Solve the equation $\sqrt{7x+15} = x+1$.

Step One: Done (radical is isolated)

Step Two, square both sides:

$$\left(\sqrt{7x+15}\right)^2 = (x+1)^2$$
$$7x + 15 = x^2 + 2x + 1$$

Step Three: Because this is a quadratic equation, you may use one of the methods for solving quadratic equations (quadratic formula, factoring, or completing the square).

$$0 = x^2 - 5x - 14$$
$$0 = (x-7)(x+2) \quad \text{This can be factored.}$$
$$x = -2, 7$$

Step Four, check the solutions:

$x = -2: \sqrt{7(-2)+15} = (-2)+1$

$\sqrt{-14+15} = -1$

$1 \neq -1$ Not a solution

$x = 7: \sqrt{7(7)+15} = (7)+1$

$\sqrt{49+15} = 8$

$\sqrt{64} = 8$ solution!

Solution Set: $\boxed{\{7\}}$

Equations with Two Radicals: To solve, we will move the radicals to opposite sides, then raise both sides to the *n*th power, where *n* is the index of the radical.

Ex 21: Solve the equation $\sqrt[4]{2x+10} - 2\sqrt[4]{x} = 0$.

Step One: Move one of the radicals to the other side. $\sqrt[4]{2x+10} = 2\sqrt[4]{x}$

Step Two: Raise both sides to the 4th power. $\left(\sqrt[4]{2x+10}\right)^4 = \left(2\sqrt[4]{x}\right)^4 \rightarrow 2x + 10 = 16x$

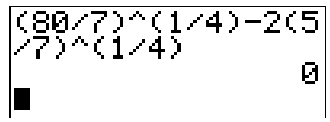
Step Three: Solve for x. $10 = 14x \rightarrow \frac{5}{7} = x$

Step Four:

$$\sqrt[4]{2\left(\frac{5}{7}\right)+10} - 2\sqrt[4]{\left(\frac{5}{7}\right)} = 0$$

$$\sqrt[4]{\frac{80}{7}} - 2\sqrt[4]{\frac{5}{7}} = 0$$

These are not perfect 4th roots, so we will check on the calculator.



The solution is $\boxed{x = \frac{5}{7}}$.

Ex 22: Solve the equation $\sqrt{x+2} + 1 = \sqrt{3-x}$.

Step One: Square both sides.

$$\left(\sqrt{x+2}+1\right)^2 = \left(\sqrt{3-x}\right)^2$$

$$x + 2 + 2\sqrt{x+2} + 1 = 3 - x$$

Step Two: Isolate the radical. $2\sqrt{x+2} = -2x \rightarrow \sqrt{x+2} = -x$

Step Three: Square each side again. $\left(\sqrt{x+2}\right)^2 = (-x)^2$

Step Four: Solve by factoring.

$$0 = x^2 - x - 2$$

$$0 = (x-2)(x+1)$$

$$x = 2, -1$$

Step Five, check the solution.

$x = 2: \sqrt{2+2} + 1 = \sqrt{3-2}$ $\quad$ $x = -1: \sqrt{-1+2} + 1 = \sqrt{3-(-1)}$

$\sqrt{4} + 1 = \sqrt{1}$ $\quad$ $\sqrt{1} + 1 = \sqrt{4}$

$3 \neq 1$ 2 is extraneous $\quad$ $2 = 2$ -1 is a solution

The solution is $\boxed{x = -1}$.

Solving Equations with Rational Exponents

- Isolate the expression with the rational exponent.
- Raise both sides to the reciprocal power (see below).
- Isolate the variable.
- Check your solution in the original equation. This is crucial, as you may obtain extraneous solutions.

To solve an equation in the form $x^{\frac{m}{n}} = a$, raise both sides to the reciprocal power: $\left(x^{\frac{m}{n}}\right)^{\frac{n}{m}} = (a)^{\frac{n}{m}} \Rightarrow x = a^{\frac{n}{m}}$

Ex 23: Solve the equation $(16x)^{\frac{3}{4}} + 44 = 556$.

Step One: Isolate the variable. $(16x)^{\frac{3}{4}} = 512$

Step Two: Raise both sides to the reciprocal power. $\left((16x)^{\frac{3}{4}}\right)^{\frac{4}{3}} = (512)^{\frac{4}{3}}$

Step Three: Simplify.

$16x = \left((512)^{\frac{1}{3}}\right)^4 \rightarrow 16x = 8^4$

$16x = 4096 \rightarrow x = 256$

Step Four: Check your answer.

$(16(256))^{\frac{3}{4}} + 44 = 556$

$\left((4096)^{\frac{1}{4}}\right)^3 = 512$

$8^3 = 512$

The solution works in the original equation so the solution is $\boxed{256}$.

