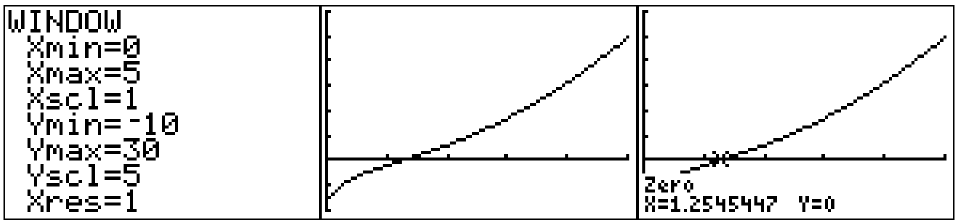
Syllabus Objective: 3.8 – The student will solve problems using the properties of definite integrals (integral as net change).

Calculating Position Given Velocity Function

Fundamental Theorem of Calculus: $\int_0^a v(t)\,dt = s(a) - s(0) \Rightarrow \boxed{s(a) = \int_0^a v(t)\,dt + s(0)}$

Ex1: $\dfrac{ds}{dt} = v(t) = t^2 - \dfrac{8}{(t+1)^2}\ \text{cm}/\text{sec},\ 0 \le t \le 5$

1. GRAPH and describe the motion.



The particle is moving left (or down) from $0 \le t < 1.25$ and right (or up) from $1.25 < t \le 5$.

2. Suppose $s(0) = 9$. Find $s(1)$ and $s(5)$.

Fundamental Theorem: $\int_0^1 v(t)\,dt = s(1) - s(0) \Rightarrow s(1) = \int_0^1 v(t)\,dt + s(0)$

$$s(1) = \int_0^1 t^2 - \frac{8}{(t+1)^2}\,dt + 9 = \left.\frac{t^3}{3} + \frac{8}{t+1}\right]_0^1 + 9 = \left(\frac{1}{3} + 4\right) - (0+8) + 9 = -\frac{11}{3} + 9 = \boxed{\frac{16}{3}}$$

$$s(5) = \int_0^5 t^2 - \frac{8}{(t+1)^2}\,dt + 9 = \left.\frac{t^3}{3} + \frac{8}{t+1}\right]_0^5 + 9 = \left(\frac{125}{3} + \frac{4}{3}\right) - (0+8) + 9 = 35 + 9 = \boxed{44}$$

DISPLACEMENT = RATE OF CHANGE x TIME $\int_0^t v(t)\,dt$

Ex2: Calculate the displacement of the particle from the example above for the time interval $0 \le t \le 5$.

Displacement: $\int_0^5 v(t)\,dt = \int_0^5 t^2 - \dfrac{8}{(t+1)^2}\,dt = \boxed{35}$

<u>TOTAL DISTANCE TRAVELED</u>: $\int_0^t |v(t)|dt$

Ex3: Calculate the total distance the particle traveled for the time interval $0 \le t \le 5$. $v(t)$ is negative on the interval $0 \le t < 1.25$ and positive on the interval $1.25 < t \le 5$.

So, the total distance traveled = $\int_0^5 \left| t^2 - \frac{8}{(t+1)^2} \right| dt = -\int_0^{1.25} t^2 - \frac{8}{(t+1)^2} dt + \int_{1.25}^5 t^2 - \frac{8}{(t+1)^2} dt$

$$\int_0^{1.25} t^2 - \frac{8}{(t+1)^2} dt = \frac{t^3}{3} + \frac{8}{t+1} \Big]_0^{1.25} \approx 4.206 - 8 = -3.793$$

$$\int_{1.25}^5 t^2 - \frac{8}{(t+1)^2} dt = \frac{t^3}{3} + \frac{8}{t+1} \Big]_{1.25}^5 \approx 43 - 4.206 = 38.794$$

$$\int_0^5 \left| t^2 - \frac{8}{(t+1)^2} \right| dt = -\int_0^{1.25} t^2 - \frac{8}{(t+1)^2} dt + \int_{1.25}^5 t^2 - \frac{8}{(t+1)^2} dt = 3.793 + 38.794 = \boxed{42.587}$$

Ex4: A car moving with initial velocity of 15 mph accelerates at the rate of $a(t) = 1.8t$ mph per second for 10 seconds.

