



## Rational Functions

### Math Background

#### Previously, you

- Simplified quadratic and polynomial functions
- Performed arithmetic operations with quadratic and polynomial functions
- Identified the domain, range and x-intercepts of real-life functions
- Graphed quadratic and polynomial functions
- Solved quadratic and polynomial functions
- Transformed parent functions of quadratic and polynomial functions

#### In this unit you will

- Identify direct, inverse and joint variation
- Perform arithmetic operations with rational expressions
- Transform rational functions
- Solve problems involving rational equations
- Create rational functions to represent real life situations

#### You can use the skills in this unit to

- Model direct, inverse and joint variation relationships.
- Interpret the domain and its restrictions of a rational function.
- Identify extraneous solutions of a rational equation.
- Describe how a rational function graph is related to its parent function.
- Model and solve real-world problems with rational functions with and without technology

#### Vocabulary

- **Direct variation** – A relationship between two variables in which one is a constant multiple of the other.
- **Domain of a rational expression** – the set of all real numbers except the value(s) of the variable that result in division by zero when substituted into the expression.
- **Extraneous Solutions** – A root of a transformed equation that is not a root of the original equation because it was excluded from the domain of the original equation.
- **Horizontal Asymptote** – A horizontal line that the graph of a function approaches as  $x$  tends to plus or minus infinity. It describes the function's end behavior.
- **Inverse variation** – A relationship between two variables in which the product is a constant.
- **Joint variation** – A direct variation between two or more variables.
- **Rational expression** – a fraction with a polynomial in the numerator and a nonzero polynomial in the denominator. Also known as an algebraic fraction.
- **Simplify a rational expression** – Write the fraction so there are no common factors other than 1 or -1.
- **Undefined** – An expression in mathematics which does not have meaning and so which is not assigned an interpretation. For example, division by zero is undefined in the field of real numbers.
- **Vertical asymptote** – A vertical line that the curve approaches more and more closely but never touches as the curve goes off to positive or negative infinity. The vertical lines correspond to the zeroes of the denominator of the rational function.



## Essential Questions

- How do you find sums, differences, products, and quotients of rational expressions?
- What knowledge and skills are required to rewrite simple rational expressions?
- How do I solve a rational equation? How are extraneous solutions generated from a rational equation?
- How are expressions used for specific scenarios with multiple constraints and conditions?

## Overall Big Ideas

The rules for addition, subtraction, multiplication and division of rational expressions are analogous to those for rational numbers.

Rational expressions can be rewritten using properties of fractions and elementary numerical algorithms.

We solve rational equations by transforming the equation into a simpler form to solve. However, this can produce solutions that do not exist in the original domain.

Complicated expressions are formed by operations on simpler expressions to provide the detail and specificity required to create non-linear functions to model certain real life situations such as those found in finance, physics, motion and three-dimensional objects.

## Skill

**To identify a relation as direct, inverse, or joint variation and to write equations modeling such relations.**

**To simplify, add, subtract, multiply, and divide rational expressions.**

**To solve equations involving rational expressions.**

**To identify vertical and horizontal asymptotes of rational functions.**

**To identify the domain and range of rational functions.**

**To graph rational functions with and without technology.**

## Related Standards

### A.APR.D.6

Rewrite simple rational expressions in different forms; write  $\frac{a(x)}{b(x)}$  in the form  $q(x) + \frac{r(x)}{b(x)}$ , where

$a(x), b(x), q(x)$ , and  $r(x)$  are polynomials with the degree of  $r(x)$  less than the degree of  $b(x)$ , using inspection, long division, or, for the more complicated examples, a computer algebra system.

### A.APR.D.7

Understand that rational expressions form a system analogous to the rational numbers, closed under addition, subtraction, multiplication, and division by a nonzero rational expression; add, subtract, multiply, and divide rational expressions.

**A.APR.A.1-2**

Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials of degree 2 and higher.

**A.SSE.A.1a**

Interpret parts of an expression, such as terms, factors, and coefficients. \*(Modeling Standard)

**A.SSE.A.1b-2**

Interpret complicated expressions, including polynomial and rational expressions, by viewing one or more of their parts as a single entity. For example, interpret  $P(1 + r)^n$  as the product of  $P$  and a factor not depending on  $P$ . \*(Modeling Standard)

**A.REI.A.2**

Solve simple rational and radical equations in one variable, and give examples showing how extraneous solutions may arise.

**A.REI.D.11-2**

Explain why the  $x$ -coordinates of the points where the graphs of the equations  $y = f(x)$  and  $y = g(x)$  intersect are the solutions of the equation  $f(x) = g(x)$ ; find solutions to  $f(x) = g(x)$  approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where  $f(x)$  and/or  $g(x)$  are linear, polynomial, rational, radical, absolute value, exponential, and logarithmic functions. \*(Modeling Standard)

**F.IF.B.4-2**

For any function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity. \*(Modeling Standard)

**F.IF.B.5-2**

Relate the domain of any function to its graph and, where applicable, to the quantitative relationship it describes. For example, if the function  $h(n)$  gives the number of person-hours it takes to assemble  $n$  engines in a factory, then the positive integers would be an appropriate domain for the function. \*(Modeling Standard)

**F.IF.C.7d**

Graph rational functions, identifying zeros and asymptotes when suitable factorizations are available, and showing end behavior. \*(Modeling Standard)



## Notes, Examples, and Exam Questions

**Unit 3.1: To identify a relation as direct, inverse, or joint variation and to write equations modeling such relations.**

**\*\*\*This is a new topic for the students. Direct, inverse and joint variation are not covered in the Algebra I, Geometry or Algebra II curriculum.**

Direct Variation:  $y$  varies directly with  $x$  if  $y = kx$ , where  $k$  is the *constant of variation* and  $k \neq 0$ . The graph of a direct variation function is a line that crosses through the origin. The constant of variation is the slope of the function. To find the constant of variation, divide  $y$  by  $x$ .

Inverse Variation:  $x$  and  $y$  show inverse variation if  $y = \frac{k}{x}$  where  $k$  is the *constant of variation* and  $k \neq 0$ . The graph of an inverse variation function does not cross the  $x$ -axis and its graph is a curve.  $y$  is said to vary proportionately with  $x$  and as  $x$  increases,  $y$  decreases. To find the constant of variation, multiply  $x$  and  $y$ .

Joint Variation: When a quantity varies directly with the product of two or more other quantities, it is said to be a joint variation and  $z = kxy$ , where  $k$  is the *constant of variation* and  $k \neq 0$ . To find the constant of variation, divide  $z$  by  $x$  and  $y$ .