Ex 24: Solve the equation $-125 = 5x^{\frac{2}{5}}$.

Step One: Isolate the variable.

$\frac{-125}{5} = \frac{5x^{\frac{2}{5}}}{5}$

$-25 = x^{\frac{2}{5}}$

Step Two: Raise both sides to the reciprocal power.

$(-25)^{\frac{5}{2}} = \left(x^{\frac{2}{5}}\right)^{\frac{5}{2}}$

$\left(\sqrt{-25}\right)^5 = x$

Because there is no real square root of -25, this equation has $\boxed{\text{no real solution}}$.

Ex 25: Solve the equation $\frac{1}{7}(x+9)^{\frac{3}{2}} = 49$.

Step One: $(x+9)^{\frac{3}{2}} = 343$

Step Two: $\left[(x+9)^{\frac{3}{2}}\right]^{\frac{2}{3}} = (343)^{\frac{2}{3}} \rightarrow x+9=49$

Step Three: $x = 40$

$\frac{1}{7}((40)+9)^{\frac{3}{2}} = 49$

Step Four: $(49)^{\frac{3}{2}} = 343$ The solution is $\boxed{x=40}$.

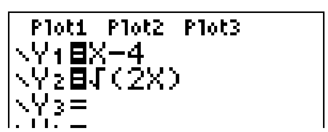
$(7)^3 = 343$

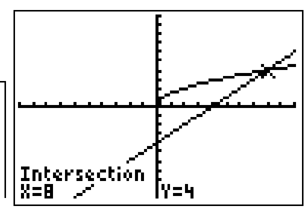
Solving a Radical Equation on the Graphing Calculator: We will solve equations by graphing. You may either graph both sides of the equation as two functions and find the *x*-coordinate of the point of intersection, or set the equation equal to zero and find the *x*-intercept of the resulting function.

Ex 26: Solve the equation $x - 4 = \sqrt{2x}$ by graphing.

Method 1: Graph both sides of the equation.

The solution is $\boxed{x=8}$.



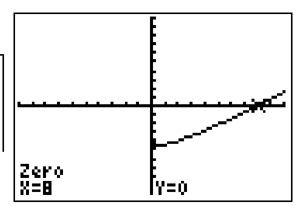


Question: What does the *y*-value represent in the point of intersection?

Method 2: Set the equation equal to zero.

$x - 4 - \sqrt{2x} = 0$

The solution is $\boxed{x=8}$.



QOD: Explain using rational exponents why raising a radical to the nth power, where n is the index of the radical, will eliminate the radical.

SAMPLE EXAM QUESTIONS

1. **Which value of x makes this equation true?** $9(x-7)^{\frac{4}{3}}=9$

   A. 1 **B.** 7 **C.** 8 **D.** 34

   Ans: C

2. **Solve for x:** $\sqrt[3]{4x+1}=5$

   A. $x=-31$ **B.** $x=6$ **C.** $x=31$ **D.** No real solution

   Ans: C

3. **Solve for x.** $x-7=\sqrt{x-1}$

   A. $x=5$ and $x=10$ **B.** $x=5$

   C. $x=10$ **D.** No real solutions

   Ans: A

4. **Solve for x.** $\sqrt{x-3}-\sqrt{x}=3$

   A. $x=4$ **B.** $x=6$

   C. $x=9$ **D.** No real solutions

   Ans: D

5. **What is the value of x in the equation** $\sqrt{2x+1}+3=6$**?**

   A. 1 **C.** 4

   B. 13 **D.** 16

   Ans: C

6. **If** $\sqrt[3]{12x+28}=4$**, what is the value of** x^3**?**

   A. -8 **B.** 3 **C.** 12 **D.** 27

   Ans: D

7. The relationship between the weight of a whale in tons, W, and the length in feet, L, is given by $W = 0.000137L^{3.18}$. Which expression below would be used to find the length of a whale that weighs 50 tons?

A. $\sqrt[3.18]{\dfrac{50}{0.000137}}$

B. $0.000137(50)^{3.18}$

C. $\dfrac{\sqrt[3.18]{50}}{0.000137}$

D. $\sqrt[3.18]{50}(0.000137)$

Ans: A

8. What is the value of x in the equation $\sqrt{2x^2+5} = \sqrt{x^2+6x}$?

E. $x = \dfrac{5}{2}, x = 3$

C. $x = 5, x = 1$

F. $x = \dfrac{5}{2}, x = -1$

D. $x = -5, x = 3$

Ans: C

<u>Unit 4.4 To derive and verify inverse functions both algebraically and graphically.</u>

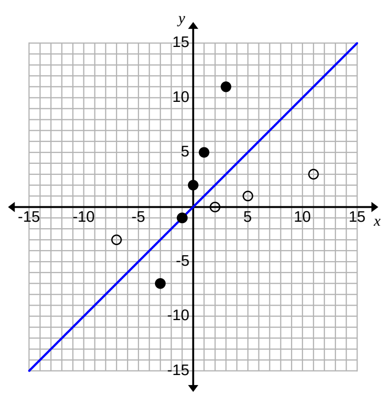
<u>Inverse Relation:</u> a mapping of the output values of a relation to its input values

Ex 27: Find the inverse relation of the relation.

