Pre-Algebra Notes – Unit 10: Measurement, Area, and Volume

Triangles, Quadrilaterals, and Polygons

Syllabus Objectives: (4.6) The student will classify polygons. (4.16) The student will validate conclusions about geometric figures and their properties.

Take this opportunity to review vocabulary and previous knowledge about angle measure for triangles—it is a good place to fill in any gaps in student knowledge. Students should know:

- The sum of the angles measures of any triangle is 180°. This can be shown with a quick demonstration:
  1) Draw and label a large triangle as shown.
  2) Cut the triangle out.
  3) Tear each angle from the triangle and place them so their vertices meet at a point.
  4) The angles form a straight line: 180°.

- acute triangle—3 acute angles
- right triangle—1 right angle
- obtuse triangle—1 obtuse angle
- equiangular triangle—3 congruent angles
- equilateral triangle—3 congruent sides
- isosceles triangle—at least 2 congruent sides
- scalene triangle—no congruent sides

A tree diagram could also be used to show the relationships.
This vocabulary can be very important when tackling some word problems.

**Example:** The ratio of the angle measures in a triangle is $2:3:5$. Classify the triangle by its angle measures.

Since we know there will be a common factor to multiply each of the angles, we can let $2x$, $3x$ and $5x$ represent the angle measures. We also know that the sum of the angle measures of a triangle is $180^\circ$. So

$$2x + 3x + 5x = 180$$

$$10x = 180$$

$$x = 18$$

Substituting $18$ into each of the expressions,

$$2x = 2(18)$$
$$3x = 3(18)$$
$$5x = 5(18)$$

$$= 36$$
$$= 54$$
$$= 90$$

Since one of the angle measures is $90^\circ$, we know that we have a right triangle.

A **polygon** is defined as a closed geometric figure formed by connecting line segments endpoint to endpoint.

<table>
<thead>
<tr>
<th>Polygons</th>
<th>Not Polygons</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1.png" alt="Polygons" /></td>
<td><img src="image2.png" alt="Not Polygons" /></td>
</tr>
</tbody>
</table>

It is considered a **convex polygon** if no segment connecting two vertices is outside the polygon. A **concave polygon** is one in which at least one segment connecting two vertices is outside the polygon. In discussions using the word *polygon*, it is usually understood to mean a convex polygon. Have students make sketches of examples of each.

<table>
<thead>
<tr>
<th>Convex Polygons</th>
<th>Concave Polygons</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image3.png" alt="Convex Polygons" /></td>
<td><img src="image4.png" alt="Concave Polygons" /></td>
</tr>
</tbody>
</table>
A **regular polygon** is one in which all the sides are of equal length and all the angles have the same measure.

Examples:

![Regular pentagon with all sides and all angles congruent](image1)

![Not a regular polygon as only 2 sides are congruent](image2)

Note: When labeling geometric figures, mark angles and segments that are equal in measure with similar marks. For example, in the pentagon all the angles are marked equal with an arc with one slash and the sides marked equal with one slash. In triangle ABC, the measure of $\overline{AB}$ and $\overline{AC}$ are shown equal with two slashes.

Polygons are named by the number of sides. We know a triangle has 3 sides. Below are the names of other polygons.

<table>
<thead>
<tr>
<th>Polygons</th>
<th>Quadrilateral</th>
<th>Pentagon</th>
<th>Hexagon</th>
<th>Heptagon</th>
<th>Octagon</th>
<th>Nonagon</th>
<th>Decagon</th>
</tr>
</thead>
<tbody>
<tr>
<td># of sides</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>10</td>
</tr>
</tbody>
</table>

Have you ever had a problem with drawing a polygon? Many times we end up with everything ‘bunched’ at the end, or not a polygon.

Here is a suggestion to more easily draw a polygon:

1. Lightly draw a circle.

![Lightly draw a circle](image3)

2. Place points on the circle to represent the endpoints of the segments of the polygon you wish to create.

![Place points on the circle](image4)

3. Connect the points to create your polygon.

![Connect the points](image5)

4. Erase the circle. You have your polygon!

![Erase the circle](image6)
Quadrilaterals are further classified; names are based on congruent sides, parallel sides and right angles.

<table>
<thead>
<tr>
<th>Quadrilateral Type</th>
<th>Definition</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parallelogram</td>
<td>Quadrilateral with both pairs of opposite sides parallel.</td>
<td><img src="image" alt="Parallelogram" /></td>
</tr>
<tr>
<td>Rhombus</td>
<td>Parallelogram with four congruent sides.</td>
<td><img src="image" alt="Rhombus" /></td>
</tr>
<tr>
<td>Rectangle</td>
<td>Parallelogram with four right angles.</td>
<td><img src="image" alt="Rectangle" /></td>
</tr>
<tr>
<td>Square</td>
<td>Parallelogram with four right angles and four congruent sides.</td>
<td><img src="image" alt="Square" /></td>
</tr>
<tr>
<td>Trapezoid</td>
<td>Quadrilateral with exactly one pair of parallel sides.</td>
<td><img src="image" alt="Trapezoid" /></td>
</tr>
</tbody>
</table>

Another way to show the relationship of the parallelograms is to complete a Venn diagram as shown below.

