Lesson 13: The Formulas for Volume

Student Outcomes

- Students develop, understand, and apply formulas for finding the volume of right rectangular prisms and cubes.

Lesson Notes

This lesson is a continuation of Lessons 11, 12, and Module 5, Topics A and B from Grade 5.

Fluency Exercise (5 minutes)

Multiplication and Division Equation with Fractions White Board Exchange

Classwork

Example 1 (3 minutes)

MP.1

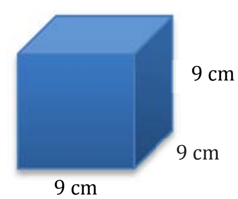
Example 1

Determine the volume of a cube with side lengths of $2\frac{1}{4}$ cm.

$$V = l\,w\,h$$
$$V = \left(2\frac{1}{4}\ cm\right)\left(2\frac{1}{4}\ cm\right)\left(2\frac{1}{4}\ cm\right)$$
$$V = \frac{9}{4}\ cm \times \frac{9}{4}\ cm \times \frac{9}{4}\ cm$$
$$V = \frac{729}{64}\ cm^3$$

Scaffolding:

Provide a visual of a cube for students to label. If needed, begin with less complex numbers for the edge lengths.



$V = (9\ cm)(9\ cm)(9\ cm)$
$V = 729\ cm^3$

Have students work through the first problem on their own and then discuss.

- Which method for determining the volume did you choose?
  - *Answers will vary. Sample response: I chose to use the $V = l\,w\,h$ formula to solve.*
- Why did you choose this method?
  - *Explanations with vary according to the method chosen. Sample response: Because I know the length, width, and height of the prism, I used $V = l\,w\,h$ instead of the other examples.*

Example 2 (3 minutes)

Example 2

Determine the volume of a rectangular prism with a base area of $\frac{7}{12}$ ft² and a height of $\frac{1}{3}$ ft.

$$V = Area\ of\ base \times height$$
$$V = \left(\frac{7}{12}\ ft^2\right)\left(\frac{1}{3}\ ft.\right)$$
$$V = \frac{7}{36}\ ft^3$$

- What makes this problem different than the first example?
  - *This example gives the area of the base instead of just giving the length and width.*
- Would it be possible to use another method or formula to determine the volume of the prism in this example?
  - *I could try fitting cubes with fractional lengths. However, I could not use the $V = l\ w\ h$ formula because I do not know the length and width of the base.*

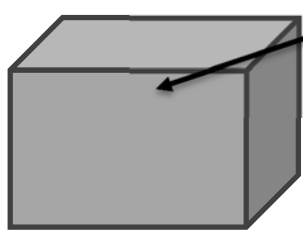
Exercises 1–5 (27 minutes)

In the exercises, students will explore how changes in the lengths of the sides affect the volume. Students can use any method to determine the volume as long as they can explain their solution. Students work in pairs or small groups.

(Please note that the relationships between the volumes will be more easily determined if the fractions are left in their original form when solving. If time allows, this could be an interesting discussion point either between partners, groups, or as a whole class when discussing the results of their work.)

Exercises 1–5

1. **Use the rectangular prism to answer the next set of questions.**



   a. **Determine the volume of the prism.**

$$V = Area\ of\ the\ base \times height$$
$$V = \left(\frac{13}{2}\ ft^2\right)\left(\frac{5}{3}\ ft.\right)$$
$$V = \frac{65}{6}\ ft^3$$

Scaffolding:

- The wording half as long may confuse some students. Explain that half as long means that the original length was multiplied by one half. A similar explanation can be used for one third as long and one fourth as long.
- Explain to students that the word doubled refers to twice as many or multiplied by two.

b. Determine the volume of the prism if the height of the prism is doubled.

$$Height \times 2 = \left(\frac{5}{3} ft. \times 2\right) = \frac{10}{3} ft.$$

$$V = Area\ of\ base \times height$$
$$V = \left(\frac{13}{2} ft^2\right)\left(\frac{10}{3} ft.\right)$$
$$V = \frac{130}{6} ft^3$$

c. Compare the volume of the rectangular prism in part (a) with the volume of the prism in part (b). What do you notice?