To find the inverse relation, we will switch the input (x) values and the output (y) values.

x	-7	-1	2	5	11
y	-3	-1	0	1	3

Note: Looking at the graph of the relation (solid points) and its inverse relation (open points), we can see that the inverse relation includes all of the points reflected over the line $y = x$.



Finding the Equation of an Inverse Relation: recall that the inverse of a relation is its reflection over the line $y = x$. Therefore, to find the equation of an inverse relation, we will reverse the *x* and *y* variables and solve for *y*.

Note: If both the relation and its inverse relation are functions, then the two relations are called **inverse functions**.

Notation for Inverse Functions: The inverse of a function f is denoted f^{-1}.

Caution: This is not to be confused with the exponent −1!!

Ex 28: Find the equation of the inverse function of $f(x) = 5x - 6$.

Step One: Switch the *x* and *y* variables. Note: $f(x) = y$. $\quad x = 5y - 6$

Step Two: Solve for *y*.

$$x + 6 = 5y$$
$$\frac{1}{5}x + \frac{6}{5} = y$$

Step Three: Write in inverse notation.

$$\boxed{f^{-1}(x) = \frac{1}{5}x + \frac{6}{5}}$$

We can verify our answer using the graph of the ordered pairs, as in example 27. However, a function and its inverse have another special relationship.

Verifying Inverse Functions: To verify that two functions are inverses, we must show that $f\left(f^{-1}(x)\right) = f^{-1}\left(f(x)\right) = x$. The function $y = x$ is the identity function, so the composition of a function and its inverse is the identity.

Ex 29: Show algebraically and graphically that the functions $f(x) = 3x + 2$ and $f^{-1}(x) = \frac{1}{3}x - \frac{2}{3}$ are inverses.

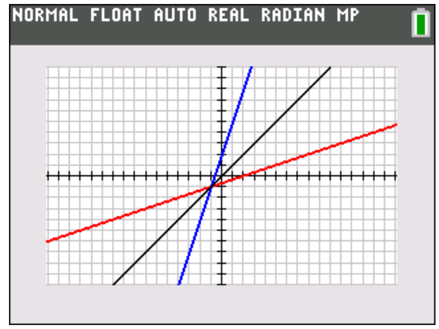
Step One: Show that $f\left(f^{-1}(x)\right) = x$.

$$f\left(f^{-1}(x)\right) = 3\left(\frac{1}{3}x - \frac{2}{3}\right) + 2$$
$$f\left(f^{-1}(x)\right) = x - 2 + 2$$
$$f\left(f^{-1}(x)\right) = x$$

Step Two: Show that $f^{-1}\left(f(x)\right) = x$.

$$f^{-1}\left(f(x)\right) = \frac{1}{3}(3x + 2) - \frac{2}{3}$$
$$f^{-1}\left(f(x)\right) = x + \frac{2}{3} - \frac{2}{3}$$
$$f^{-1}\left(f(x)\right) = x$$

Step Three: Graph the functions and show that they are a reflection over the line $y = x$.



In this graph, the line $y = x$ is in black.

$f(x)$ is in blue and it's inverse is in red.

We have shown that these functions are inverses.

Ex 30: Find the inverse of the function $y = x^2$ for $x \geq 0$.

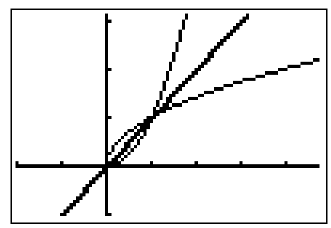
Step One: Switch the *x* and *y*. $x = y^2$

Step Two: Solve for *y*. $y = \pm\sqrt{x}$

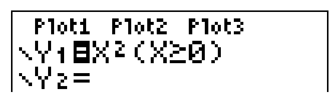
Because $x \geq 0$, we only need the positive square root. $y = \sqrt{x}$

Step Three: Rewrite in inverse notation. $\boxed{y^{-1} = \sqrt{x}}$

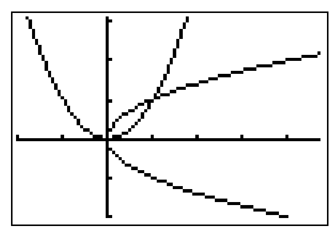
Let's take a look at the graphs of these two functions. They are reflections of each other over the line $y = x$.