1. How fast is the car going when the 10 seconds are up?

Fundamental Theorem: $\int_0^{10} a(t)dt = v(10) - v(0) \Rightarrow v(10) = \int_0^{10} a(t)dt + v(0)$

$$v(10) = \int_0^{10} 1.8t dt + 15 = 0.9t^2 \Big]_0^{10} + 15 = 0.9(10)^2 - 0.9(0)^2 + 15 = \boxed{105 \text{ mph}}$$

Check the units: $\int a(t)dt \Rightarrow \frac{\text{mph}}{\text{sec}} \cdot \text{sec} = \text{mph}$

2. How far did the car travel during the 10 seconds?

Total Distance Traveled: $\int_0^{10} |v(t)|dt = \int_0^{10} |0.9t^2| dt = 0.3t^3 \Big]_0^{10} = 300$ miles

Check the units: $\int v(t)dt \Rightarrow \frac{\text{mi}}{\text{hr}} \cdot \text{sec} \ne \text{mi}$ $\qquad$ $1 \text{ sec} = \frac{1}{60}\text{hr}$, so $d = 300 \cdot \frac{1}{60} = \boxed{5 \text{ mi}}$

Accumulation Over Time: given a varying rate of change, the total accumulation is the integral of the rate of change

Accumulation given the rate function $r(t)$ over an interval $[a,b]$: $\int_a^b r(t)\,dt$

"Given a rate, INTEGRATE!" ☺

Ex5: If, *t* days into the surf season, the water temperature at your favorite surf spot is changing at a rate of $0.05t^2+4$ degrees per day, then what is the total change in the water temperature from the start of the season $(t=0)$ to the start of day 10?

Total Change in Water Temperature: $\int_0^{10}\left(0.05t^2+4\right)dt=\frac{170}{3}\approx\boxed{56.667^\circ}$

Using a Table of Values

Ex6: A pump connected to a generator operates at a varying rate, depending on how much power is being drawn from the generator to operate other machinery. The rate (gallons per minute) at which the pump operates is recorded at 5-minute intervals for one hour as shown in the table. How many gallons were pumped during that hour?

Time (min)	Rate (gal/min)
0	58
5	60
10	65
15	64
20	58
25	57
30	55
35	55
40	59
45	60
50	60
55	63
60	63

Total Gallons Pumped = $\int_0^{60} R(t)\,dt$

Use the trapezoidal rule to approximate the integral:

$$\int_0^{60} R(t)\,dt\approx\frac{1}{2}(5)\left(58+2(60)+2(65)+\ldots+2(60)+2(63)+63\right)=3582.5\approx\boxed{3582\text{ gallons}}$$

You Try: Over the period 1990-2000, the number of pagers in use in the United States was changing at a rate of approximately $-0.40t^2+3t+0.75$ million pagers per year, where *t* is time in years since 1990. In 1995 approximately 32 million pagers were in use. How many were in use in 2000?

QOD: What is the difference between position, displacement, and distance traveled?

Sample AP Calculus AB Exam Question(s):

1. The rate of change of the altitude of a hot-air balloon is given by $r(t) = t^3 - 4t^2 + 6$ for $0 \le t \le 8$. Which of the following expressions gives the change in altitude of the balloon during the time the altitude is decreasing?

   (A) $\int_{1.572}^{3.514} r(t)\,dt$

   (B) $\int_{0}^{8} r(t)\,dt$

   (C) $\int_{0}^{2.667} r(t)\,dt$

   (D) $\int_{1.572}^{3.514} r'(t)\,dt$

   (E) $\int_{0}^{2.667} r'(t)\,dt$

2. A particle moves along the *x*-axis so that at any time $t > 0$, its acceleration is given by $a(t) = \ln\left(1 + 2^t\right)$. If the velocity of the particle is 2 at time $t = 1$, then the velocity of the particle at time $t = 2$ is

(A) 0.462 (B) 1.609 (C) 2.555 (D) 2.886 (E) 3.346

Sample AP Free Response Question (2009):

The rate at which people enter an auditorium for a rock concert is modeled by the function *R* given by $R(t) = 1380t^2 - 675t^3$ for $0 \le t \le 2$ hours; $R(t)$ is measured in people per hour. No one is in the auditorium at time $t = 0$, when the doors open. The doors close and the concert begins at time $t = 2$.