**Ex 1:** Do the following show direct variation, inverse variation, joint variation or neither?

a.  $xy = 4.8$  Solve for  $y$ .  $y = \frac{4.8}{x}$  Inverse Variation

b.  $x + y = 8$  Solve for  $y$ .  $y = 8 - x$  Neither

c.  $z = -3y\sqrt{x}$  Joint Variation

d.  $\frac{y}{x} = -3$  Solve for  $y$ .  $y = -3x$  Direct Variation

**Ex 2:**  $x$  and  $y$  vary inversely and  $y = 6$ , when  $x = 1.5$ . Write an equation that relates  $x$  and  $y$ .

Then, find  $y$  when  $x = \frac{3}{4}$ .

Step 1: Use the equation  $y = \frac{k}{x}$ . Substitute the values for  $x$  and  $y$ .  $6 = \frac{k}{1.5}$

Step 2: Solve for  $k$ .  $k = 6(1.5) = 9$



Step 3: Write the equation that relates  $x$  and  $y$ .

$$y = \frac{9}{x}$$

Step 4: Use the equation to find  $y$  when  $x = \frac{3}{4}$ .

$$y = \frac{9}{\frac{3}{4}} = 9 \cdot \frac{4}{3} = 12$$

**Ex 3:** The variable  $y$  varies jointly with  $x$  and  $z$ . Use the given values to write an equation relating  $x$ ,  $y$ , and  $z$ .

$$x = 1, y = \frac{1}{3}, z = -5$$

Use the equation  $y = kxz$ . Substitute the values in for  $x$ ,  $y$ , and  $z$ .

$$\frac{1}{3} = k(1)(5)$$

Solve for  $k$ .

$$k = \frac{1}{15}$$

Write the equation that relates  $x$ ,  $y$ , and  $z$ .

$$y = \frac{1}{15}xz$$

**Ex 4:** Write an equation for the following.

a.  $z$  varies jointly with  $x^2$  and  $y$ .

$$z = kx^2y$$

b.  $y$  varies inversely with  $x$  and  $z$ .

$$y = \frac{k}{xz}$$

c.  $y$  varies directly with  $x^3$  and inversely with  $z$ .

$$y = \frac{kx^3}{z}$$

#### Application Problems:

**Ex 5:** The volume of gas in a container varies inversely with the amount of pressure. A gas has volume  $75 \text{ in.}^3$  at a pressure of  $25 \text{ lb/in.}^2$ . Write a model relating volume and pressure.

Use the equation  $V = \frac{k}{P}$ . Substitute the values of  $V$  and  $P$ .

$$75 = \frac{k}{25}$$

Solve for  $k$ .

$$k = 75(25) = 1875$$

Write the equation that relates  $V$  and  $P$ .

$$V = \frac{1875}{P}$$



**Ex 6:** The weight of an object on Mars varies directly with its weight on Earth. An object that weighs 70 pounds on Mars weighs 210 pounds on Earth. If an object weighs 160 pounds on Earth, how much would it weigh on Mars?

This is a direct variation example. Let  $x$  = weight on Earth and  $y$  = weight on Mars. Solve for the constant of variation.

$$k = \frac{70}{210} = \frac{1}{3} \quad \text{Our model is: } y = \frac{1}{3}x$$

$$\text{Weight on Mars: } y = \frac{1}{3}(160) \approx \boxed{53.3 \text{ pounds}}$$

**Ex 7:** The ideal gas law states that the volume  $V$  (in liters) varies directly with the number of molecules  $n$  (in moles) and temperature  $T$  (in Kelvin) and varies inversely with the pressure  $P$  (in kilopascals). The constant of variation is denoted by  $R$  and is called the universal gas constant. Write an equation for the ideal gas law. Then, estimate the universal gas constant if  $V = 251.6$  liters;  $n = 1$  mole;  $t = 288$  K;  $P = 9.5$  kilopascals.

Step 1: Set up the model: 
$$V = \frac{nRT}{P}$$

Step 2: Substitute and solve for  $R$ : 
$$251.6 = \frac{(1)R(288)}{9.5} \rightarrow 2390.2 = 288R \rightarrow \boxed{R \approx 8.3}$$

**Ex 8:** The volume of wood in a tree ( $V$ ) varies directly as the height ( $h$ ) and inversely as the square of the girth ( $g$ ). If the volume of a tree is 144 cubic meters when the height is 20 meters and the girth is 1.5 meters, what is the height of a tree with a volume of 1000 and girth of 2 meters?

Step 1: Set up the model: 
$$V = \frac{kh}{g^2}$$

Step 2: Substitute and solve for  $k$ : 
$$144 = \frac{20k}{(1.5)^2} \rightarrow 20k = 144(2.25) \rightarrow k = \frac{144(2.25)}{20} = 16.2$$

Step 3: Substitute new values, solve for  $h$ : 
$$h = \frac{1000(2^2)}{16.2} = \boxed{246.1 \text{ meters}}$$

### Checking Data for Inverse Variation

**Ex 9:** Do these data show inverse variation? If so, find a model.

First, notice that as  $x$  increases,  $y$  decreases, so this could be an inverse variation. If  $H = \frac{k}{W}$ , then there exists a constant,  $k$ , such that  $k = HW$ .

$W$	2	4	6	8	10
$H$	9	4.5	3	2.25	1.8

Check the products  $HW$  from the table.  $k = 2 \cdot 9 = 4 \cdot 4.5 = 6 \cdot 3 = 8 \cdot 2.25 = 10 \cdot 1.8 = 18$

The product  $k$  is constant, so  $H$  and  $W$  vary inversely.

$$\boxed{H = \frac{18}{W}}$$



**Ex 10:** Determine whether  $x$  and  $y$  show direct variation, inverse variation or neither. If there is a relationship, find the model.

Notice as  $x$  increases,  $y$  increases, so this could be a direct variation. If

$y = kx$ , then there exists a constant,  $k$ , such that  $k = \frac{y}{x}$ .

$x$	1	1.5	2	5
$y$	0.86	1.29	1.72	4.30

Check the ratios from the table.  $\frac{y}{x} = \frac{0.86}{1} = \frac{1.29}{1.5} = \frac{1.72}{2} = \frac{4.30}{5} = 0.86$

The product  $k$  is constant, so  $x$  and  $y$  vary directly.

$$y = 0.86x$$

**QOD:** Suppose  $x$  varies inversely with  $y$  and  $y$  varies inversely with  $z$ . How does  $x$  vary with  $z$ ? Justify your answer algebraically.