Calculator Note: To graph $y = x^2$ on its restricted domain, use parentheses after the function.



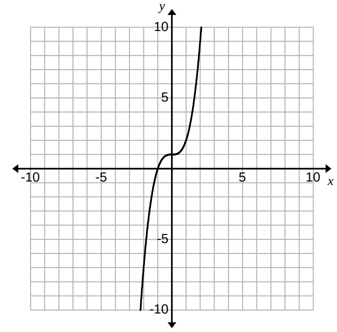
What if the domain of $y = x^2$ was not restricted? Let's take a look at the graphs. You can see that the inverse of $y = x^2$ is not a function.



By looking at a function's graph, we can see if it has an inverse function using the Horizontal Line Test.

<u>Horizontal Line Test</u>: If a horizontal line intersects the graph of a function *f* not more than once, then the inverse of *f* is a function.

Ex 31: Determine whether $y = x^3 + 1$ is a function. If it is, determine if it has an inverse function. If it does, find the inverse function and graph to verify.



Step One: Look at the graph of $y = x^3 + 1$ and use the vertical line test to verify it is a function.

It passes the vertical line test, so it is a function.

Step Two: Look at the graph of $y = x^3 + 1$ and use the horizontal line test to verify it has an inverse function.

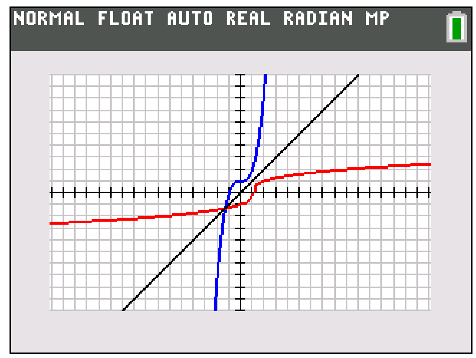
It passes the horizontal line test, so it has an inverse function.

Step Three: Find the inverse function.

$$x = y^3 + 1$$
$$x - 1 = y^3$$
$$\sqrt[3]{x-1} = y$$
$$\boxed{y^{-1} = \sqrt[3]{x-1}}$$

Step Four: Graph the two functions.

The inverse is a function, and it is the reflection of the original function over the line $y = x$.



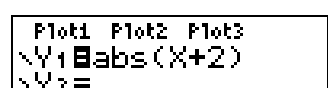
<u>Power Function:</u> a function of the form $y = ax^b$, where *a* is a real number and *b* is a rational number

Note: As shown in the examples above, the inverse of a power function is a **radical function**.

<u>Graphing Inverses on the Graphing Calculator</u>

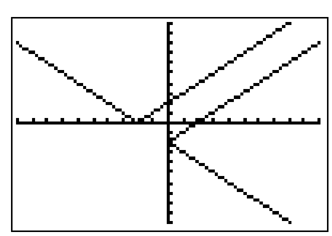
Your calculator cannot find the inverse of a function, but it can draw the inverse.

Ex 32: Use a graphing calculator to graph the inverse of $y = |x+2|$.



Step One: Enter the function into Y1. (For absolute value, use "abs" in the MATH menu.)

Step Two: On the home screen choose 8:DrawInv from the Draw menu, then choose Y1 from the VARS menu. Press Enter, and the calculator will automatically draw the inverse.



Is the inverse a function? Did you know before the calculator graphed it?