(a) How many people are in the auditorium when the concert begins?

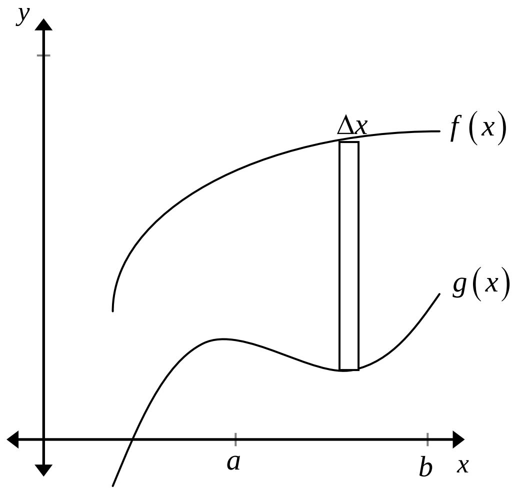
(b) Find the time when the rate at which people enter the auditorium is a maximum. Justify your answer.

(c) The total wait time for all the people in the auditorium is found by adding the time each person waits, starting at the time the person enters the auditorium and ending when the concert begins. The function *w* models the total wait time for all the people who enter the auditorium before time *t*. The derivative of *w* is given by $w'(t) = (2 - t)R(t)$. Find $w(2) - w(1)$, the total wait time for those who enter the auditorium after time $t = 1$.

(d) On average, how long does a person wait in the auditorium for the concert to begin? Consider all people who enter the auditorium after the doors open, and use the model for total wait time from part (c).

Syllabus Objective: 3.11 – The student will find the area between two or more curves.

<u>Area Between Two Curves:</u>



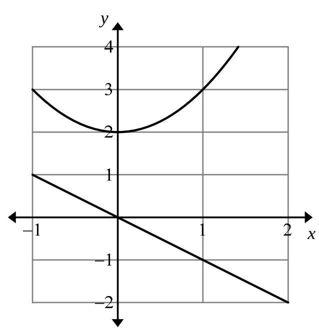
Rectangle Area = $\left[f(c_k) - g(c_k)\right]\Delta x$ Area of Region (Riemann Sum): $\sum\left[f(c_k) - g(c_k)\right]\Delta x$

Limit as $\Delta x \to 0$: $\int_a^b \left[f(x) - g(x)\right]dx$

<u>Area Between Two Curves</u>: Let *f* and *g* be continuous functions, with $f(x) \le g(x)$; $A = \int_a^b \left[f(x) - g(x)\right]dx$

Ex1: Find the area between the curves $y = x^2 + 2,\ y = -x,\ x = 0,$ and $x = 1$.

Graph: Upper Curve: $f(x) = x^2 + 2$ Lower Curve: $g(x) = -x$



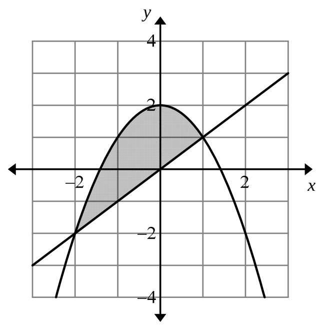
$A = \int_a^b \left[f(x) - g(x)\right]dx$:

$$A = \int_0^1 \left[\left(x^2 + 2\right) - (-x)\right]dx = \int_0^1 \left(x^2 + x + 2\right)dx = \left.\frac{x^3}{3} + \frac{x^2}{2} + 2x\right]_0^1 = \left(\frac{1}{3} + \frac{1}{2} + 2\right) - (0) = \boxed{\frac{17}{6}}$$

<u>Area of a Region Between Intersecting Curves</u>: you must calculate the intersection points

Ex2: Find the area of the region bounded by the graphs of $f(x) = 2 - x^2$ and $g(x) = x$.