When the height of the rectangular prism is doubled, the volume is also doubled.

MP.2

d. Complete and use the table below to determine the relationships between the height and volume.

Height in Feet	Volume in Cubic Feet
$\frac{5}{3}$	$\frac{65}{6}$
$\frac{10}{3}$	$\frac{130}{6}$
$\frac{15}{3}$	$\frac{195}{6}$
$\frac{20}{3}$	$\frac{260}{6}$

What happened to the volume when the height was tripled?

The volume tripled.

What happened to the volume when the height was quadrupled?

The volume quadrupled.

What conclusions can you make when the base area stays constant and only the height changes?

Answers will vary but should include the idea of a proportional relationship. Each time the height is multiplied by a number, the original volume will be multiplied by the same amount.

MP.7

2. a. If A represents the area of the base and h represents the height, write an expression that represents the volume.

Ah

b. If we double the height, write an expression for the new height.

$2h$

c. Write an expression that represents the volume with the doubled height.

$A2h$

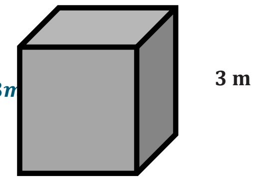
d. Write an equivalent expression using the commutative and associative properties to show the volume is twice the original volume.

$2(Ah)$

3. Use the cube to answer the following questions.

   a. Determine the volume of the cube.

$$V = l\,w\,h$$
$$V = (3m)(3m)(3m)$$
$$V = 27\ m^3$$



   b. Determine the volume of a cube whose side lengths are half as long as the side lengths of the original cube.

$$V = l\,w\,h$$
$$V = \left(\frac{3}{2}\ m\right)\left(\frac{3}{2} m\right)\left(\frac{3}{2}\ m\right)$$
$$V = \frac{27}{8}\ m^3$$

   c. Determine the volume if the side lengths are one fourth as long as the original cube's side lengths.

$$V = l\,w\,h$$
$$V = \left(\frac{3}{4}\ m\right)\left(\frac{3}{4}\ m\right)\left(\frac{3}{4}\ m\right)$$
$$V = \frac{27}{64}\ m^3$$

   d. Determine the volume if the side lengths are one sixth as long as the original cube's side length.

$$V = l\,w\,h$$
$$V = \left(\frac{3}{6}\ m\right)\left(\frac{3}{6} m\right)\left(\frac{3}{6}\ m\right)$$
$$V = \frac{27}{216}\ m^3$$

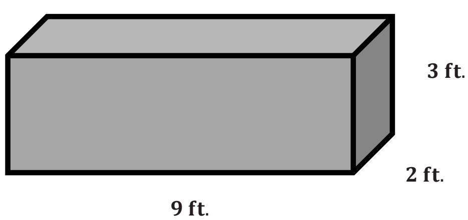
OR

$$V = \frac{1}{8}\ m^3$$

   e. Explain the relationship between the side lengths and the volumes of the cubes.

If each of the sides are changed by the same fractional amount $\left(\frac{1}{a}\right)$ of the original, then the volume of the new figure will be $\left(\frac{1}{a}\right)^3$ of the original volume. For example, if the sides are $\frac{1}{2}$ as long, then the volume will be $\left(\frac{1}{2}\right)^3 = \frac{1}{8}$ as much.

4. Check to see if the relationship you found in Exercise 1 is the same for rectangular prisms.



   a. Determine the volume of the rectangular prism.

$$V = l\,w\,h$$
$$V = (9\ ft.)(2\ ft.)(3\ ft.)$$
$$V = 54\ ft^3$$

b. Determine the volume if all of the sides are half as long as the original lengths.

$$V = l\,w\,h$$
$$V = \left(\frac{9}{2}\,ft.\right)\left(\frac{2}{2}\,ft.\right)\left(\frac{3}{2}\,ft.\right)$$
$$V = \frac{54}{8}\,ft^3$$

OR

$$V = \frac{27}{4}\,ft^3$$

c. Determine the volume if all of the sides are one third as long as the original lengths.

$$V = l\,w\,h$$
$$V = \left(\frac{9}{3}ft.\right)\left(\frac{2}{3}\,ft.\right)\left(\frac{3}{3}\,ft.\right)$$
$$V = \frac{54}{27}\,ft^3$$

OR

$$V = 2\,ft^3$$

d. Is the relationship between the side lengths and the volume the same as the one that occurred in Exercise 1? Explain your answer.