### SAMPLE EXAM QUESTIONS

**1. The value of  $s$  varies jointly with  $m$  and  $p$ . If  $s = 10$  when  $m = 3$  and  $p = 4$ , what is the value of  $s$  when  $m = 7$  and  $p = 2$ ?**

- A.  $\frac{47}{6}$
- B.  $\frac{81}{5}$
- C.  $\frac{49}{5}$
- D.  $\frac{35}{3}$

Ans: D

**2. Do the data in the table represent a direct or an inverse variation? Write an equation to model the data in the table.**

- A. inverse variation;  $xy = 1.5$
- B. direct variation;  $y = \frac{2}{3}x$
- C. direct variation;  $y = \frac{3}{2}x$
- D. inverse variation;  $xy = 54$

$x$	6	8	12	20
$y$	9	12	18	30

Ans: C



3. The pitch  $P$  of a musical tone varies inversely as its wavelength  $W$ . One tone has a pitch of 321 vibrations per second and a wavelength of 10.5 ft. Find the wavelength of another tone that has a pitch of 374 vibrations per second.

- A. 0.000087 ft
- B. 9 ft
- C. 0.11 ft
- D. 11,433.7 ft

Ans: B

**Unit 3.2: To simplify, add, subtract, multiply, and divide rational expressions**

\*\*\*Unit 3.2 is review for CPM students and should be a familiar topic. This learning target was covered in Algebra II, Unit 6.

Rational Function: a function of the form  $f(x) = \frac{p(x)}{q(x)}$ , where  $p(x)$  and  $q(x)$  are polynomial functions.

Simplified Form of a Rational Expression: a rational expression in which the numerator and denominator have no common factors other than 1.

**Simplifying a Rational Expression**

1. Factor the numerator and denominator, if you can
2. Divide out any common factors

Ex 11: Simplify the expression  $\frac{2x^2 - 5x - 3}{x^2 - 9}$ .

Factor. 
$$\frac{(2x+1)(x-3)}{(x+3)(x-3)}$$

Divide out common factors. 
$$= \frac{(2x+1)\cancel{(x-3)}}{(x+3)\cancel{(x-3)}} = \frac{2x+1}{x+3}$$

Ex 12: Simplify the expression  $\frac{y^4 + 3y^3 + 2y^2}{2y^3 - 4y^2 - 16y}$ .

Factor. 
$$\frac{y^2(y^2 + 3y + 2)}{2y(y^2 - 2y - 8)} = \frac{y^2(y+2)(y+1)}{2y(y-4)(y+2)}$$

Divide out common factors. 
$$= \frac{\cancel{y^2}(y+2)\cancel{(y+1)}}{2\cancel{y}(y+2)(y-4)} = \frac{y(y+1)}{2(y-4)}$$





## Multiplying and Dividing Rational Expressions

### Multiplying Rational Expressions

1. Factor numerators and denominators (if necessary).
2. Divide out common factors.
3. Multiply numerators and denominators.

**Ex 13:** Multiply  $\frac{45x^2}{x-9} \cdot \frac{x^2-5x-36}{3x^3+12x^2}$ .

Factor. 
$$\frac{45x^2}{x-9} \cdot \frac{(x+4)(x-9)}{3x^2(x+4)}$$

Divide out common factors. 
$$= \frac{45x^2}{\cancel{(x-9)}} \cdot \frac{\cancel{(x+4)} \cancel{(x-9)}}{3x^2 \cancel{(x+4)}}$$

Divide. 
$$= \frac{45}{3} = \boxed{15}$$

**Ex 14:** Find the product.  $\frac{x+2}{27x^3+8} \cdot (9x^2-6x+4)$

Factor. 
$$\frac{(x+2)}{(3x+2)(9x^2-6x+4)} \cdot \frac{(9x^2-6x+4)}{1}$$

Divide out common factors. 
$$= \frac{(x+2)}{(3x+2) \cancel{(9x^2-6x+4)}} \cdot \frac{\cancel{(9x^2-6x+4)}}{1} = \boxed{\frac{x+2}{3x+2}}$$

### Multiplying Rational Expressions with Monomials

Use the properties of exponents to multiply numerators and denominators, then divide.

**Ex 15:** Multiply  $\frac{12c^3d^2}{21ab} \cdot \frac{14a^2b}{8c^2d}$ .

Use the properties of exponents and simplify. 
$$\frac{12c^3d^2}{21ab} \cdot \frac{14a^2b}{8c^2d} = \frac{168a^2bc^3d^2}{168abc^2d} = \boxed{acd}$$

### Dividing Rational Expressions

Multiply the first expression by the reciprocal of the second expression and simplify.



**Ex 16:** Divide.  $\frac{3x^2 - 17x - 6}{4x^2 - 20x - 24} \div \frac{6x^2 - 7x - 3}{2x^2 - x - 3}$

Multiply by the reciprocal.  $\frac{3x^2 - 17x - 6}{4x^2 - 20x - 24} \cdot \frac{2x^2 - x - 3}{6x^2 - 7x - 3}$

Factor and simplify.  $= \frac{\cancel{(3x+1)} \cancel{(x-6)}}{4 \cancel{(x-6)} \cancel{(x+1)}} \cdot \frac{\cancel{(2x-3)} \cancel{(x+1)}}{\cancel{(2x-3)} \cancel{(3x+1)}} = \boxed{\frac{1}{4}}$

**Ex 17:** Find the quotient of  $\frac{8x^2 + 10x - 3}{4x^2}$  and  $4x^2 - x$ .

Multiply by the reciprocal.  $\frac{8x^2 + 10x - 3}{4x^2} \cdot \frac{1}{4x^2 - x}$

Factor and simplify.  $\frac{\cancel{(4x-1)} (2x+3)}{4x^2} \cdot \frac{1}{x \cancel{(4x-1)}} = \boxed{\frac{2x+3}{4x^3}}$

**Ex 18:** Simplify.  $\frac{x^2 - 4x}{x^2 - 16} \cdot (15xy^2 + 5y^3) \div \frac{3x^2 + xy}{x^2 + 5x + 4}$

Multiply by the reciprocal.  $\frac{x^2 - 4x}{x^2 - 16} \cdot \frac{15xy^2 + 5y^3}{1} \cdot \frac{x^2 + 5x + 4}{3x^2 + xy}$

Factor and simplify.  $\frac{\cancel{x} \cancel{(x-4)}}{\cancel{(x+4)} \cancel{(x-4)}} \cdot \frac{5y^2 \cancel{(3x+y)}}{1} \cdot \frac{\cancel{(x+4)} (x+1)}{\cancel{x} \cancel{(3x+y)}} = \boxed{5y^2(x+1)}$