Graph: Upper Curve: $f(x) = 2 - x^2$ Lower Curve: $g(x) = x$



Points of Intersection: $2 - x^2 = x \Rightarrow x^2 + x - 2 = 0 \Rightarrow (x+2)(x-1) = 0 \Rightarrow x = -2, 1$

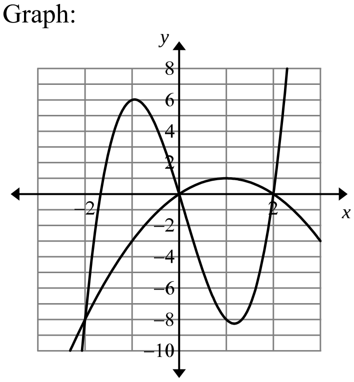
Area: $A = \int_{-2}^{1} \left[\left(2 - x^2\right) - (x) \right] dx = \int_{-2}^{1} \left(2 - x^2 - x\right) dx = 2x - \frac{x^3}{3} - \frac{x^2}{2} \Big]_{-2}^{1} = \left(2 - \frac{1}{3} - \frac{1}{2}\right) - \left(-4 + \frac{8}{3} - 2\right) = \boxed{\frac{9}{2}}$

<u>Curves That Intersect at More Than Two Points</u>

Watch out for the upper curve and lower curve switching places!

Ex3: Find the area of the region between the graphs of $f(x) = 3x^3 - x^2 - 10x$ and $g(x) = -x^2 + 2x$.

Graph:



Points of Intersection: $3x^3 - x^2 - 10x = -x^2 + 2x \Rightarrow 3x^3 - 12x = 0 \Rightarrow 3x\left(x^2 - 4\right) = 0 \Rightarrow x = 0, -2, 2$

On the interval $[-2, 0]$: Upper Curve: $f(x) = 3x^3 - x^2 - 10x$ Lower Curve: $g(x) = -x^2 + 2x$

On the interval $[0, 2]$: Upper Curve: $g(x) = -x^2 + 2x$ Lower Curve: $f(x) = 3x^3 - x^2 - 10x$

$$A = \int_{-2}^{0} \left[\left(3x^3 - x^2 - 10x\right) - \left(-x^2 + 2x\right) \right] dx + \int_{0}^{2} \left[\left(-x^2 + 2x\right) - \left(3x^3 - x^2 - 10x\right) \right] dx$$

Area: $= \int_{-2}^{0} \left(3x^3 - 12x\right) dx + \int_{0}^{2} \left(-3x^3 + 12x\right) dx = \frac{3}{4}x^4 - 6x^2 \Big]_{-2}^{0} + \left[-\frac{3}{4}x^4 + 6x^2 \right]_{0}^{2}$

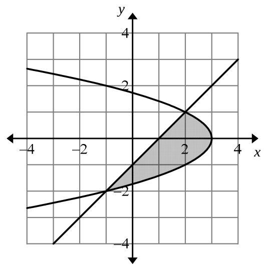
$= \left[0 - (12 - 24)\right] + \left[(-12 + 24) - 0\right] = \boxed{24}$

Using Horizontal Rectangles: Use the equations of the curves in terms of y.

Note: Everything must be in terms of y, including the limits of integration.
Instead of integrating Upper Curve – Lower Curve, integrate Right Curve – Left Curve.

Ex4: Find the area of the region bounded by the graphs of $x = 3 - y^2$ and $x = y + 1$.