![Venn Diagram](image)

Again, vocabulary will become very important when trying to solve word problems.
Example: A quadrilateral has both pairs of opposite sides parallel. One set of opposite angles are congruent and acute. The other set of angles is congruent and obtuse. All four sides are NOT congruent. Which name below best classifies this figure?

A. parallelogram  
B. rectangle  
C. rhombus  
D. trapezoid

We have both pairs of opposite sides parallel, so it cannot be the trapezoid. Since the angles are not 90° in measure, we can rule out the rectangle. We are told that the 4 sides are not congruent, so it cannot be the rhombus. Therefore, we have a parallelogram.

We have looked at classifying quadrilaterals, but what about their angle measure? We can take this opportunity to get students thinking of how many degrees exist in the sum of the interior angles of a quadrilateral (saving the next extension to all polygons for a later unit—Chapter 13).

Make a drawing of any quadrilateral.

Draw all the diagonals from one vertex.  
(For a quadrilateral, there is only one diagonal.)

We now have 2 triangles. Since we know that the angle measure in each triangle is 180°, we can conjecture that the sum of the angle measures in a quadrilateral is 180° + 180° = 360°.

Example: Find the value of x in the figure.

\[ 80 + 70 + 90 + x = 360 \]
\[ \text{sum of the angle measures in a quadrilateral is 360°} \]
\[ 240 + x = 360 \]
\[ \text{combine like terms} \]
\[ x = 120 \]
\[ \text{subtract 240 from each side} \]

The missing angle x would have a measure of 120°.
Perimeter

Syllabus Objective: (4.7) The student will find the perimeter of plane figures.

The perimeter of a polygon is the sum of the lengths of the segments that make up the sides of the polygon.

Example: Find the perimeter of the regular pentagon.

Since the pentagon is regular, we know all five sides have a measurement of 2 meters. So we simply multiply $5 \cdot 2$ for an answer of 10 meters for the perimeter.

Example: The perimeter of an isosceles triangle is 42 feet. The length of the base (not one of the congruent sides) is 12 feet. What are the lengths of the other 2 sides?

An isosceles triangle has at least 2 equal sides (by definition). A sketch often helps. We know the base is not one of the equal sides; we can label the unknown lengths as “$x$” and determine an equation to solve.

\[
x + x + 12 = 42
\]
\[
2x + 12 = 42
\]
\[
-12 = -12
\]
\[
2x = 30
\]
\[
x = 15
\]

The lengths of the other two sides are 15 feet.

Example: Find the perimeter of the figure below. Classify the quadrilateral.

Since both pairs of opposite sides are parallel and all 4 angles are right angles, we have a rectangle. The opposite sides are also equal, so $P = 16 + 4 + 16 + 4$ or 40.

The figure is a rectangle and it has a perimeter of 40 feet.
Area of Parallelograms, Triangles and Trapezoids

**Syllabus Objective: (4.8) The student will find the area of plane figures.**

One way to describe the size of a room is by naming its dimensions. A room that measures 12 ft. by 10 ft. would be described by saying it’s a 12 by 10 foot room. That’s easy enough.

There is nothing wrong with that description. In geometry, rather than talking about a room, we might talk about the size of a rectangular region.

For instance, let’s say I have a closet with dimensions 2 feet by 6 feet. That’s the size of the closet.

![Example of a rectangle with dimensions 2 ft. by 6 ft.]

Someone else might choose to describe the closet by determining how many one foot by one foot tiles it would take to cover the floor. To demonstrate, let me divide that closet into one foot squares.

![Example of a closet divided into one foot squares]

By simply counting the number of squares that fit inside that region, we find there are 12 squares.

If I continue making rectangles of different dimensions, I would be able to describe their size by those dimensions, or I could mark off units and determine how many equally sized squares can be made.

Rather than describing the rectangle by its dimensions or counting the number of squares to determine its size, we could multiply its dimensions together.

Putting this into perspective, we see the number of squares that fits inside a rectangular region is referred to as the **area**. A shortcut to determine that number of squares is to multiply the base by the height.

The **area of a rectangle** is equal to the product of the length of the base and the length of a height to that base.

That is $A = bh$. Most books refer to the longer side of a rectangle as the length ($l$), the shorter side as the width ($w$). That results in the formula $A = lw$. The answer in an area problem is always given in square units because we are determining how many squares fit inside the region.
Example: Find the area of a rectangle with the dimensions \(3\text{ m by 2 m}\).

\[
A = lw
\]

\[
A = 3 \cdot 2
\]

\[
A = 6
\]

*The area of the rectangle is \(6\text{ m}^2\).*

Example: Find the area of the rectangle.