### SAMPLE EXAM QUESTIONS

1. Simplify the expression.

$$\frac{3x^3 - 27x}{x^3 - 4x^2 + 3x}$$

A.  $\frac{3(x+3)}{x-1}$

C.  $\frac{3(x+3)(x-3)}{x^2 - 4x + 3}$

B.  $\frac{3(x-3)}{x+1}$

D.  $\frac{x^2 + 3x + 9}{x(x-1)}$

Ans: A



2. Simplify the expression:  $\frac{x^2 - 3x - 4}{x^2 - 16} \cdot \frac{x^2 + 4x}{x^2 - x - 2}$

- A. 1  
 B.  $\frac{x-2}{x+4}$   
 C.  $\frac{1}{x-4}$   
 D.  $\frac{x}{x-2}$

Ans: D

3. Multiply. Simplify your answer.

$$\frac{8x^4y^2}{3z^3} \cdot \frac{9xy^2z^6}{4y^4}$$

- A.  $6x^5y^8z^9$   
 B.  $6x^4yz^2$   
 C.  $6x^5z^3$   
 D.  $\frac{3}{2}x^3y^2z$

Ans: C

4. Which expression represents the quotient?  $\frac{8x^6z^4 + 4x^4z^2}{4x^2z}$

- A.  $2x^4z^3 + x^2z$   
 B.  $4x^4z^3 + 3x^2z$   
 C.  $2x^3z^4 + x^2z^2$   
 D.  $4x^3z^4 + 3x^2z^2$

Ans: A

5. Divide. Simplify your answer.  $\frac{2x-6}{10x^2-24x-18} \div \frac{8x+56}{5x^2+33x+18}$

- A.  $\frac{x+6}{8(x+7)}$   
 B.  $\frac{x-4}{5(x+2)}$   
 C.  $\frac{x-3}{3x}$   
 D.  $\frac{x+4}{6(x+1)}$

Ans: A

### Adding and Subtracting Rational Expressions

#### Adding and Subtracting Rational Expressions with Like Denominators

Add or subtract the numerators. Keep the common denominator. Simplify the sum or difference.



$$\text{Ex 19: Subtract. } \frac{3}{2x} - \frac{7}{2x} = \frac{3-7}{2x} = \frac{-4}{2x} = \boxed{\frac{-2}{x}}$$

$$\text{Ex 20: Add. } \frac{3x}{x+2} + \frac{6}{x+2} = \frac{3x+6}{x+2} = \frac{3\cancel{(x+2)}}{\cancel{(x+2)}} = \boxed{3}$$

### Adding and Subtracting Rational Expressions with Unlike Denominators

1. Find the least common denominator (in factored form).
2. Rewrite each fraction with the common denominator.
3. Add or subtract the numerators, keep the common denominator, and simplify.

$$\text{Ex 21: Find the sum. } \frac{x-8}{4x^2+21x+5} + \frac{6}{12x+3}$$

$$\text{Factor and find the LCD. } \frac{x-8}{(4x+1)(x+5)} + \frac{6}{3(4x+1)} \quad \text{LCD} = 3(4x+1)(x+5)$$

Rewrite each fraction with the LCD.

$$\frac{x-8}{(4x+1)(x+5)} \cdot \frac{3}{3} + \frac{6}{3(4x+1)} \cdot \frac{(x+5)}{(x+5)} = \frac{3x-24}{3(4x+1)(x+5)} + \frac{6x+30}{3(4x+1)(x+5)}$$

$$\text{Add the fractions and simplify. } = \frac{9x+6}{3(4x+1)(x+5)} = \frac{\cancel{3}(3x+2)}{\cancel{3}(4x+1)(x+5)} = \boxed{\frac{3x+2}{(4x+1)(x+5)}}$$

\*\*\*We can leave the denominator in factored form.

$$\text{Ex 22: Subtract: } \frac{x+1}{x^2+4x+4} - \frac{6}{x^2-4}$$

$$\text{Factor and find the LCD. } \frac{x+1}{(x+2)^2} - \frac{6}{(x+2)(x-2)} \quad \text{LCD} = (x+2)^2(x-2)$$

Rewrite each fraction with the LCD.

$$\frac{(x+1)}{(x+2)^2} \cdot \frac{(x-2)}{(x-2)} - \frac{6}{(x+2)(x-2)} \cdot \frac{(x+2)}{(x+2)} = \frac{x^2-x-2}{(x+2)^2(x-2)} - \frac{6x+12}{(x+2)^2(x-2)}$$

$$\text{Subtract the fractions. } = \frac{x^2-x-2-6x-12}{(x+2)^2(x-2)} = \boxed{\frac{x^2-7x-14}{(x+2)^2(x-2)}}$$



**Ex 23:** Perform the indicated operations and simplify.  $\frac{4}{x+1} + \frac{5}{x-1} - \frac{3}{x}$

Find the LCD.  $\text{LCD} = x(x+1)(x-1)$

Rewrite each fraction with the LCD.  $\frac{4}{x+1} \cdot \frac{x(x-1)}{x(x-1)} + \frac{5}{x-1} \cdot \frac{x(x+1)}{x(x+1)} - \frac{3}{x} \cdot \frac{x^2-1}{x^2-1}$

Add and subtract the fractions.

$$\frac{4x^2-4}{x(x-1)(x+1)} + \frac{5x^2+5}{x(x-1)(x+1)} - \frac{3x^2-3}{x(x-1)(x+1)} = \frac{4x^2+5x^2-3x^2-4+5+3}{x(x-1)(x+1)} = \frac{6x^2+4}{x(x-1)(x+1)}$$

\*\*Note: Denominator can be left in factored form.

QOD: Explain if the following is a true statement. The LCD of two rational expressions is the product of the denominators.

### SAMPLE EXAM QUESTIONS

1. Which expression is equivalent to  $\frac{2x}{x+4} - \frac{x^2+4}{x^2-16}$ ?

A.  $\frac{1}{x+4}$

C.  $\frac{(x+2)(x-2)}{(x+4)(x-4)}$

B.  $\frac{x^2-8x-4}{(x+4)(x-4)}$

D.  $\frac{3x^2-8x+4}{(x+4)(x-4)}$

Ans: B

2. Add and simplify if possible.  $\frac{x^2+7x+10}{x^2+x-2} + \frac{x^2+4x-5}{x^2+9x+20}$

A.  $\frac{2x^2+11x+5}{2x^2+10x+18}$

C.  $\frac{2x^2+7x+21}{(x-1)(x+4)}$

B.  $\frac{x^2+7x+21}{(x-1)(x+4)}$

D.  $\frac{2x^2+11x+5}{(x-1)(x+4)}$

Ans: C



### Unit 3.3 Solve equations involving rational expressions

\*\*\*Unit 3.3 is review for CPM students and should be a familiar topic. This learning target was covered in Algebra II, Unit 6.

Rational Equation: an equation in which one or more of the terms is a rational expression.