SAMPLE EXAM QUESTIONS

1. What is the inverse of $f(x) = 2x + 9$?

A. $f^{-1}(x) = \frac{x}{2} + 9$ **C.** $f^{-1}(x) = \frac{x-9}{2}$

B. $f^{-1}(x) = \frac{1}{2x+9}$ **D.** $f^{-1}(x) = \frac{2}{x-9}$ Ans: C

2. If $f^{-1}(x) = \frac{4}{3}x + 8$, what is $f(x)$?

A. $f(x) = \frac{3}{4}(x-8)$ **C.** $f(x) = \frac{4}{3}x - 6$

B. $f(x) = \frac{3}{4}x - 8$ **D.** $f(x) = \frac{4}{3}(x-8)$ Ans: A

3. If the point (a, b) lines on the graph *y= f(x)*, the graph of $y = f^{-1}(x)$ must contain point

A. (0, b) **B.** (a, 0) **C.** (b, a) **D.** (-a, -b) Ans: B

4. The inverse function of {(2, 6), (-3, 4), (7, -5)} **is**

A. {(-2,6),(3,4),(-7,-5)} **C.** {(6,2),(4,-3),(-5,7)}

B. {(2,-6),(-3,-4),(7,5)} **D.** {(-6,-2),(-4,3),(5,-7)} Ans: C

5. Given the relation *A*: {(3, 2), (5, 3), (6, 2), (7,4)}

A. Both A and A^{-1} are functions. **C.** Only A^{-1} is a function.

B. Neither A nor A^{-1} are functions. **D.** Only A is a function. Ans: D

6. The inverse of the function $2x + 3y = 6$ is

A. $y = -\frac{2}{3}x + 2$ **C.** $y = \frac{3}{2}x + 2$

B. $y = -\frac{3}{2}x + 3$ **D.** $y = \frac{2}{3}x + 3$ Ans: B

7. Which is the inverse of the function $y = \dfrac{\sqrt{2x-3}}{3}$?

C. $y = \dfrac{x^2+3}{2}$, where $x \geq 0$

D. $y = \dfrac{3x^2+3}{2}$, where $x \geq 0$

E. $y = \dfrac{9x^2+3}{2}$, where $x \geq 0$

F. $y = \dfrac{x^2}{2}$, where $x \geq 0$

Ans: B

8. Which statement must be true if f and g are inverses of one another?

A. $(f \circ g)(x) + (g \circ f)(x) = x$

B. $(f \circ g)(x) = f(x) \bullet g(x) = (g \circ f)(x) = g(x) \bullet f(x) = x$

C. $(f \circ g)(x) = f(g(x)) = (g \circ f)(x) = g(f(x)) = x$

D. $(f \circ g)(x) = \dfrac{1}{(g \circ f)(x)} = x$

Ans: C

Unit 4.5 To graph square root and cube root equations using "parent" functions, inverse functions and transformations.

Parent Function - Standard Form for a Radical Function: $y = a\sqrt[n]{x-h} + k$

In this section, we will concentrate on graphing the square root and cube root functions, when the index is 2 or 3. The square root function is the inverse of a quadratic function, and the cube root function is the inverse of a cubic function.

We will use the "parent functions" $y = \sqrt{x}$ and $y = \sqrt[3]{x}$.

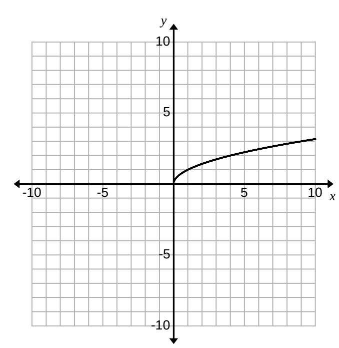
Ex 33: Graph of $y = \sqrt{x}$ or $y = x^{\frac{1}{2}}$:

Domain: $x \geq 0$; Range: $y \geq 0$

x-intercept, y-intercept: (0, 0)

End behavior: $f(x) \to +\infty$ as $x \to +\infty$

x	$f(x) = \sqrt{x}$
0	0
1	1
4	2
9	3



Ex 34: Graph of $y=\sqrt[3]{x}$ or $y=x^{\frac{1}{3}}$:

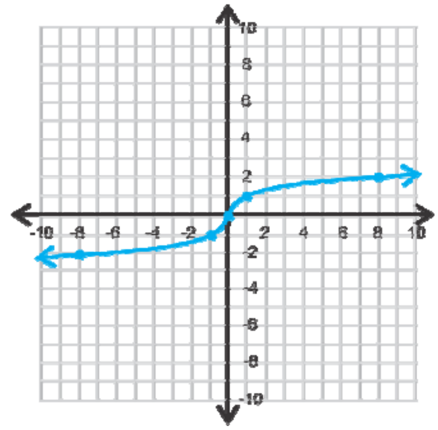
Domain: all real numbers

Range: all real numbers

x-intercept, y-intercept: (0, 0)

End behavior: $f(x)\to+\infty$ as $x\to+\infty$
$f(x)\to-\infty$ as $x\to-\infty$

X	Y1
-5	-1.71
-4	-1.587
-3	-1.442
-2	-1.26
-1	-1
0	0
1	1
2	1.2599
3	1.4422
4	1.5874
5	1.71



<u>Explore on a graphing calculator</u>: Graph the following on the graphing calculator and make note of the changes to the appropriate parent function.

$y=-\sqrt{x}$ $\quad y=2\sqrt{x}$ $\quad y=-3\sqrt{x}$ $\quad y=\frac{1}{3}\sqrt{x}$

$y=\sqrt{x+3}$ $\quad y=\sqrt{x-4}$ $\quad y=\sqrt{x}+5$ $\quad y=\sqrt{x}-2$

Do the same activity with cube roots in place of the square roots.