Be careful! Area of a rectangle is easy to find, and students may quickly multiply to get an answer of 18. This is wrong because the measurements are in different units. We must first convert feet into yards, or yards into feet.

\[
\frac{\text{yards}}{\text{feet}} \rightarrow \frac{1}{3} = \frac{x}{9}
\]

\[
9 = 3x
\]

\[
x = 3
\]

We now have a rectangle with dimensions \(3\text{ yd. by 2 yd.}\).

\[
A = lw
\]

\[
A = (3)(2)
\]

\[
A = 6
\]

*The area of our rectangle is \(6\text{ square yards}\).*

If I were to cut one corner of a rectangle and place it on the other side, I would have the following:

We now have a parallelogram. Notice, to form a parallelogram, we cut a piece of a rectangle from one side and placed it on the other side. Do you think we changed the area? The answer is no. All we did was rearrange it; the area of the new figure, the parallelogram, is the same as the original rectangle.
So we have the **area of a parallelogram = bh**.

![Parallelogram Diagram]

**Example:** The height of a parallelogram is twice the base. If the base of the parallelogram is 3 meters, what is its area?

*First, find the height. Since the base is 3 meters, the height would be twice that or 2(3) or 6 m. To find the area,*

\[ A = bh \]

\[ A = 3 \cdot 6 \]

\[ A = 18 \]

*The area of the parallelogram is 18 m².*

Standardized testing (CRT and NHSPE included) often ask students to address what will happen to a value for the area when dimensions are changed. Let’s look at an example.

**Syllabus Objective: (4.13) The student will describe how changes in the value of one variable effect the values of the remaining variables in a relationship.**

Using graph paper, draw rectangles with the following dimensions: 1 by 2 units, 2 by 4 units, and 4 by 8 units. Determine the perimeter and area for all 3 rectangles.

- **Example:**

  - **1 by 2 units:**
    - Perimeter: \( P = 6 \) units
    - Area: \( A = 2 \) units²
  - **2 by 4 units:**
    - Perimeter: \( P = 12 \) units
    - Area: \( A = 8 \) units²
  - **4 by 8 units:**
    - Perimeter: \( P = 24 \) units
    - Area: \( A = 32 \) units²

- **When you double the base and the height, what happens to the perimeter? (doubles)**
  - **Area? (quadruples)**

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- When you quadruple the base and the height, what happens to the perimeter? (quadruples) Area? (16 times)
- How might you show this relationship for area algebraically?

\[
A = bh
\]

If both base and height are doubled:
\[
A = 2b \cdot 2h
\]
\[
A = 4bh
\]

The new area is 4 times the original area.

We have established that the area of a parallelogram is \( A = bh \). Let’s see how that helps us to understand the area formula for a triangle and trapezoid.

For this parallelogram, its base is 4 units and its height is 3 units. Therefore, the area is \( 4 \cdot 3 = 12 \text{ units}^2 \).

If we draw a diagonal, it cuts the parallelogram into 2 triangles. That means one triangle would have one-half of the area or 6 units\(^2\). Note the base and height stay the same. So for a triangle,
\[
A = \frac{1}{2} bh, \text{ or } \frac{1}{2}(4)(3) = 6 \text{ units}^2
\]

For this parallelogram, its base is 8 units and its height is 2 units. Therefore, the area is \( 8 \cdot 2 = 16 \text{ units}^2 \).

If we draw a line strategically, we can cut the parallelogram into 2 congruent trapezoids. One trapezoid would have an area of one-half of the parallelogram’s area (8 units\(^2\)). Height remains the same. The base would be written as the sum of \( b_1 \) and \( b_2 \). For a trapezoid:
\[
A = \frac{1}{2}(b_1 + b_2)h, \text{ or } \frac{1}{2}(2 + 6)2 = 8 \text{ units}^2
\]
Circles: Circumference and Area

Syllabus Objectives:  
(4.9) The student will find the circumference of circles.  
(4.10) The student will find the area of circles.

A circle is defined as all points in a plane that are equal distance (called the radius) from a fixed point (called the center of the circle). The distance across the circle, through the center, is called the diameter. Therefore, a diameter is twice the length of the radius, or \( d = 2r \).

We called the distance around a polygon the perimeter. The distance around a circle is called the circumference. There is a special relationship between the circumference and the diameter of a circle. Let’s get a visual to approximate that relationship. Take a can with 3 tennis balls in it. Wrap a string around the can to approximate the circumference of a tennis ball. Then compare that measurement with the height of the can (which represents three diameters). You will discover that the circumference of the can is greater than the three diameters (height of the can).

You can make an exercise for students to discover an approximation for this circumference/diameter relationship which we call \( \pi \). Have students take several circular objects, measure the circumference \( (C) \) and the diameter \( (d) \). Have students determine \( \frac{C}{d} \) for each object; have groups average their results. Again, they should arrive at answers a little bigger than 3. This should help convince students that this ratio will be the same for every circle.

We can then introduce that \( \frac{C}{d} = \pi \) or \( C = \pi d \). Since \( d = 2r \), we can also write \( C = 2\pi r \). Please note that \( \pi \) is an irrational number (never ends or repeats). Mathematicians use \( \pi \) to represent the exact value of the circumference/diameter ratio.