To solve a rational equation, **we can eliminate** the fractions by multiplying every term in the equation by the LCD. Then check your solution(s) in the original equation. When solving a rational equation, any possible solution that results in a zero in the denominator must be excluded from your list of solutions. These **extraneous solutions** are solutions that do not satisfy the original equation.

**Ex 24:** Solve the equation.  $\frac{1}{2x} + \frac{3}{x+7} = \frac{-1}{x}$

Step One: Identify values that are excluded from our solutions: 0 and -7

Step Two: Find the LCD and multiply every term by the LCD. LCD =  $2x(x+7)$

$$2x(x+7) \cdot \frac{1}{2x} + 2x(x+7) \cdot \frac{3}{x+7} = 2x(x+7) \cdot \frac{-1}{x}$$

$$(x+7) + 6x = -2x - 14$$

$$7x + 7 = -2x - 14$$

Step Three: Solve the equation.

$$9x = -21 \rightarrow x = -\frac{21}{9} = \boxed{-\frac{7}{3}}$$

Step Four: Check the solution:  $\frac{1}{2\left(-\frac{7}{3}\right)} + \frac{3}{\left(-\frac{7}{3} + 7\right)} = -\frac{1}{\left(-\frac{7}{3}\right)} \rightarrow -\frac{3}{14} + \frac{9}{14} = \frac{3}{7} \rightarrow \frac{6}{14} = \frac{3}{7}$

**Ex 25:** Solve the equation  $\frac{3}{x} - \frac{1}{2} = \frac{12}{x}$

Step One: Values that are excluded: 0

Step Two: Multiply every term by the LCD =  $2x$

$$2\cancel{x} \cdot \frac{3}{\cancel{x}} - \cancel{2}x \cdot \frac{1}{\cancel{2}} = 2\cancel{x} \cdot \frac{12}{\cancel{x}}$$

$$6 - x = 24$$

Step Three: Solve the equation.

$$-x = 18$$

$$x = -18$$

Step Four: Check.

$$\frac{3}{-18} - \frac{1}{2} = \frac{12}{-18} \Rightarrow -\frac{1}{6} - \frac{1}{2} = -\frac{2}{3} \text{ true, so } \boxed{x = -18}$$



**Ex 26:** Solve the equation  $\frac{5x}{x+1} = 4 - \frac{5}{x+1}$

Step One: Identify excluded values: -1

Step Two: Multiply every term by the LCD =  $x + 1$

$$\cancel{(x+1)} \cdot \frac{5x}{\cancel{(x+1)}} = (x+1) \cdot 4 - \cancel{(x+1)} \cdot \frac{5}{\cancel{(x+1)}}$$

$$5x = 4x + 4 - 5$$

Step Three: Solve the equation.  $x = -1$

Step Four: Check.  $\frac{5(-1)}{(-1)+1} = 4 - \frac{5}{(-1)+1}$  **False**

This solution leads to division by zero in the original equation. Therefore, it is an *extraneous solution*. This equation has **no solution**. This was discovered in Step One.

**Ex 27:** Solve the equation  $\frac{5}{x} - 2 = \frac{2}{x+3}$

Step One: Identify excluded values: 0, -3

Step Two: Multiply every term by the LCD =  $x(x+3)$

$$\cancel{x} \cancel{(x+3)} \cdot \frac{5}{\cancel{x}} - x(x+3) \cdot 2 = x \cancel{(x+3)} \cdot \frac{2}{\cancel{(x+3)}}$$

$$5x + 15 - 2x^2 - 6x = 2x$$

Step Three: Solve the equation.  $0 = 2x^2 + 3x - 15$

$$x = \frac{-3 \pm \sqrt{9 - 4(2)(-15)}}{4}$$

$$x = \frac{-3 \pm \sqrt{129}}{4}$$

Step Four: Check. (Use Technology) Solution:  $x = \frac{-3 \pm \sqrt{129}}{4}$

**Ex 28:** Solve the equation  $\frac{6}{x-3} = \frac{8x^2}{x^2-9} - \frac{4x}{x+3}$

Step One: Identify excluded values: 3, -3



Step Two: Multiply every term by the LCD =  $(x-3)(x+3)$

$$\cancel{(x-3)}(x+3) \cdot \frac{6}{\cancel{(x-3)}} = \cancel{(x-3)}(x+3) \cdot \frac{8x^2}{\cancel{(x-3)}(x+3)} - \cancel{(x-3)}(x+3) \cdot \frac{4x}{\cancel{(x+3)}}$$

$$6x+18 = 8x^2 - 4x^2 + 12x$$

Step Three: Solve the equation.

$$6x+18 = 8x^2 - 4x^2 + 12x \quad (2x-3)(x+3) = 0$$

$$0 = 4x^2 + 6x - 18 \quad x = -3, \frac{3}{2}$$

$$0 = 2x^2 + 3x - 9$$

Step Four: Check. -3 is extraneous (See Step One)

$$\left(\frac{3}{2}\right) - 3 = \frac{8\left(\frac{3}{2}\right)^2}{\left(\frac{3}{2}\right)^2 - 9} - \frac{4\left(\frac{3}{2}\right)}{\frac{3}{2} + 3} \quad \text{Solution: } x = \frac{3}{2}$$

$$\frac{12}{-3} = -\frac{8}{3} - \frac{4}{3} \quad \text{true}$$

Solving a Rational Equation by Cross Multiplying: This technique can be used when each side of the equation is a single rational expression.

**Ex 29:** Solve the equation  $\frac{3}{x^2 + 4x} = \frac{1}{x+4}$ .

Step One: Identify excluded values: 0, -4

Step Two: Cross multiply.

$$3(x+4) = 1(x^2 + 4x)$$

$$3x+12 = x^2 + 4x$$

Step Three: Solve.

$$0 = x^2 + x - 12 \quad x = -4, 3$$

$$0 = (x+4)(x-3)$$

Step Four: Check.

$$\frac{3}{(-4)^2 + 4(-4)} = \frac{1}{(-4) + 4} \quad \frac{3}{(3)^2 + 4(3)} = \frac{1}{(3) + 4}$$

$$\frac{3}{0} = \frac{1}{0} \quad \text{division by zero!} \quad \frac{3}{21} = \frac{1}{7} \quad \text{true}$$

Solution:  $x = 3$  (Note: -4 is an extraneous solution as identified in Step One.)