<u>Summary of the Transformations on Square Root and Cube Root Functions:</u>

$y=a\sqrt{x-h}+k$ $\qquad y=a\sqrt[3]{x-h}+k$

a: If $a<0$, the graph is reflected over the *x*-axis. If $0<|a|<1$, the graph is compressed vertically (or stretched horizontally. If $|a|>1$, the graph is stretched vertically.

h: The graph is shifted *h* units horizontally.

k: The graph is shifted *k* units vertically.

For the Square root function:

The domain is: $\{x\,|\,x\geq h\}$ *The range is*: If $a>0$, then the range is $\{f(x)\,|\,f(x)\geq k\}$

If $a<0$, then the range is $\{f(x)\,|\,f(x)\leq k\}$

For the Cube root function:

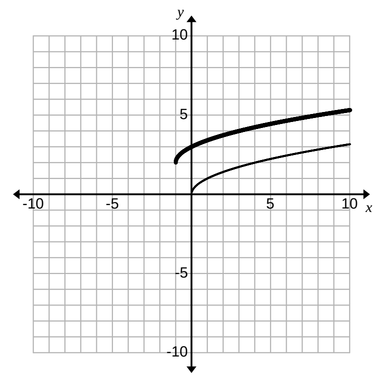
The domain is: All Real Numbers *The range is:* All Real Numbers

Ex 35: Describe how the graph of $y=\sqrt{x+1}+2$ compares to $y=\sqrt{x}$. Then, sketch the graph and state its domain and range.

$h=-1$ and $k=2$, so the graph will be shifted left 1 and up 2.

$a=1$, so the graph will not be stretched.

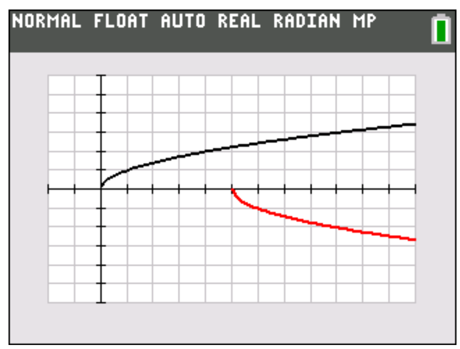
Note: It helps to sketch the graph of the parent function first. The graph of $y=\sqrt{x+1}+2$ is the bold graph.



Domain: $x\geq -1$; Range: $y\geq 2$ (Note: These can be found by looking at the graph.) They can also be found algebraically. The radicand must always be positive, so set $\sqrt{x+1}\geq 0$ and solve the inequality, so we get $x+1\geq 0 \rightarrow x\geq -1$. To find the range algebraically, let the radicand equal zero and solve for y:
$y=\sqrt{x+1}+2 \rightarrow y=0+2 \rightarrow y=2$ and since a is positive, we have $y\geq 0$.

Ex 36: Graph $y_1=\sqrt{x}$ and $y_2=-\sqrt{x-5}$ on the same coordinate plane. Describe the transformation.

Horizontal Shift RIGHT 5 units, reflect over the x-axis. (graph is in red)

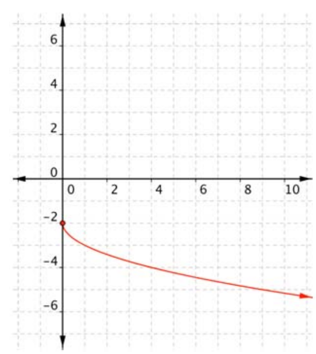


Domain: $x\geq 5$; Range: $y\leq 0$

x-intercept: $0=\sqrt{x-5} \rightarrow 0=x-5 \rightarrow \boxed{5=x}$

y-intercept: $y=\sqrt{0-5} \rightarrow$ No y-intercept

Ex 37: Sketch the graph of $f(x)=-\sqrt{x}-2$.



Transformations: Reflect over the x-axis, shift down 2 units. Note: It helps to do the reflection first before doing other translations.

Domain: $x\geq 0$; Range: $y\leq -2$

Work: $\sqrt{x}\geq 0$, so $x\geq 0$. $y=-\sqrt{x}-2 \rightarrow y\leq 0-2 \rightarrow y\leq -2$.

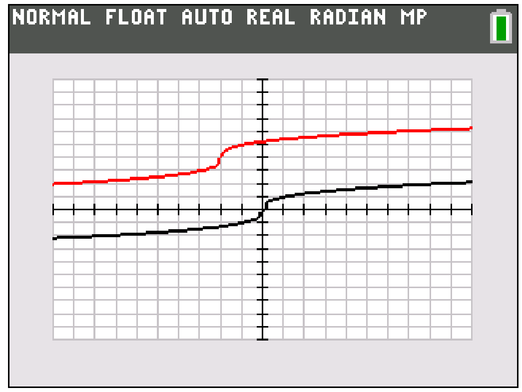
Note that it is $\leq$ and not $\geq$ because the graph is reflected over the x-axis.

x-intercept: None

y-intercept: -2

Ex 38: Graph $y_1 = \sqrt[3]{x}$ and $y_2 = \sqrt[3]{x+2} + 4$ on the same coordinate plane. Describe the transformation.