Graph: Right Curve: $x = 3 - y^2$ Left Curve: $x = y + 1$



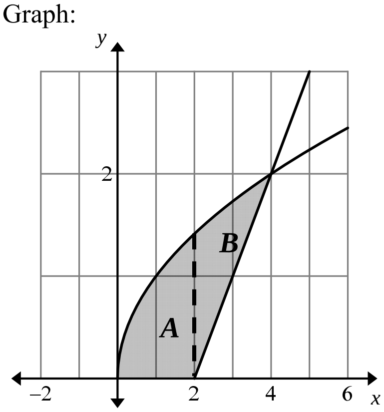
Points of Intersection: $3 - y^2 = y + 1 \Rightarrow y^2 + y - 2 = 0 \Rightarrow (y+2)(y-1) = 0 \Rightarrow y = -2, 1$

Area: $\int_{-2}^{1} \left[\left(3 - y^2\right) - \left(y + 1\right) \right] dy = \int_{-2}^{1} \left(2 - y^2 - y\right) dy = 2y - \frac{y^3}{3} - \frac{y^2}{2} \Big]_{-2}^{1} = \left(2 - \frac{1}{3} - \frac{1}{2}\right) - \left(-4 + \frac{8}{3} - 2\right) = \boxed{\frac{9}{2}}$

Using Subregions

Ex5: Find the area of the region in the first quadrant bounded above by $y = \sqrt{x}$ and below by the x-axis and the line $y = x - 2$.

Graph:



Region A: Interval $[0, 2]$; Area under the curve $y = \sqrt{x}$

Region B: Interval $[2, 4]$; Area between the curves $y = \sqrt{x}$ and $y = x - 2$

Area =

$$\int_0^2 \sqrt{x}\,dx + \int_2^4 \left[\sqrt{x} - (x - 2) \right] dx = \frac{2}{3} x^{\frac{3}{2}} \Big]_0^2 + \left[\frac{2}{3} x^{\frac{3}{2}} - \frac{x^2}{2} + 2x \right]_2^4 = \frac{2}{3} \cdot 2\sqrt{2} + \left(\frac{2}{3} \cdot 8 - 8 + 8 \right) - \left(\frac{2}{3} \cdot 2\sqrt{2} - 2 + 4 \right) = \boxed{\frac{10}{3}}$$

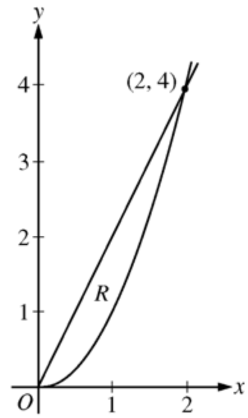
Note: This problem could also be done by writing the functions in terms of y. $A = \int_0^2 \left[(y + 2) - y^2 \right] dy$

<u>You Try:</u> The sine and cosine curves intersect infinitely many times, bounding regions of equal areas. Find the area of one of these regions.

<u>QOD:</u> How do you decide whether to integrate with respect to *x* or *y* when completing an area problem?

<u>Sample AP Free Response Question (2009):</u>

No calculator is allowed for this problem.



Let *R* be the region in the first quadrant enclosed by the graphs of $y = 2x$ and $y = x^2$, as shown in the figure above.

Find the area of *R*.

Syllabus Objectives: 3.12 – The student will find the volume of a solid of revolution. 3.13 – The student will find the volume of a solid with known cross sections.

Teacher Note: To help students visualize the 3-dimensional shapes formed by revolving a region, I recommend using Calculus in Motion. This is a collection of pre-made activities for the Geometer's Sketchpad. It can be purchased at www.calculusinmotion.com

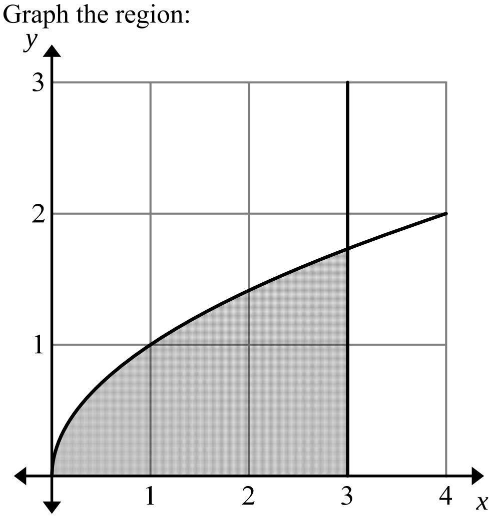
Disk Method: Volume of a Solid of Revolution

Horizontal Axis of Revolution: A disk is a circle. To find the volume, calculate the sum of an infinite number of disks (circles) using an integral. $V=\pi\int_a^b\left[R(x)\right]^2dx$, where $R(x)$ is the radius of one disk, represented by a function.