**Ex 30:** Solve the equation  $\frac{x}{x^2-2} = \frac{-1}{x}$ .

Step One: Identify excluded values:  $0, \pm\sqrt{2}$

Step Two: Cross multiply.  $x(x) = -1(x^2 - 2)$   
 $x^2 = -x^2 + 2$

Step Three: Solve.  $2x^2 = 2$   
 $x^2 = 1$   
 $x = \pm 1$

Step Four: Check.  $\frac{-1}{(-1)^2 - 2} = \frac{-1}{(-1)}$        $\frac{1}{(1)^2 - 2} = \frac{-1}{(1)}$   
 $1 = 1$  true       $-1 = -1$  true

Solutions:  $x = \pm 1$

**Ex 31:** Solve the rational equation.  $\frac{7x+1}{2x+5} + 1 = \frac{10x-3}{3x}$

This is not a proportion. We cannot cross multiply. We must find the LCD and multiply.

Step One: Multiply every term by the LCD =  $3x(2x+5)$

$$3x(2x+5) \cdot \frac{7x+1}{2x+5} + 3x(2x+5) \cdot 1 = 3x(2x+5) \cdot \frac{10x-3}{3x}$$

$$3x(7x+1) + 3x(2x+5) = (2x+5)(10x-3)$$

Step Two: Solve the equation.  $21x^2 + 3x + 6x^2 + 15x = 20x^2 + 44x - 15$   
 $7x^2 - 26x + 15 = 0 \Rightarrow (7x-5)(x-3) = 0$

$$x = 3, \frac{5}{7}$$

Step Three: Check the solution:  $\frac{21+1}{6+5} + 1 = \frac{30-3}{9}$  is true. 3 is a solution.

$$\frac{5+1}{\frac{10}{7}+5} + 1 = \frac{\frac{50}{7}-3}{\frac{15}{7}}$$
 is true.  $\frac{5}{7}$  is also a solution.

QOD: When is cross-multiplying an appropriate method for solving a rational equation?



## SAMPLE EXAM QUESTIONS

1. Solve  $7 = \frac{1}{3n} + \frac{1}{4n}$ .

A.  $n = \frac{1}{12}$

C.  $n = \frac{2}{49}$

B.  $n = \frac{1}{42}$

D.  $n = \frac{1}{84}$

Ans: A

2. Solve the rational equation for y:

$$\frac{5}{y-5} - \frac{5y-10}{y^2-8y+15} = \frac{5}{y-3}$$

A.  $y = -2$

C.  $y = 0$

B.  $y = 2$

D.  $y = 4$

Ans: C

3. Solve the equation  $\frac{x}{x^2-49} + \frac{7}{x-7} = \frac{1}{x+7}$ .

A. 8

C. 7

B. -8

D. No solution

Ans: B

4. Solve:  $\frac{1}{x^2+4x} = \frac{5}{x+4}$

A.  $x = \frac{1}{5}$

C.  $x = 5$

B.  $x = -4, x = \frac{1}{5}$

D.  $x = -4, x = 5$

Ans: A





**Unit 3.4 To identify vertical and horizontal asymptotes of rational functions. Unit 3.5 To identify the domain and range of rational functions. Unit 3.6 To graph rational functions with and without technology.**

\*\*\*Spend more time on these last three units. Although Unit 6.1 in Algebra II had the students transforming the graph of  $y = \frac{1}{x}$ , students did not take a close look at the asymptotes or domain and range.

**Ex 32:** Use a table of values to graph the function  $y = \frac{1}{x}$ .

x	-4	-3	-2	-1	$-\frac{1}{2}$	$\frac{1}{2}$	1	2	3	4
y	$-\frac{1}{4}$	$-\frac{1}{3}$	$-\frac{1}{2}$	-1	-2	2	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$

Note: The graph of  $y = \frac{1}{x}$  has two branches. The  $x$ -axis ( $y = 0$ ) is a horizontal asymptote, and the  $y$ -axis ( $x = 0$ ) is a vertical asymptote. Domain and Range: All real numbers not equal to zero.

A **horizontal asymptote** is a horizontal line that the graph of a function approaches as  $x$  tends to plus or minus infinity. As  $x$  grows very large or very small, the function approaches this value. A **vertical asymptote** is a vertical line near which the function grows to infinity or negative infinity. It is a place where the function is undefined.

**Exploration:** Graph each of the functions on the graphing calculator. Describe how the graph compares to the graph of  $y = \frac{1}{x}$ . Include horizontal and vertical asymptotes and domain and range in your answer.



1.  $y = \frac{2}{x}$

2.  $y = -\frac{2}{x}$

3.  $y = \frac{1}{x-3}$

4.  $y = \frac{1}{x+5}$

5.  $y = \frac{1}{x} - 4$

6.  $y = \frac{1}{x} + 3$

**Hyperbola:** the graph of the function:  $y = \frac{a}{x-h} + k$

\*\*\*From the exploration above, describe the asymptotes, domain and range, and the effects of  $a$  on the general equation of a hyperbola  $y = \frac{a}{x-h} + k$ . Horizontal Asymptote:  $y = k$  Vertical Asymptote:  $x = h$

Domain: All real numbers not equal to  $h$ . Range: All real numbers not equal to  $k$ .

As  $|a|$  gets bigger, the branches move farther away from the origin. If  $a > 0$ , the branches are in the first and third quadrants. If  $a < 0$ , the branches are in the second and fourth quadrants.





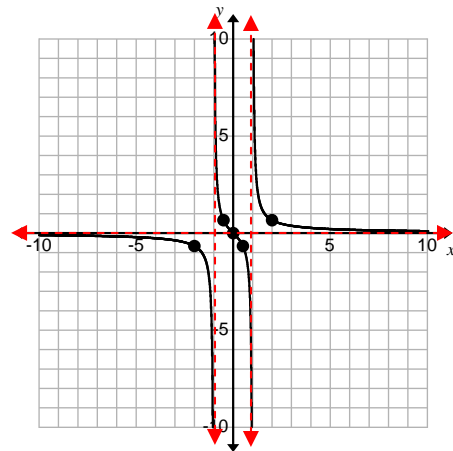
Summarize the Graphs of Rational Functions:  $f(x) = \frac{p(x)}{q(x)}$

- $x$ -intercepts: the zeros of  $p(x)$
- Vertical Asymptotes: occur at the zeros of  $q(x)$
- Horizontal Asymptote: describes the END BEHAVIOR of the graph (as  $x \rightarrow -\infty$  and  $x \rightarrow \infty$ )
  - If the degree of  $p(x)$  is less than the degree of  $q(x)$ , then  $y = 0$  is a horizontal asymptote.
  - If the degree of  $p(x)$  is equal to the degree of  $q(x)$ , then  $y =$  (the ratio of the leading coefficients) is a horizontal asymptote.
  - If the degree of  $p(x)$  is greater than the degree of  $q(x)$ , then the graph has no horizontal asymptote. It will have a slant asymptote.

