TAKE IT TO THE MAT

A NEWSLETTER ADDRESSING THE FINER POINTS OF MATHEMATICS INSTRUCTION

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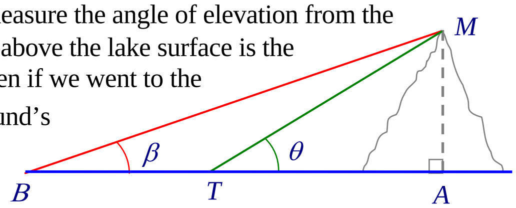
The subject of this month's *Take It to the MAT* is a question from a local science teacher. The teacher asks, "If I'm in a boat out on a lake, is there a way that I can determine the height of a distant object, like an island or mountain? How accurate would I be?" The short answers are, "Yes," and, "It depends." This is a great investigation for students.

Let's take the scenario of being on a lake and finding the elevation of a distant island's peak. Consider an observer on a boat to be located at point *B* and the top of the island to be point *M*. We'll call the point directly under the peak at the same elevation as the observer to be point *A*. Triangle *BAM* is a right triangle.

We could use a simple angle measuring device, like a clinometer, to measure the angle of elevation from the boat to the top of the island. Call that angle β. The height of the peak above the lake surface is the distance $AM = BA\tan\beta$. The problem is that we don't know *BA*. Even if we went to the island, we couldn't directly measure *BA* since point *A* is under the ground's surface! Students should recognize this and investigate a solution.



We need a second observation point. We'll pilot our boat from point *B* directly toward point *A* for a given distance to point *T*. The angle of elevation from *T* to *M* will be θ. (How we measure the distance *BT* on the lake is left to the imagination of the students.)

Triangle *TAM* is also a right triangle, where $AM = TA\tan\theta$. We know that $BA = BT + TA$, so we have enough information to find *AM*. Step 1: find *BA* and *TA* in terms of what we know, β and θ, and what we want to know, *AM*. Step 2: substitute the expressions for *BA* and *TA* into the equation $BA = BT + TA$. Step 3: solve for *AM*.

Step 1

$$AM = BA\tan\beta \Rightarrow BA = \frac{AM}{\tan\beta}$$

$$AM = TA\tan\theta \Rightarrow TA = \frac{AM}{\tan\theta}$$

Step 2

$$BA = BT + TA$$

$$\frac{AM}{\tan\beta} = BT + \frac{AM}{\tan\theta}$$

Step 3

$$\frac{AM}{\tan\beta} - \frac{AM}{\tan\theta} = BT$$

$$AM\left(\frac{1}{\tan\beta} - \frac{1}{\tan\theta}\right) = BT$$

$$AM = \frac{BT}{\frac{1}{\tan\beta} - \frac{1}{\tan\theta}}$$

$$AM = \frac{BT}{\cot\beta - \cot\theta}$$

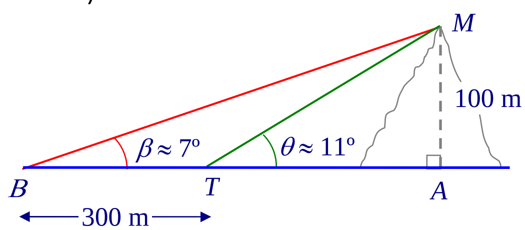
We now have an equation that relates *AM* to our measured quantities:

$AM = \dfrac{BT}{\cot\beta - \cot\theta}$. (Students should also be able to show that

$\dfrac{BT}{\cot\beta - \cot\theta} = \dfrac{BT\sin\beta\sin\theta}{\sin(\theta - \beta)}$, since it's easier to use the sine key than $\dfrac{1}{\tan}$.)

What kids really need to consider, though, is how good their estimates of the island's height will be with this method. Let's set up a situation and then determine how far off we might be.

Say we have an island that rises 100. m above the surface of the lake ($AM = 100$) and that the correct angle of elevation from point B is 7º. Let's also assume that if we proceed toward the island 300. m, the angle of elevation is now 11º. (In the real world, we wouldn't know the angles, but we need to here to evaluate possible errors.)