Yes, the relationship that was found in the problem with the cubes still holds true with this rectangular prism. When I found the volume of a prism with side lengths that were one-third the original, the volume was $\left(\frac{1}{3}\right)^3 = \frac{1}{27}$ the original.

MP.2 & MP.7

5. a. If e represents a side length of the cube, create an expression that shows the volume of the cube.

e^3

b. If we divide the side lengths by three, create an expression for the new edge length.

$\frac{1}{3}e$ *or* $\frac{e}{3}$

c. Write an expression that represents the volume of the cube with one third the side length.

$\left(\frac{1}{3}e\right)^3$ *or* $\left(\frac{e}{3}\right)^3$

d. Write an equivalent expression to show that the volume is $\frac{1}{27}$ of the original volume.

$$V = \left(\frac{1}{3}e\right)^3$$
$$V = \left(\frac{1}{3}e\right)\left(\frac{1}{3}e\right)\left(\frac{1}{3}e\right)$$
$$V = \left(\frac{1}{9}e^2\right)\left(\frac{1}{3}e\right)$$
$$V = \frac{1}{27}e^3$$

Closing (2 minutes)

- How did you determine which method to use when solving the exercises?
  - *If I were given the length, width, and height, I have many options for determining the volume. I could use $V = l\ w\ h$. I could also determine the area of the base first and then use $V =$ Area of the base $\times$ height. I could also use a unit cube and determine how many cubes would fit inside.*
  - *If I was given the area of the base and the height, I could use the formula $V =$ Area of the base $\times$ height, or I could also use a unit cube and determine how many cubes would fit inside.*
- What relationships did you notice between the volume and changes in the length, width, or height?
  - *Answers will vary. Students may mention that if the length, width, or height is changed by a certain factor, the volume will be affected by that same factor.*
  - *They may also mention that if all three dimensions are changed by the same factor, the volume will change by that factor cubed. For example, if all the sides are $\frac{1}{2}$ as long as the original, the volume will be $\left(\frac{1}{2}\right)^3$ as large as the original.*

Exit Ticket (5 minutes)

Name ______________________________ Date ______________

Lesson 13: The Formulas for Volume

Exit Ticket

1. A new company wants to mail out samples of its hair products. The company has a sample box that is a rectangular prism with a rectangular base with an area of $23\frac{1}{3}$ in². The height of the prism is $1\frac{1}{4}$ in.

   Determine the volume of the sample box.

2. A different sample box has a height that is twice as long as the original. What is the volume of this sample box? How does the volume of this sample box compare to the volume of the sample box in Problem 1?

Exit Ticket Sample Solutions

1. A new company wants to mail out samples of its hair products. The company has a sample box that is a rectangular prism with a rectangular base with an area of $23\frac{1}{3}$ in². The height of the prism is $1\frac{1}{4}$ in. Determine the volume of the sample box.

$V = Area\ of\ base \times height$

$V = \left(23\frac{1}{3}\ in^2\right)\left(1\frac{1}{4}\ in.\right)$

$V = \frac{70}{3}\ in^2 \times \frac{5}{4}\ in.$

$V = \frac{350}{12}\ in^3$

OR

$V = \frac{175}{6}\ in^3$

2. A different sample box has a height that is twice as long as the original. What is the volume of this sample box? How does the volume of this sample box compare to the volume of the sample box in Problem 1?