**Ex 35:** Graph the function  $y = \frac{x}{x^2 - 1}$ .

- $x$ -intercepts:  $x = 0$
- Vertical Asymptotes:  $x^2 - 1 = 0$   
 $x = -1, x = 1$
- Horizontal Asymptote: Since the degree of  $p(x)$  is less than the degree of  $q(x)$ , then  $y = 0$
- Domain: all real numbers except -1 and 1
- Range: all real numbers
- Plot points between and outside the vertical asymptotes

$x$	-2	$-\frac{1}{2}$	0	$\frac{1}{2}$	2
$y$	$-\frac{2}{3}$	$\frac{2}{3}$	0	$-\frac{2}{3}$	$\frac{2}{3}$



▲ **Note:** The point  $(0, 0)$  lies on a **horizontal asymptote**. This is valid since horizontal asymptotes describe end behavior. A graph can NEVER have a point exist on a **vertical asymptote** since vertical asymptotes represent where the function does not exist.

Local (Relative) Extrema: the *local (relative) maximum* is the largest value of the function in a local area, and the *local (relative) minimum* is the smallest value of the function in a local area



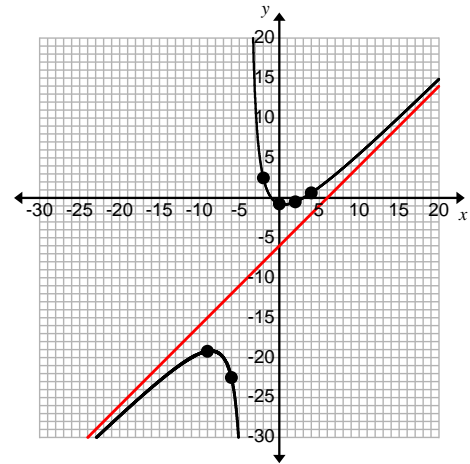
**Ex 36:** Sketch the graph of  $f(x) = \frac{x^2 - 2x - 3}{x + 4}$ . Use the graphing calculator to check your answer and to find the local extrema.

$$x^2 - 2x - 3 = 0$$

- $x$ -intercepts:  $(x - 3)(x + 1) = 0$   
 $x = 3, -1$

- Vertical Asymptotes:  $x + 4 = 0$   
 $x = -4$
- Horizontal Asymptote: The degree of  $p(x)$  is greater than the degree of  $q(x)$ , so there is no horizontal asymptote
- Slant Asymptote:  $y = x - 6$
- Domain: all real numbers except  $-4$
- Range:  $(-\infty, -19.165] \cup [-0.8348, \infty)$
- Plot points to the left and right of the vertical asymptote

$x$	-9	-6	-2	0	2	4
$y$	-19.2	-22.5	2.5	-0.75	-0.5	0.63



$$x + 4 \overline{) x^2 - 2x - 3}$$

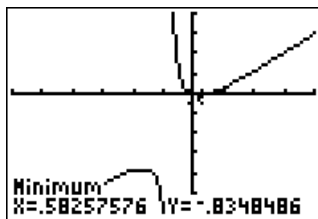
Using long division:  $\frac{x^2 + 4x}{-6x - 3}$  The quotient is:  $x - 6 + \frac{21}{x + 4}$ .

$$\frac{6x - 24}{21}$$

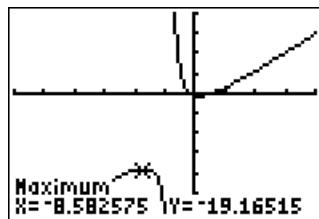
Using transformations in this form, it tells us there is a slant asymptote  $(x - 6)$ , to vertically stretch the graph (vertical stretch factor of 21), and go left four units.

To find the range, use technology:

Local Minimum:  $-0.8348$



Local Maximum:  $-19.165$

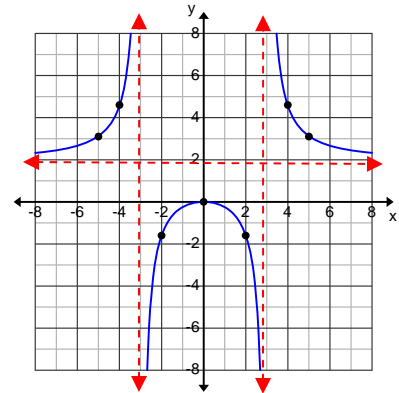




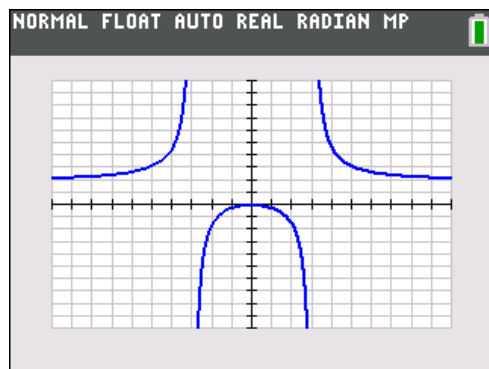
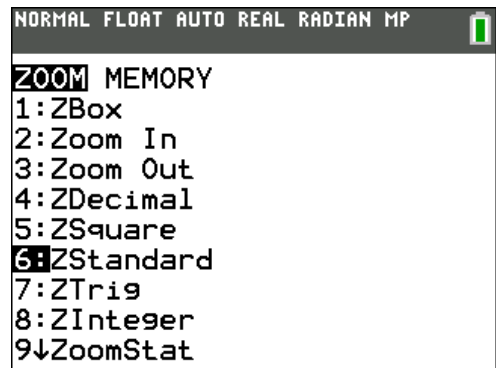
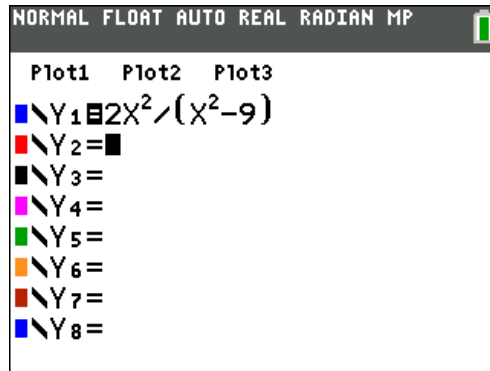
**Ex 37: Graph**  $y = \frac{2x^2}{x^2 - 9}$ . Graph with and without technology.