Transformations: Shift left 2 and shift up 4 units. Graph is in red.



Domain: All Reals; Range: All Reals

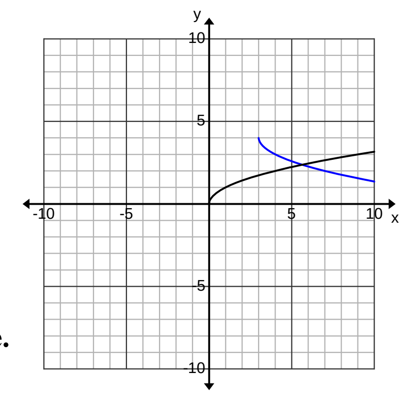
x-intercept: $0 = \sqrt[3]{x+2} + 4 \rightarrow -4 = \sqrt[3]{x+2}$

$-64 = x + 2 \rightarrow x = -66$

y-intercept: $y = \sqrt[3]{0+2} + 4 \approx 5.26$

Ex 39: Sketch the graph the function $y = 4 - (x-3)^{\frac{1}{2}}$ and state its domain and range.

(Hint: Write it in the form $y = a\sqrt{x-h} + k$ first.)



Standard Form: $f(x) = -\sqrt{x-3} + 4$

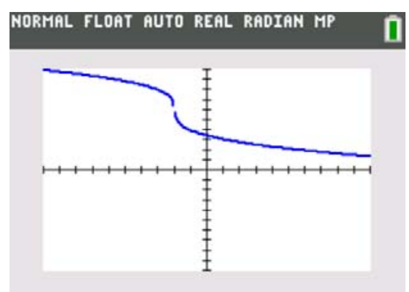
Transformations: Reflect over the x-axis, shift right 3 and down 4 units

Domain: $x \geq 3$

Range: $y \leq 4$

Parent function in black, transformed in blue.

Ex 40: Using your findings from the exploration and knowledge of transformations, predict how the graph of $y = -2\sqrt[3]{x+2} + 6$ would compare to the graph of $y = \sqrt[3]{x}$. Use the graphing calculator to verify your conjecture.



It would be reflected over the *x*-axis, stretched vertically by 2, shifted to the left 2 and shifted up 6.

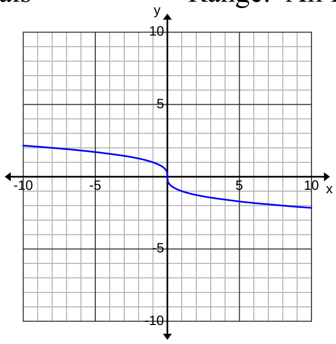
Ex 41: Sketch the graph the function $y = \sqrt[3]{4-x}$ and state its domain and range.

(Hint: Write it in the form $y = a\sqrt[3]{x-h} + k$ first.)

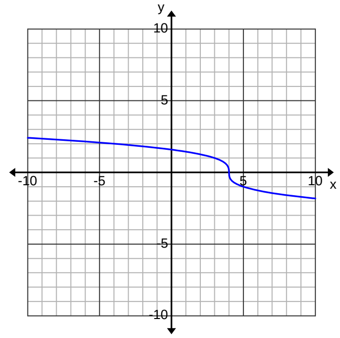
Standard Form: $f(x) = \sqrt[3]{-(x-4)}$