$V = Area\ of\ base \times height$

$V = \left(23\frac{1}{3}\ in^2\right)\left(2\frac{1}{2}\ in.\right)$

$V = \left(\frac{70}{3}\ in^2\right)\left(\frac{5}{2}\ in.\right)$

$V = \frac{350}{6}\ in^3$

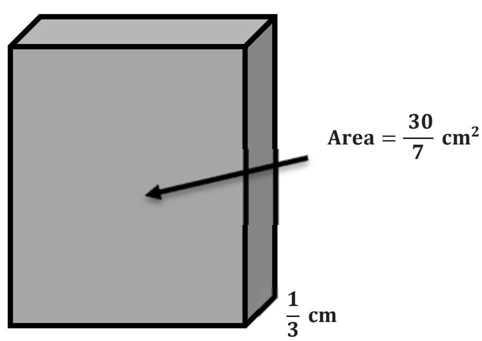
OR

$V = \frac{175}{3}\ in^3$

By doubling the height, we have also doubled the volume.

Problem Set Sample Solutions

1. Determine the volume of the rectangular prism.



$V = Area\ of\ base \times height$

$V = \left(\frac{30}{7}\ cm^2\right)\left(\frac{1}{3}\ cm\right)$

$V = \frac{30}{21}\ cm^3$

OR

2. Determine the volume of the rectangular prism in Problem 1 if the height is quadrupled (multiplied by four). Then determine the relationship between the volumes in Problem 1 and this prism.

$V = Area\ of\ base \times height$

$V = \left(\frac{30}{7}\ cm^2\right)\left(\frac{4}{3} cm\right)$

$V = \frac{120}{21}\ cm^3$

OR

$V = \frac{40}{7}\ cm^3$

When the height was quadrupled, the volume was also quadrupled.

3. The area of the base of a rectangular prism can be represented by A, and the height is represented by h.

a. Write an expression that represents the volume of the prism.

$V = Ah$

b. If the area of the base is doubled, write an expression that represents the volume of the prism.

$V = 2Ah$

c. If the height of the prism is doubled, write an expression that represents the volume of the prism.

$V = A2h = 2Ah$

d. Compare the volume in parts (b) and (c). What do you notice about the volumes?

The expressions in part (b) and part (c) are equal to each other.

e. Write an expression for the volume of the prism if both the height and the area of the base are doubled.

$V = 2A2h = 4Ah$

4. Determine the volume of a cube with a side length of $5\frac{1}{3}$ in.

$$V = l\,w\,h$$
$$V = \left(5\frac{1}{3} in.\right)\left(5\frac{1}{3}\ in.\right)\left(5\frac{1}{3} in.\right)$$
$$V = \frac{16}{3}\ in. \times \frac{16}{3}\ in. \times \frac{16}{3}\ in.$$
$$V = \frac{4096}{27}\ in^3$$

5. Use the information in Problem 4 to answer the following:

a. Determine the volume of the cube in Problem 4 if all of the side lengths are cut in half.

$$V = l\,w\,h$$
$$V = \left(2\frac{2}{3}\ in.\right)\left(2\frac{2}{3} in.\right)\left(2\frac{2}{3} in.\right)$$
$$V = \frac{8}{3}\ in. \times \frac{8}{3}\ in. \times \frac{8}{3}\ in.$$
$$V = \frac{512}{27}\ in^3$$

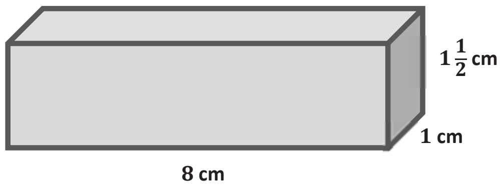
b. **How could you determine the volume of the cube with the side lengths cut in half using the volume in Problem 4?**

Because each side is half as long, I know that the volume will be $\frac{1}{8}$ the volume of the cube in Problem 4. This is because the length, the width, and the height were all cut in half.

$$\frac{1}{2}l \times \frac{1}{2}w \times \frac{1}{2}h = \frac{1}{8}lwh$$

$$\frac{1}{8} \times \frac{4,096}{27} in^3 = \frac{512}{27}\ in^3$$

6. **Use the rectangular prism to answer the following questions.**



a. **Complete the table.**

Length	Volume
$l = 8$ cm	$12\ cm^3$
$\frac{1}{2}l =\ 4$ cm	$6\ cm^3$
$\frac{1}{3}l = \frac{8}{3}$ cm	$4\ cm^3$
$\frac{1}{4}l = 2$ cm	$3\ cm^3$
$2l =\ 16$ cm	$24\ cm^3$
$3l =\ 24$ cm	$36\ cm^3$
$4l =\ 32$ cm	$48\ cm^3$

b. **How did the volume change when the length was one third as long?**

4 is one third of 12. Therefore, when the length is one third as long, the volume is one third as much also.