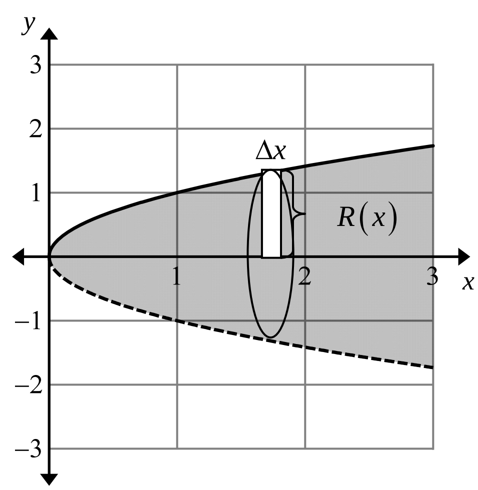
Revolving About the *x*-Axis

Ex1: Find the volume of the solid formed by revolving the region bounded by the graph of $y=\sqrt{x}$, the *x*-axis, and the line $x=3$ about the *x*-axis.

Graph the region:



Revolve about the *x*-axis and sketch one disk (circle):



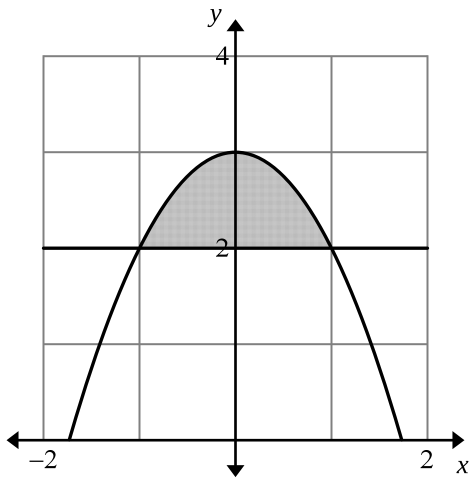
Write an expression for $R(x)$: $R(x)=\sqrt{x}$

Calculate the volume using the Disk Method: $V=\pi\int_0^3\left(\sqrt{x}\right)^2dx=\pi\int_0^3 xdx=\pi\left[\frac{x^2}{2}\right]_0^3=\pi\left(\frac{9}{2}-0\right)=\boxed{\frac{9\pi}{2}}$

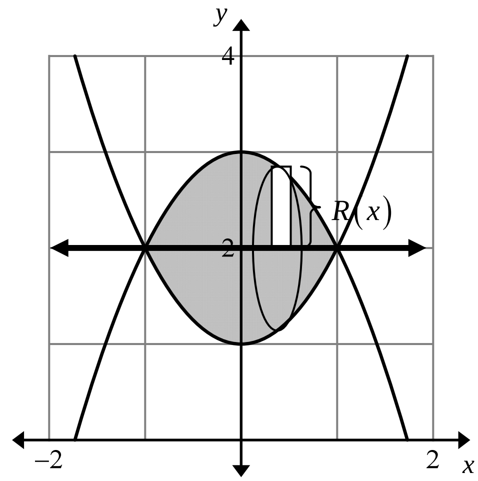
Revolving About a Horizontal Line (not the *x*-axis)

Ex2: Find the volume of the solid formed by revolving the region bounded by $f(x)=3-x^2$ and $g(x)=2$ about the line $y=2$.

Graph the region:



Revolve about the line $y=2$ and sketch one disk (circle):



Write an expression for $R(x)$: $R(x)=f(x)-2=\left(3-x^2\right)-2=1-x^2$

Calculate the volume using the Disk Method:

$$V=\pi\int_{-1}^{1}\left(1-x^2\right)^2dx=\pi\int_{-1}^{1}\left(1-2x^2+x^4\right)dx=\pi\left[x-\frac{2x^3}{3}+\frac{x^5}{5}\right]_{-1}^{1}=\pi\left[\left(1-\frac{2}{3}+\frac{1}{5}\right)-\left(-1+\frac{2}{3}-\frac{1}{5}\right)\right]=\boxed{\frac{16\pi}{15}}$$

Note: Because of the symmetry of the graph, we could have used $V=2\pi\int_{0}^{1}\left(1-x^2\right)^2dx$.