Transformations: Reflect over the **y-axis**, shift 4 units right

Domain: All Reals Range: All Reals



$f(x) = \sqrt[3]{-x}$



$f(x) = \sqrt[3]{-(x-4)}$

SAMPLE EXAM QUESTIONS

1. Identify the x **and** y **intercepts of the function** $f(x) = \sqrt[3]{x-8}$ **.**

A. (8,0) and (0,-2)

B. (2,0) and (0,2)

C. (8,0) and (0,8)

D. (-2,0) and (0,8)

Ans: A

2. Which is the domain of the function $f(x) = 5\sqrt{x-4} + 3$ **?**

A. $\{x \mid x \geq 4\}$

B. $\{x \mid x \geq 3\}$

C. $\{x \mid x \geq 0\}$

D. $\{x \mid x \in \mathbb{R}\}$

Ans: A

3. Compare the graph of $y = 6 - \sqrt[3]{x}$ with the graph of its parent function $f(x) = \sqrt[3]{x}$.

A. Shifts 6 units down

B. Reflects across the x-axis and shifts 6 units down

C. Reflects across the x-axis and shifts 6 units up

D. Reflects across the y-axis and shifts 6 units up

Ans: C

4. Compare the graph of $y = 6 - \sqrt[3]{x}$ with the graph of its parent function $f(x) = \sqrt[3]{x}$.

A. Shifts 6 units down

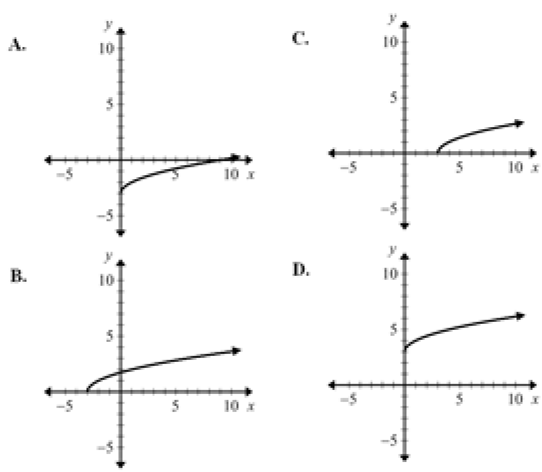
B. Reflects across the x-axis and shifts 6 units down

C. Reflects across the x-axis and shifts 6 units up

D. Reflects across the y-axis and shifts 6 units up

Ans: C

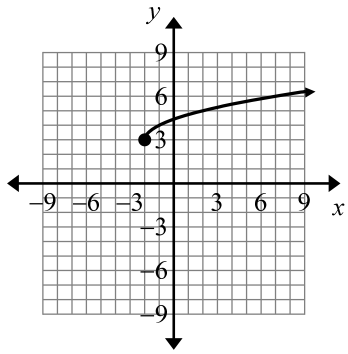
5. What is the graph of $y = \sqrt{x} - 3$?



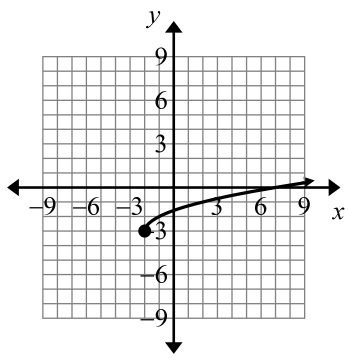
Ans: A

6. Which is the graph of $f(x) = \sqrt{x-2} + 3$?

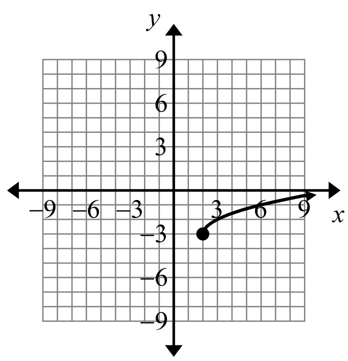
A.



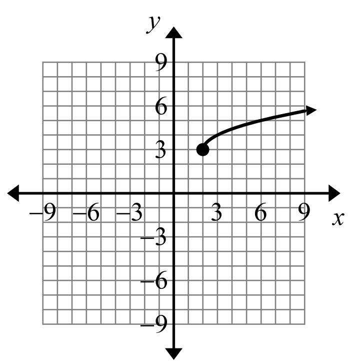
B.



C.



D.



Ans: D

7. Compare the graph of $y = 6 - \sqrt[3]{x}$ with the graph of its parent function $f(x) = \sqrt[3]{x}$.

A. Shifts 6 units down

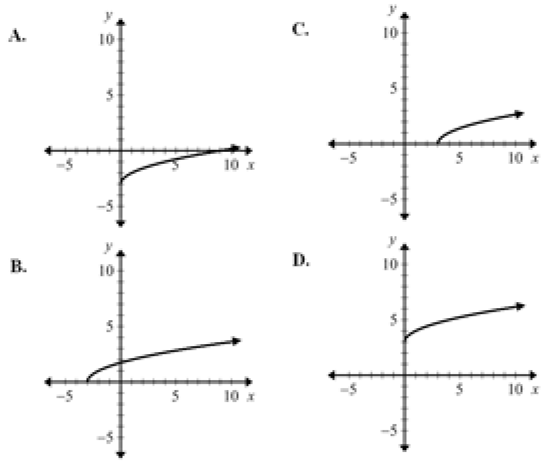
B. Reflects across the x-axis and shifts 6 units down

C. Reflects across the x-axis and shifts 6 units up

D. Reflects across the y-axis and shifts 6 units up

Ans: C

8. What is the graph of $y = \sqrt{x} - 3$?



Ans: A