Example: If a circle has a diameter of 4 m, what is the circumference? Use 3.14 to approximate \( \pi \). State your answer to the nearest 0.1 meter.

Using the formula:
\[
C = \pi d
\]
\[
C \approx (3.14)(4)
\]
\[
C \approx 12.56
\]

The circumference is about 12.6 meters.

Many standardized tests (including the CRT and the district common exams) ask students to leave their answers in terms of \( \pi \). Be sure to practice this!

Example: If a circle has a radius of 5 feet, find its circumference. Do not use an approximation for \( \pi \).

Using the formula:
\[
C = 2\pi r
\]
\[
C = 2 \cdot \pi \cdot 5
\]
\[
C = 10\pi
\]

The circumference is about 10\( \pi \) feet.
Example: If a circle has a circumference of 12\pi inches, what is the radius?

Using the formula: \[ C = 2\pi r \]
\[ 12\pi = 2\pi r \]
\[ \frac{12\pi}{2\pi} = \frac{2\pi r}{2\pi} \]
\[ 6 = r \]

The radius is 6 inches.

Example: A circle has a circumference of 24 m. Using \( \pi \approx 3.14 \), find the diameter. Round your answer to the nearest whole number.

Using the formula: \[ C = \pi d \]
\[ 24 \approx (3.14)d \]
\[ \frac{24}{3.14} \approx \frac{(3.14)d}{3.14} \]
\[ 7.6 \approx d \]

The diameter is about 8 meters.

You can demonstrate the formula for finding the area of a circle. First, draw a circle; cut it out. Fold it in half; fold in half again. Fold in half two more times, creating 16 wedges when you unfold the circle. Cut along these folds.

Rearrange the wedges, alternating the pieces tip up and down (as shown), to look like a parallelogram.
The more wedges we cut, the closer it would approach the shape of a parallelogram. No area has been lost (or gained). Our “parallelogram” has a base of \( \pi r \) and a height of \( r \). We know from our previous discussion that the area of a parallelogram is \( bh \). So we now have the area of a circle:

\[
A = \pi r^2
\]

**Example:** Find the area of the circle to the nearest square meter if the radius of the circle is 12 m. Use \( \pi \approx 3.14 \).

*Using the formula:*

\[
A = \pi r^2
\]

\[
A \approx (3.14)(12)^2
\]

\[
A \approx 452.16
\]

*The area of the circle is about 452 square meters.*

**Example:** Find the area of the circle if the diameter is 10 inches. Leave your answer in terms of \( \pi \).

*Using the formula:*

\[
A = \pi r^2
\]

\[
A = \pi (10)^2
\]

\[
A = 100\pi
\]

*The area of the circle is 100\( \pi \) square inches.*

**Example:** If the area of a circle is 70 square meters, find the radius to the nearest meter.

*Using the formula:*

\[
A = \pi r^2
\]

\[
70 \approx (3.14)r^2
\]

\[
\frac{70}{3.14} \approx \frac{(3.14)r^2}{3.14}
\]

\[
22.3 \approx r^2
\]

\[
\sqrt{22.3} \approx r
\]

\[
4.7 \approx r
\]

*The radius of the circle is about 5 meters.*
Example: The dimensions of a church window are shown below. Find the area of the window to the nearest square foot.

First, find the area of the rectangle.

\[ A = bh \]
\[ A = 11 \cdot 8 \]
\[ A = 88 \]

Next, we have half of a circle. We are given the diameter, so the radius would be half of the 11 feet or 5.5 feet. To find the area of half of a circle with radius 5.5,

\[ A = \frac{1}{2} \pi r^2 \]
\[ A \approx \frac{1}{2} (3.14)(5.5)^2 \]
\[ A \approx 47.4925 \]

To find the total area we add the two areas we found: \( 88 + 47.4925 \approx 135.4925 \)

The area of the church window is about 136 square feet.

Surface Area

Syllabus Objective: (4.11) The student will find the surface area of prisms, cylinders, pyramids and cones.

The surface area of a solid is the sum of the areas of all the surfaces that enclose that solid. To find the surface area, draw a diagram of each surface as if the solid was cut apart and laid flat. Label each part with the dimensions. Calculate the area for each surface. Find the total surface area by adding the areas of all of the surfaces. If some of the surfaces are the same, you can save time by calculating the area of one surface and multiplying by the number of identical surfaces.

As a class activity, you may want to take common boxes (ie. cereal box, etc.) to have students actually cut apart the solids and lay them flat. An oatmeal container can be used to illustrate what the cylinder looks like when it is flattened. You could use a paper towel or toilet paper cardboard roll to quickly show how the lateral surface of a cylinder flattens into a rectangle.
**Example:** Find the surface area of the prism shown. All surfaces are rectangles.

Divide the prism into its parts. Label the dimensions.