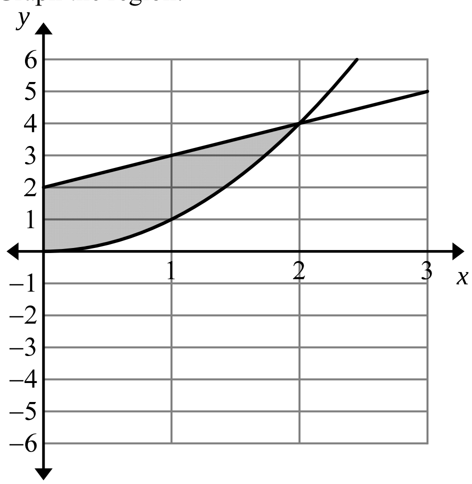
The Washer Method: used when there is a "hole" in the disk

Area of a Washer = $\pi\left[(\text{outer radius})^2 - (\text{inner radius})^2\right] = \pi\left[\left[R(x)\right]^2 - \left[r(x)\right]^2\right]$

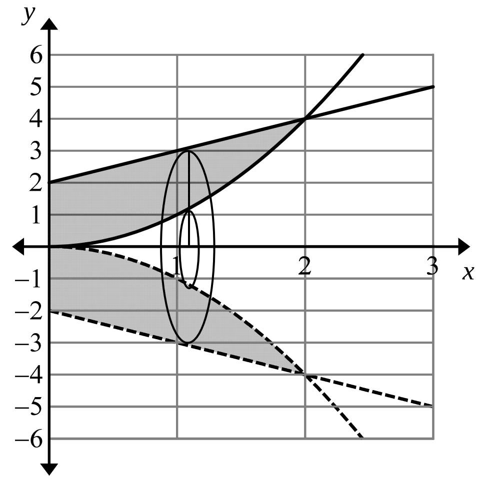
Volume of a Solid of Revolution: Washer Method: $V = \pi\int_a^b \left[\left[R(x)\right]^2 - \left[r(x)\right]^2\right]dx$

Ex: Find the volume of the solid formed by revolving the region bounded by the graphs of $y = x+2,\ y = x^2,$ and $x = 0$ about the *x*-axis.

Graph the region:



Revolve about the *x*-axis and sketch one washer:



Write an expression for $R(x)$ and $r(x)$: $R(x) = x+2,\ r(x) = x^2$

Calculate the volume using the washer method:

$$V = \pi\int_0^2 \left[\left[R(x)\right]^2 - \left[r(x)\right]^2\right]dx = \pi\int_0^2 \left[(x+2)^2 - \left(x^2\right)^2\right]dx = \pi\int_0^2 \left(x^2 + 4x + 4 - x^4\right)dx$$

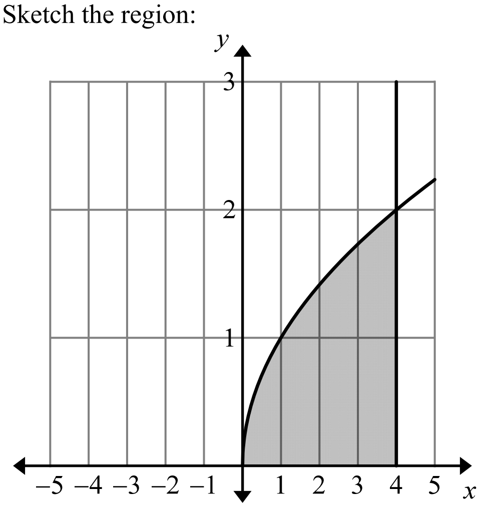
$$= \pi\left[\frac{x^3}{3} + 2x^2 + 4x - \frac{x^5}{5}\right]_0^2 = \pi\left(\frac{8}{3} + 8 + 8 - \frac{32}{5} - 0\right) = \boxed{\frac{184\pi}{15}}$$

Vertical Axis of Revolution: A disk is a circle. To find the volume, calculate the sum of an infinite number of disks (circles) using an integral. $V=\pi\int_a^b\left[R(y)\right]^2dy$, where $R(y)$ is the radius of one disk, represented by an equation in terms of *y*.