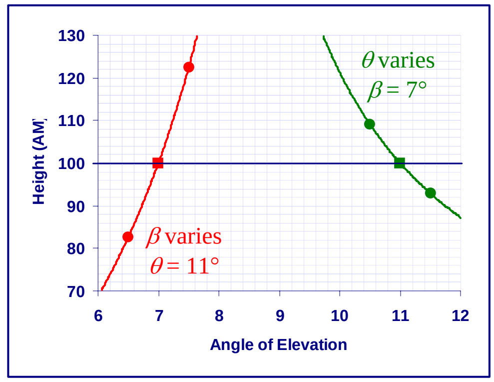


We'll ignore any measurement error of the distance *BT* and concentrate on the angle measurements. Assuming that our clinometer is well constructed, any error should be from the reading of the angles of elevation. If the precision of our instrument is 1º, then the maximum error is ±0.5º. Let's see what happens if we measure β accurately and θ is either of the three values 10.5º, 11.0º, or 11.5º.

β	θ	*AM*	Error
7.0º	10.5º	109 m	9 %
7.0º	11.0º	100 m	0
7.0º	11.5º	92.9 m	–7 %

We get greater errors when we underestimate the value of θ by a given amount than overestimate. This can be seen in the graph on the next page.

The green curve is the function $AM = \dfrac{300}{\cot 7^\circ - \cot\theta}$.



Now, let's measure θ accurately and let β be 6.5º, 7.0º, or 7.5º.

β	θ	AM	Error
6.5º	11.0º	82.6 m	–17 %
7.0º	11.0º	100. m	0
7.5º	11.0º	122 m	22 %

Now, greater errors occur when we overestimate the value of β by a given amount. The red curve in the graph at right is the function

$AM = \dfrac{300}{\cot\beta - \cot 11.0^\circ}$. We also notice that the errors in the value of AM are greater for a given error of β than of θ.

β	θ	AM	Error
6.5º	11.5º	77.7 m	–23.3 %
6.5º	10.5º	88.7 m	–11.3 %
7.0º	11.0º	100. m	0
7.5º	11.5º	112 m	12 %
7.5º	10.5º	136 m	36 %

Finally, consider the greatest possible errors of β and θ when varying together; $AM = \dfrac{300}{\cot\beta - \cot\theta}$. Not surprisingly, the largest error occurs when we have the smallest value of θ and the largest value of β.

β	θ	AM	Error
10.5º	25.5º	90.9 m	–9.1 %
10.5º	24.5º	93.7 m	–7.3 %
11.0º	25.0º	100. m	0
11.5º	25.5º	106 m	6 %
11.5º	24.5º	110 m	10 %

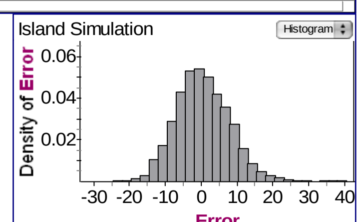
The next question we might ask students is what happens if we move closer. Let's assume that the angles of elevation are now $\beta = 11°$ and $\theta = 25°$, and that BT is still 300 m. As we move closer, the error decreases. This would seem to make sense, because as angle increases, the slope of cotangent decreases, so we would have smaller differences between the cotangent of the measured angle and the cotangent of the actual angle.

So, why don't we get as close as we can? That's the general idea, but it all depends on how close we can get our boat to the island.

Island Simulation

	Beta	Theta	AM	Error	AbsErr
1	7.2	11	108.3	8.25	8.25
2	7.1	11.2	100.7	0.74	0.74
3	6.9	10.8	99.3	-0.71	0.71
4	6.9	11.2	93.4	-6.63	6.63
5	6.8	10.8	95.4	-4.58	4.58

Finally, students could do more advanced analysis of the errors. Let's assume that errors in measurement are normally distributed with a mean error of zero. Thus, the probability of a given error decreases as the error increases and we are just as likely to overestimate as we are to underestimate. This can be modeled using the Random Normal (randNorm) function on a graphing calculator or with computer software like *Fathom*. The display at right is a *Fathom* simulation of 25,000 runs of the scenario where the actual angle measurements are 7º and 11º and $BT = 300$.



Because of the way that β and θ behave, we are more likely to overestimate the value of AM than underestimate it. This can be seen in the fact that the mean error is about +0.45%. However, the mean absolute error is on the order of 6%. A simulation with the second set of angles (11º and 25º, not shown) reduces those errors to about a third of what they are with the smaller angles.

Island Simulation

AM	100.45369
Error	0.45369019
AbsErr	5.8216636

S1 = mean ( )

We have taken a simple trigonometry problem and explored it from several angles by including analysis, probability, and "what if?" situations. To make such a problem non-trivial, we should have our students do the same.