<table>
<thead>
<tr>
<th>Bases</th>
<th>Lateral Faces</th>
</tr>
</thead>
<tbody>
<tr>
<td>top</td>
<td>front</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
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<tr>
<td>15</td>
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<tr>
<td>bottom</td>
<td>back</td>
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<td>2</td>
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<td>15</td>
<td>15</td>
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<td>side</td>
<td>side</td>
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<td>4</td>
<td>4</td>
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<td>2</td>
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</tbody>
</table>

Find the area of all the surfaces.

<table>
<thead>
<tr>
<th>Bases</th>
<th>Lateral Faces</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A = bh$</td>
<td>$A = bh$</td>
</tr>
<tr>
<td>$A = 15 \cdot 2$</td>
<td>$A = 15 \cdot 4$</td>
</tr>
<tr>
<td>$A = 30$</td>
<td>$A = 60$</td>
</tr>
<tr>
<td>$A = bh$</td>
<td>$A = bh$</td>
</tr>
<tr>
<td>$A = 15 \cdot 2$</td>
<td>$A = 2 \cdot 4$</td>
</tr>
<tr>
<td>$A = 30$</td>
<td>$A = 8$</td>
</tr>
</tbody>
</table>

Surface Area = $30 + 30 + 60 + 60 + 8 + 8$

= **196**

**The surface area of the prism is 196 cm².**

Note: Since some of the faces were identical, we could multiply by 2 instead of adding the value twice. That work would look like

Surface Area = $2(30) + 2(60) + 2(8)$

= $60 + 120 + 16$

= **196**
We need to define the three-dimensional figures with which we will be working.

**cylinder**: a solid with two congruent circular bases that lie in parallel planes.

**prism**: a solid formed by two parallel polygonal congruent bases that lie in parallel planes—the other faces are rectangles.

**pyramid**: a solid whose base may be any polygon, with the other faces triangles.

**cone**: a solid with one circular base.

**Example**: Find the surface area of the cylinder shown (to the nearest meter).

The bases are identical. The area of one circle would be

\[ A = \pi r^2 \]

\[ A \approx (3.14)(4.5)^2 \]

\[ A \approx 63.585 \]

Therefore, the area for both circles would be

\[ 2(63.585) \text{ or } 127.17 \]

Surface area \(= 127.17 + 141.3 = 268.47 \)

The surface area of the cylinder is about \(269 \text{ m}^2\).
Example: Find the surface area of the square pyramid shown (to the nearest inch).

The area of the base would be $12 \cdot 12 = 144$.

The area of one of the lateral faces would be $A = \frac{1}{2}bh$

But since there are 4 lateral faces, $4(48) = 192$. This is the surface area of the lateral faces.

Surface area would be $144 + 192 = 336$.

The surface area of the square-based pyramid is 336 square inches.

Finding the surface area of a cone is related to finding the surface area of a pyramid. To find the lateral area of the square pyramid, we took the sum of the four triangles that made up the lateral surface. If we wished to find the surface area of hexagonal pyramid, we would find the sum of the areas of the six triangles.

What if we wanted to use a pyramid with regular polygonal base of 100 sides? 1000 sides? What would this polygon begin to look like? As the number of sides of a regular polygon increases, the polygon’s shape approaches that of a circle. As the number of faces increases, the lateral faces’ shapes would approach that of a cone.

We do not want to confuse the slant height of a pyramid with the actual height of the pyramid, so we use the variable $l$ rather than $h$ for slant height. The lateral surface area of the square pyramid can be shown as:

$$\text{Lateral surface area} = \frac{1}{2}bl + \frac{1}{2}bl + \frac{1}{2}bl + \frac{1}{2}bl$$

$$= \frac{1}{2}l(b + b + b + b)$$

$$= \frac{1}{2}l(\text{perimeter})$$

For the cone, we will substitute circumference for perimeter, and get

$$\text{Lateral surface area for a cone} = \frac{1}{2}l(C) = \frac{1}{2}l(2\pi r) = \pi rl$$

So to find the surface area of a cone is the sum of the area of the base ($A = \pi r^2$) and the lateral surface area, or surface area of a cone: $\pi r^2 + \pi rl$. 

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Example: Find the surface area of the cone. Leave your answer in terms of $\pi$.

Using the formula:

\[
S = \pi r^2 + \pi rl
= \pi (3)^2 + \pi (3)8
= 9\pi + 24\pi
= 33\pi
\]

The surface area of the cone is $33\pi \text{ cm}^2$.

Volume

Syllabus Objective: (4.12) The student will find the volume of prisms, cylinders, pyramids and cones.

If you were to buy dirt for your yard, it’s typically sold in cubic yards—that’s describing volume. If you were laying a foundation for a house or putting in a driveway, you’d want to buy cement, and cement is often sold by the cubic yard. Carpenters, painters and plumbers all use volume relationships.

The volume of a three dimensional figure measures how many cubes will fit inside it. It’s easy to find the volume of a solid if it is a rectangular prism with whole number dimensions. Let’s consider a figure $3 \text{ m} \times 2 \text{ m} \times 4 \text{ m}$.