- x-intercepts:  $x^2 = 0$   
 $x = 0$
- Vertical Asymptotes:  $x^2 - 9 = 0$   
 $x = -3, x = 3$
- Horizontal Asymptote: The numerator and denominator have the same degree, so the horizontal asymptote is the ratio of the leading coefficients:  $y = \frac{2}{1} = 2$
- Domain: all real numbers except -3, 3
- Range:  $(-\infty, 0] \cup (2, \infty)$
- Plot points to the left and right of the vertical asymptotes

x	-5	-4	-2	2	4	5
y	3.1	4.6	-1.6	-1.6	4.6	3.1



Using technology:

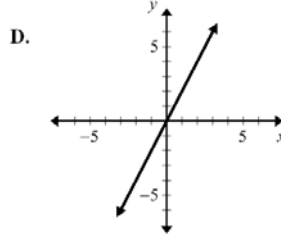
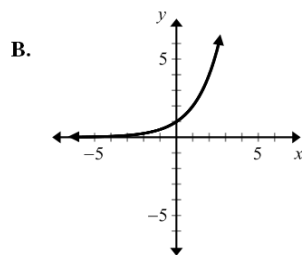
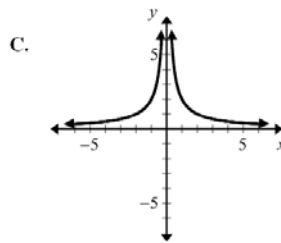
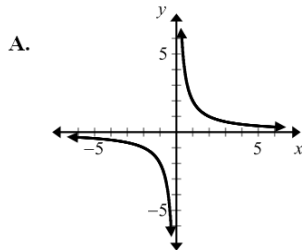






## SAMPLE EXAM QUESTIONS

1. What is the graph of  $f(x) = \frac{2}{x}$ ?



Ans: A

2. What are the asymptotes of the function:  $f(x) = \frac{-x^2}{x^2 - 25}$ ?

- A.  $y = 0, x = -5$
- B.  $y = -1, x = -5, x = 5$
- C.  $y = -1, x = -5$
- D.  $y = 0, x = -5, x = 5$

Ans: B

3. Which intervals correctly define the domain of  $f(x) = \frac{1}{x+4} - 2$ ?

- A.  $(-\infty, 4)$  and  $(4, \infty)$
- B.  $(-\infty, -4)$  and  $(-4, \infty)$
- C.  $(-\infty, -4)$  and  $(-4, \infty)$
- D.  $(-\infty, -4)$  and  $(-2, \infty)$

Ans: B



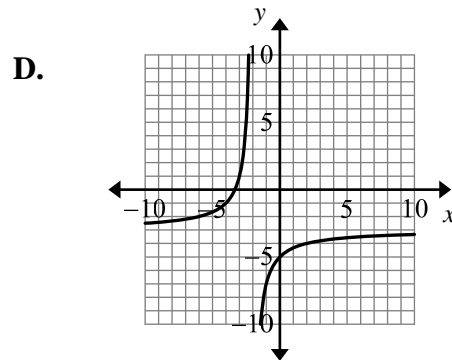
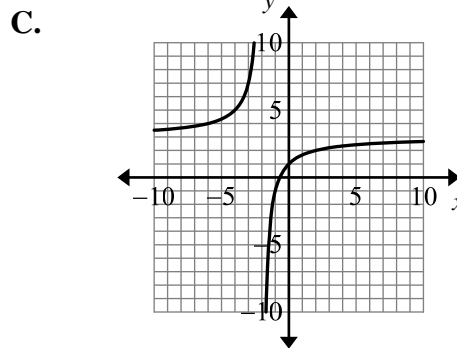
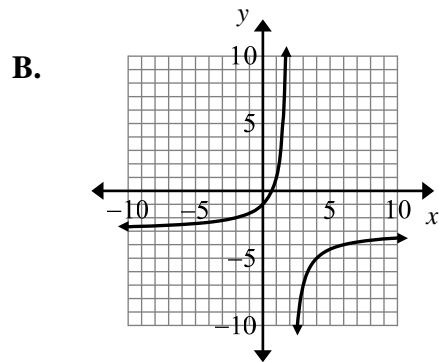
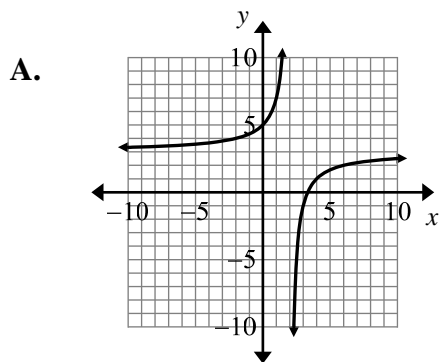
4. Which set contains all the real numbers that are not part of the domain

of  $f(x) = \frac{x+4}{x^2+4x-32}$  ?

- A. {8}
- B. {-4, 8}
- C. {-4}
- D. {-8, 4}

**Ans: D**

5. Which is the graph of  $f(x) = \frac{-4}{x+2} - 3$  ?



**Ans: D**



6. What are the asymptotes of the function  $f(x) = \frac{-x^2}{x^2 + x - 30}$ ?

- A.  $x = -6, x = 5, y = -1$                       C.  $x = -6, x = 5, y = 1$   
 B.  $x = -5, x = 6, y = -1$                       D.  $x = -5, x = 6, y = 1$

Ans: A

7. Which intervals correctly define the domain of  $f(x) = \frac{1}{x+4} - 2$

- A.  $(-\infty, 4)$  and  $(4, \infty)$                       C.  $(-\infty, -4)$  and  $(4, \infty)$   
 B.  $(-\infty, -4)$  and  $(-4, \infty)$                       D.  $(-\infty, -4)$  and  $(-2, \infty)$

Ans: C

8. Which statement is true for the function  $f(x) = \frac{1}{x+4}$ ?

- A. 4 is not in the range of the function.  
 B. 4 is not in the domain of the function.  
 C. -4 is not in the range of the function.  
 D. -4 is not in the domain of the function.

Ans: D

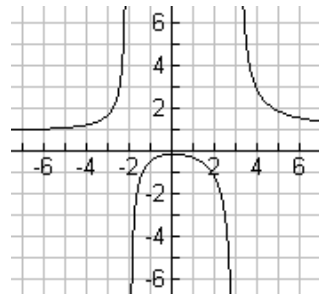
9. Which of the following functions has a slant asymptote?

- A.  $y = \frac{x^2 - 3x - 4}{x^2 - 4}$                       C.  $y = \frac{x^2 - 4}{x + 3}$   
 B.  $y = \frac{x + 3}{x^2 - 5x - 14}$                       D.  $y = \frac{(x - 3)^2}{(x - 3)(x + 4)}$

Ans: C

10. What is the domain of the following function?

- A.  $\{x \mid x < 0 \cup x > 1, x \in \mathbb{R}\}$   
 B.  $\{x \mid x \neq -3, 2, x \in \mathbb{R}\}$   
 C.  $\{x \mid x \in \mathbb{R}\}$   
 D.  $\{x \mid x \neq -2, 3, x \in \mathbb{R}\}$



Ans: D