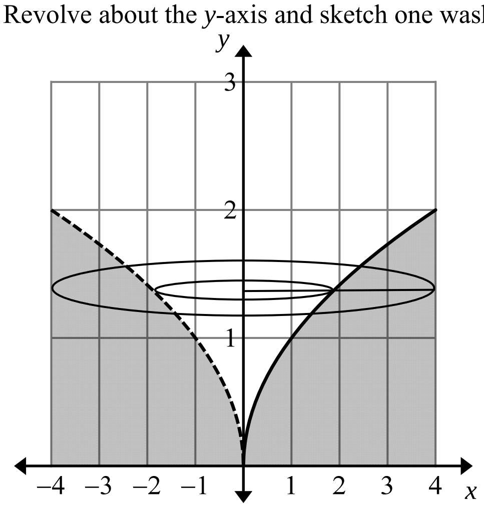
Revolving About the *y*-Axis

Ex: Find the volume of the solid generated by revolving the region bounded by the graphs of $y=\sqrt{x}$, $y=0$, and $x=4$ about the *y*-axis.

Sketch the region:



Revolve about the *y*-axis and sketch one washer.



Write an expression for $R(y)$ and $r(y)$: $R(y)=4,\ r(y)=y^2$ (Solve for *x* in terms of *y*.)
Calculate the volume using the washer method:

Caution: Be sure to use *y*-coordinates as the limits of integration!

$$V=\pi\int_0^2\left[\left[R(y)\right]^2-\left[r(y)\right]^2\right]dy=\pi\int_0^2\left[(4)^2-\left(y^2\right)^2\right]dy=\pi\int_0^2\left(16-y^4\right)dy=\pi\left[16y-\frac{y^5}{5}\right]_0^2=\pi\left(32-\frac{32}{5}-0\right)=\boxed{\frac{128\pi}{5}}$$

You Try: Let *R* be the region in the first and second quadrants bounded above by the graph of $y=\dfrac{20}{1+x^2}$ and below by the horizontal line $y=2$.

a) Find the volume of the solid generated when *R* is rotated about the *x*-axis.

b) The region *R* is the base of a solid. For this solid, the cross sections perpendicular to the *x*-axis are semicircles. Find the volume of this solid.

QOD: Explain how to determine whether to use the disk or washer method when calculating a volume of a solid of revolution.

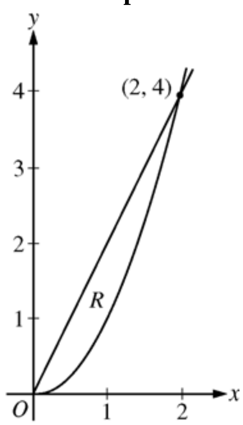
Sample AP Calculus AB Exam Question(s):


The base of a solid is the region in the first quadrant bounded by the y-axis, the graph of $y = \tan^{-1} x$, the horizontal line $y = 3$, and the vertical line $x = 1$. For this solid, each cross section perpendicular to the *x*-axis is a square. What is the volume of the solid?

(A) 2.561
(B) 6.612
(C) 8.046
(D) 8.755
(E) 20.773

Sample AP Free Response Question (2009):

No calculator is allowed for this problem.



Let *R* be the region in the first quadrant enclosed by the graphs of $y = 2x$ and $y = x^2$, as shown in the figure above.

(a) The region *R* is the base of a solid. For this solid, at each *x* the cross section perpendicular to the *x*-axis has area $A(x) = \sin\left(\frac{\pi}{2}x\right)$. Find the volume of the solid.

(b) Another solid has the same base *R*. For this solid, the cross sections perpendicular to the *y*-axis are squares. Write, but do not evaluate, an integral expression for the volume of the solid.