We can count the cubes measuring $1 \text{ meter}$ on an edge. The bottom layer is $3 \times 2$—there are $6$ square meter cubes on the bottom layer.

We have three more layers stacked above it (for a total of $4$ layers), or $6 + 6 + 6 + 6 = 24$.

Now we can reason that if I know how many cubes are in the first layer ($6$), then to find the total number of cubes in the stack, you simply multiply the number on the first layer by the height of the stack ($6 \cdot 4 = 24$).

This is a way of finding volume. We find the area of the base ($B$) and multiply it times the height ($h$) of the object.

For prisms and cylinders, $V = Bh$, where $B$ is the area of the base and $h$ is the height.

Leaving the reasoning to a later course in geometry, we can also state that for pyramids and cones, $V = \frac{1}{3} Bh$, where $B$ is the area of the base and $h$ is the height.
Example: Find the volume of the prism shown.

The bases of the prism are the triangles, so to find the area of the base we will use the formula \( A = \frac{1}{2}bh \). The height will be the distance between the two bases (4). We have:

\[
A = Bh \\
A = \frac{1}{2}(8)(6)(4) \\
A = 96
\]

The volume of the triangular prism is 96 cubic meters.

Example: Find the volume of a chocolate cake that has a diameter of 24 cm and a height of 14 cm. Use \( \frac{22}{7} \) as an approximation for \( \pi \).

\[
V = Bh . \quad \text{Our base is a circle, so we will need the radius. The radius is one-half the diameter so } r = \frac{1}{2}(24) \text{ or 12. We would now have} \\
V = Bh \\
V = \left( \frac{22}{7} \right)(12)^2(14) \\
V = 6336
\]

The volume of the chocolate cake is 6336 cubic centimeters.

Example: Find the volume of the square pyramid.

\[
V = \frac{1}{3}Bh \\
V = \frac{1}{3}(12)(12)(8) \\
V = 384
\]

The volume of the square pyramid is 384 cubic inches.
Example: Find the diameter of the cone if you know the volume of the cone is $18\pi$ cubic centimeters.

\[
V = \frac{1}{3} Bh
\]

\[
18\pi = \frac{1}{3}\pi r^3 (6)
\]

\[
18\pi = 2\pi r^2
\]

\[
\frac{18\pi}{2\pi} = \frac{2\pi r^2}{2\pi}
\]

\[
9 = r^2
\]

\[
3 = r
\]

Since the diameter is twice the radius, the diameter of the cone would be $2(3)$ or $6$ cm.

Example: The solid shown to the right is a cube with a cylinder-shaped hole in it. Find the volume of the solid. Round your answer to the nearest whole number. Use $\pi = 3.14$.

Our plan will be to find the volume of the cube, then the volume of the cylinder. To find the volume of the solid, we will then subtract the cylinder’s volume from that of the cube.

\[
V_{\text{cube}} = Bh
\]

\[
V_{\text{cylinder}} = Bh
\]

\[
V_{\text{cube}} = (5)(5)(5)
\]

\[
V_{\text{cylinder}} \approx (3.14)(0.5)^2 (5)
\]

\[
V_{\text{cube}} = 125
\]

\[
V_{\text{cylinder}} \approx 3.925
\]

\[
V_{\text{cube}} - V_{\text{cylinder}} \approx 125 - 3.925
\]

\[
\approx 121.075
\]

The volume of the solid is approximately 121 cubic feet.
Precision, Accuracy, Error and Tolerance

Syllabus Objective: (4.15) The student will demonstrate an understanding of precision, error, and tolerance when using appropriate measurement tools.

We often use numbers that are not exact. Measurements are approximate—there is no such thing as a perfect measurement. The precision of a number refers to its exactness—to the level of detail to which the tool can measure. Measurements cannot be more precise than the measuring tool. This is very important in science!

Example: To what degree of precision can you measure a length using this ruler?

![Ruler](image)

To the nearest \( \frac{1}{8} \) of an inch

To the nearest mm

The smaller the unit of measurement, the more precise the measure. Understanding this simple statement will help students to answer a type of question that often occurs on the CRT. Consider some measures of time, such as 15 seconds and 15 hours. A measure of 15 seconds implies it is precise to the nearest second, or a time interval between 14.5 and 15.5 seconds. The time of 15 hours is far less precise: it suggests a time between 14.5 and 15.5 hours. The potential error in the first interval is 0.5 seconds; the potential error in the 15 hours scenario is 0.5 hours or 1800 seconds. Because the potential for error is greater, the 15-hour-measure is less precise.

Example: Choose the more precise measurement in the given pair.

(a) 3 m, 35 km
(b) 12 inches, 1 foot
(c) 1 pound, 1 ounce

3 m is more precise (meters are smaller than km)
12 inches (inches are smaller than a foot)
1 ounce (an ounce is smaller than a pound)

The number of decimal places in a measurement can also affect precision. Using time again, a measure of 5.1 seconds is more precise than 5 seconds. The 5.1 measurement implies a measure precise to the nearest tenth of a second. The potential error in 5.1 seconds is 0.05 seconds, compared to the potential error of 0.5 seconds with the measure of 5 seconds.
Example: Choose the more precise measurement in the given pair.

