TAKE IT TO THE MAT

A NEWSLETTER ADDRESSING THE FINER POINTS OF MATHEMATICS INSTRUCTION

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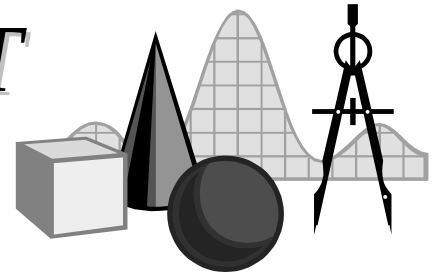

A topic that appears throughout high school mathematics is *absolute value*. It seems straightforward on the surface, but can be quite tricky. In this issue of *Take It to the MAT* we will examine how we often treat absolute value and how forgetting one part of a definition can quickly get students into trouble.

The most common definition of *absolute value* is, "It's always positive." We begin with examples like $|5| = 5$ and $|-5| = 5$, showing that "the sign" doesn't matter. Then we expand our definition to, "$|x|$ is always positive no matter what the value of x," or "$|y-4|$ is always positive." Those statements, while not formal definitions, are useful.

The next step is the solving of simple absolute value equations. For example, solve these three equations: $|x| = 5$, $|y-4| = 5$, and $|z| = -1$. The solutions, respectively, are: $x = \pm 5$, $y = -1$ or $y = 9$, and no solution.

The explanation given when solving the first equation is usually, "Since the absolute value of x is five, then x must be either five or negative five." Very true. The second equation just requires one more step: "The absolute value of $y - 4$ is five, so $y - 4$ must be either five or negative five." Thus, we must now essentially solve two simpler equations: $y - 4 = 5$ and $y - 4 = -5$. The last of the three equations has no solution because the absolute value of z "must be positive." Since negative one is not positive, there is no value of z that could make the statement true.

As an aside, at some time a curious student may ask about the absolute value of zero. Since zero is neither positive or negative, "dropping the sign" is not an option. It is at this time that we introduce the absolute value as "always positive *or zero*." More formally, we would say that **absolute value is non-negative**. This is a point that cannot be overlooked.

We tell students that when encountering equations of the form $|f(x)| = g(x)$ that $f(x) = \pm g(x)$. We may not say it exactly like that, but the "positive or negative" component is always present and usually emphasized. It is a valuable technique and provides a quick road to the solution set. Try it on this one: $|y-4| = y-4$.

Easy! Using the "positive or negative" technique we get two simpler equations to solve: $y - 4 = +(y-4)$ or $y - 4 = -(y-4)$. Solving each for y, the first is reflexive and therefore y is any real number; the second yields $y = 4$. The quick conclusion is that since four is a real number, the solution is all real numbers.

But didn't we forget that since the absolute value of $y - 4$ must be non-negative so must $y - 4$? That was the first part of our informal definition. Therefore, (from the right side of the equation) if $y - 4 \geq 0$, then the only solutions we can consider are when $y \geq 4$. The actual solution is all real numbers greater than or equal to four. Neglecting the "always positive" condition led us to an incorrect answer.

Students should definitely see the more formal definition $|x| = x$ if $x \geq 0$, $|x| = -x$ if $x < 0$. They should also apply it to equations like those given above. The informal definitions and techniques discussed here are useful, but can lead to difficulty if students are not vigilant and teachers fail to completely address them.