c. **How did the volume change when the length was tripled?**

36 is three times as much as 12. Therefore, when the length is three times as long, the volume is also three times as much.

d. **What conclusion can you make about the relationship between the volume and the length?**

When only the length is changed, and the width and height stay the same, the change in the volume is proportional to the change in the length.

7. The sum of the volumes of two rectangular prisms, Box A and Box B, are 14.325 cm³. Box A has a volume of 5.61 cm³.

a. Let B represent the volume of Box B in cubic centimeters. Write an equation that could be used to determine the volume of Box B.

$14.325 = 5.61 + B$

b. Solve the equation to determine the volume of Box B.

$B = 8.715\ cm^3$

c. If the area of the base of Box B is 1.5 cm² write an equation that could be used to determine the height of Box B. Let h represent the height of Box B in centimeters.

$8.715 = 1.5h$

d. Solve the equation to determine the height of Box B.

$h = 5.81\ cm$

White Board Exchange: Multiplication and Division Equations with Fractions

Progression of Exercises:	**Answers:**
1. $5y = 35$	$y = 7$
2. $3m = 135$	$m = 45$
3. $12k = 156$	$k = 13$
4. $\frac{f}{3} = 24$	$f = 72$
5. $\frac{x}{7} = 42$	$x = 298$
6. $\frac{c}{13} = 18$	$c = 234$
7. $\frac{2}{3}g = 6$	$g = 9$
8. $\frac{3}{5}k = 9$	$k = 15$
9. $\frac{3}{4}y = 10$	$y = \frac{40}{3} = 13\frac{1}{3}$
10. $\frac{5}{8}j = 9$	$j = \frac{72}{5} = 14\frac{2}{5}$
11. $\frac{3}{7}h = 13$	$h = \frac{91}{3} = 30\frac{1}{3}$
12. $\frac{m}{4} = \frac{3}{5}$	$m = \frac{12}{5} = 2\frac{2}{5}$
13. $\frac{f}{3} = \frac{2}{7}$	$f = \frac{6}{7}$
14. $\frac{2}{5}p = \frac{3}{7}$	$p = \frac{15}{14} = 1\frac{1}{14}$
15. $\frac{3}{4}k = \frac{5}{8}$	$k = \frac{20}{24} = \frac{5}{6}$

Fluency work such as this exercise should take 5–12 minutes of class.

How to Conduct a White Board Exchange:

All students will need a personal white board, white board marker, and a means of erasing their work. An economical recommendation is to place card stock inside sheet protectors to use as the personal white boards and to cut sheets of felt into small squares to use as erasers.

It is best to prepare the problems in a way that allows you to reveal them to the class one at a time. For example, use a flip chart or PowerPoint presentation; write the problems on the board and cover with paper beforehand, allowing you to reveal one at a time; or, write only one problem on the board at a time. If the number of digits in the problem is very low (e.g., 12 divided by 3), it may also be appropriate to verbally call out the problem to the students.

The teacher reveals or says the first problem in the list and announces, "Go." Students work the problem on their personal white boards, holding their answers up for the teacher to see as soon as they have them ready. The teacher gives immediate feedback to each student, pointing and/or making eye contact with the student and responding with an affirmation for correct work such as, "Good job!", "Yes!", or "Correct!" For incorrect work, respond with guidance such as "Look again!", "Try again!", or "Check your work!"

If many students have struggled to get the answer correct, go through the solution of that problem as a class before moving on to the next problem in the sequence. Fluency in the skill has been established when the class is able to go through each problem in quick succession without pausing to go through the solution of each problem individually. If only one or two students have not been able to get a given problem correct when the rest of the students are finished, it is appropriate to move the class forward to the next problem without further delay; in this case, find a time to provide remediation to that student before the next fluency exercise on this skill is given.