(a) 5.4 m, 5.67 m  5.67 is more precise (hundredths of a meter smaller than tenths)
(b) 3 yards, 3.6 yards  3.6 is more precise (tenths of a yard more precise than yards)

This leads us to a discussion about significant digits. All the digits that are known with certainty are called significant digits. Below are the rules for determining significant digits. The only tricky digits are the zeros:

- All non-zero digits are significant digits.
  - 3 has one significant digit
  - 2.5 has two significant digits
  - 356.491 has six significant digits
- Zeros that occur between significant digits are significant digits.
  - 207 has 3 significant digits
  - 6.005 has 4 significant digits
  - 20.006 has 5 significant digits
- Zeros to the right of the decimal point AND to the right of a non-zero digit are significant digits.
  - 0.10 has 2 significant digits (the 0 before the decimal is not significant while the 0 to the right of the decimal point and the digit 1 are significant)
  - 0.0040 has 2 significant digits (just the last two)
  - 4.60 has 3 significant digits
  - 460 has 2 significant digits (zero is to the left of the decimal point)
  - 46.00 has 4 significant digits
  - 460.00 has 5 significant digits (the two zeros to the right of the decimal point are significant—this makes the zero to the left of the decimal point significant because it lies between significant digits)

Example: Determine the number of significant digits in each measurement.

(a) 32.75  4 significant digits (all nonzero digits)
(b) 43.023  5 significant digits (zero is between significant digits)
(c) 0.0240  3 significant digits (zero after the last nonzero digit and to the right of the decimal point is significant)
(d) 0.007  1 significant digit

Differing levels of precision can cause us a problem when dealing with arithmetic operations. Suppose I wish to add 11.1 seconds to 13.47 seconds. The answer, 24.57 seconds, is misleading. That is:

11.1 seconds implies the time is between 11.05 and 11.15 seconds
13.47 seconds implies the time is between 13.465 and 13.475 seconds
The sum should imply the time is between 24.515 and 24.625 seconds
But the sum 24.57 seconds implies the time is between 24.565 and 24.575, which is more precise than the actual result.
So it is generally accepted that *when you add or subtract, you report your answer to the same precision as the least precise measure.* In other words, the answer should have the same number of digits to the right of the decimal point as the measurement with the least number of digits to the right of the decimal point. In our example, we would report 24.57 as 24.6 seconds.

**Example:** Calculate. Use the correct number of significant digits.

(a) \[4.5 + 2.17 = 6.67, \quad \approx 6.7 \quad \text{Answer rounded to 1 digit to the right of the decimal (tenths)}\]

(b) \[15 - 5.6 = 9.4 \quad \approx 9 \quad \text{Answer rounded to no digits to the right of the decimal (units)}\]

Multiplying or dividing measures creates a different type of problem. For instance, I want to find the area of a rectangle that measures 2.7 cm by 4.6 cm. When I multiply, I obtain the answer 12.42 cm^2. However, 2.7 implies 2.65 cm to 2.75 cm and 4.6 implies 4.55 cm to 4.65 cm. The product should imply 12.0575 cm^2 to 12.7875 cm^2. But the product 12.42 cm^2 implies 12.415 cm^2 to 12.425 cm^2, which is more precise than the actual result.

The accepted practice when multiplying or dividing is *to report the result using the fewest number of significant digits in the original measures given.* Or, when you multiply or divide measurements, the answer should have the same number of significant digits as the measurement with the least number of significant digits. In our example, there are two significant digits in 2.7 cm and 4.6 cm, so the result is rounded to two significant digits, 12 cm^2.

**Example:** Calculate. Use the correct number of significant digits.

(a) Find the area of a parallelogram with height of 12.1 inches and base of 6 inches. \[\text{Area} = (12.1)(6), \quad \text{which is } 72.6. \quad \text{However, the lowest number of significant digits is one, so we would round the answer to 70 square inches.}\]

(b) \[14.2 \div 0.05 \quad 14.2 \div 0.05 = 284. \quad \text{However, the lowest number of significant digits is one, so we would round the answer to 300.}\]

Another concept that has to do with measurement is *accuracy.* Many people think precision and accuracy are the same thing. They are *not!* The *accuracy* of a measurement refers to *how close the measured value is to the true or accepted value.* For example, if you are in a lab and you obtain a weight measurement of 4.7 kg for an object, but the actual or known weight is 10 kg, then your measurement is not *accurate* (your measurement is not close to the accepted value). However, if you weigh the object five times, and get 4.7 kg each time, then your measurement is precise—each measurement was the same as the previous. Precision is independent of accuracy. In this case, you were very precise, but inaccurate.

Another way to explain it:
Imagine a basketball player shooting baskets. If the player shoots with *precision*, his aim will always take the ball to the same location, which may or may not be close the basket. If the player shoots with *accuracy*, his aim will always take the ball close to or into the basket. A good player will be both precise and accurate: shoot the ball the same way each time and make the basket.

If you are a soccer player, and you always hit the left goal post (instead of scoring), what can you conclude? You are precise, but not accurate!

A dartboard analogy is often used to help us understand the difference between accuracy and precision. Imagine a person throwing darts, trying to hit the bull’s eye. There are 4 scenarios:

- **Not Precise/Not Accurate**
  It is a random pattern:
  darts are not clustered and are not near bull’s eye.

- **Precise/Not Accurate**
  Darts are clustered together but did not hit the bull’s eye.

- **Not Precise/Accurate**
  Darts are not clustered together, but their “average” hit the bull’s eye.

- **Precise/Accurate**
  Darts are clustered together and their “average” position hit the bull’s eye.
It is easy to confuse precision and accuracy. The tool that you use affects both the precision and accuracy of your measurement. Measuring with a millimeter tape allows greater precision than measuring with an inch tape. Because the error using the millimeter tape should be less than the inch tape, accuracy also improves.

Suppose that a tape measure is used to measure the diameter of two circles. Let’s suppose you measure the first circle to be 15 cm, and a second circle to be 201 cm. The two measures are equally precise (both measured to the nearest cm). However, their accuracy may be quite different. Let’s further suppose that the accepted values for the measurements are 16 cm and 202 cm. The errors for these measurements are \( \frac{1}{16} = 0.0625 \) or 6.25% and \( \frac{1}{202} = 0.0049504 \) or about 0.5%. The second measurement is more accurate because the error is smaller.

One more way to think of this: accuracy implies that a measurement is basically right, given a margin of error. Precision is the level of detail; or typically the number of digits after a decimal point. For instance, I ask how far it is to the store. You could tell me “about 3 miles”, while a GPS device might tell me “2.85 miles”. About 3 miles is a pretty accurate, but 2.85 is both accurate and precise. Now, you could have told me “15.345 miles”, which would make you very precise, but not accurate.

Most of the time in our everyday life, we want accuracy; precision is not as useful. But in science and engineering, both precision and accuracy are important.


Here is a powerpoint regarding this material: [http://www.cced.net/octcomhs/math/acc_prec/accuracy_printfile.pdf](http://www.cced.net/octcomhs/math/acc_prec/accuracy_printfile.pdf)

The last concept we need to cover is that of tolerance. When you buy a bag or box of anything at the store that is sold by weight, you do not get the *exact* amount. Instead, you purchase an amount that is *about* the weight listed. If it is produce like potatoes or apples in a bag, the store sets a limit to what the bag can weigh and still be sold. This *range* of weight is the acceptable lower limit (any less and the consumer does not get enough) to the upper limit (any more and the store is giving away too much).

*Tolerance* is the greatest range of variation that can be allowed. How much error is occurring or is acceptable? *Tolerance* is expressed as the *average of the limits plus or minus one-half of range*.

We are going to limit our discussion to that of Nevada standards for grade 8: understanding tolerance when using measurement tools. Please note that the CRT will only use metric units and that the ± symbol will not be used.
**Error in measurement** may be represented by a **tolerance interval** (margin of error). Machines used in manufacturing often set tolerance intervals, or ranges in which product measurements will be tolerated or accepted before they are considered flawed.

To determine the **tolerance interval** in a measurement, add and subtract one-half of the precision of the measuring instrument to the measurement. Any measurements within this range are "tolerated" or perceived as correct.

**Example:** A measurement of 5.6 cm is made with a metric ruler that has a precision of 0.1 cm. Find the tolerance interval.

One-half of the precision is \( \frac{1}{2}(0.1) \) or 0.05. So the tolerance interval in this measurement is 5.6 ± 0.05 cm, or from 5.55 cm to 5.65 cm.

**Example:** The weights of 12 sacks of potatoes range from 14.75 pounds to 15.15 pounds. If \( P \) is the weight, in pounds, of one of these sacks, which of the following must be true?

A. \( |P – 14.75| \leq 0.2 \)
B. \( |P – 14.95| \leq 0.2 \)
C. \( |P + 14.95| \leq 0.2 \)
D. \( |P – 0.2| \leq 0.2 \)
E. \( |P – 12| \leq 0.2 \)

To find the range, take the upper limit (larger measurement, here: 15.15) and subtract the lower limit (smaller measurement, here: 14.75).

Range \( \rightarrow \) 15.15 – 14.75 = 0.4

The tolerance is expressed as the average of the limits plus or minus one-half of range.

Average of the limits \( \rightarrow \frac{15.15 + 14.75}{2} = 14.95 \)

Expressed in algebra, using absolute value, an individual potato sack would be:

\( |P – (\text{average of the limits})| \leq \text{range}/2 \)

So our solution is: B. \( |P – 14.95| \leq 0.2 \)

The tolerance is 14.95 ± 0.2 pounds, or from 14.75 to 15.15 pounds as stated in the problem.