Syllabus Objectives: 12.1 – The student will understand and apply the concept of the limit of a function at given values of the domain. 12.2 – The student will find the limit of a function at given values of the domain.

Limit – how the outputs of a function behave as the inputs **approach** some value

Notation: $L = \lim_{x \to c} f(x)$ "The limit (*L*) as *x* approaches *c* of $f(x)$."

Finding a Limit

I. Table

Ex: $\lim_{x \to 0} \frac{\sin x}{x}$

Use the table to choose values of *x* close to zero (from the left and right).

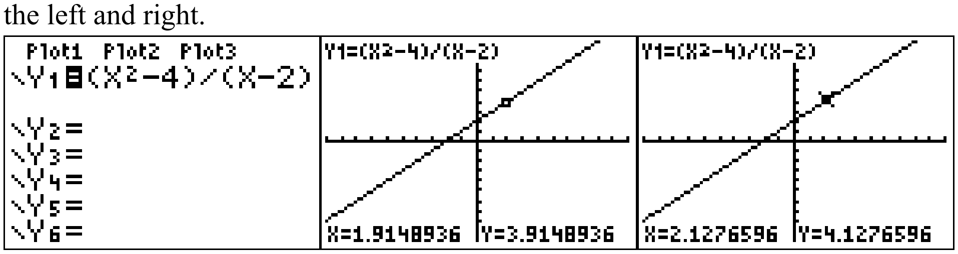
As *x* approaches 0, it appears the function is approaching 1.

$\lim_{x \to 0} \frac{\sin x}{x} = \boxed{1}$

II. Graphically

Ex: $\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$

Graph the function and use the TRACE feature to see function values as we approach 2 from the left and right.



As *x* approaches 2, it appears the function is approaching 4.

$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \boxed{4}$

III. Analytically

a. Direct Substitution

Ex: $\lim_{x \to 2} \frac{x^2 - 2x + 8}{x + 2}$

Substitute $x = 2$ into the function: $\lim_{x \to 2} \frac{x^2 - 2x + 8}{x + 2} = \frac{(2)^2 - 2(2) + 8}{2 + 2} = \boxed{2}$

b. Algebraic Manipulation

Ex: $\lim_{x\to -3} \frac{9-x^2}{x+3}$

We cannot use direct substitution, because of division by zero. So we must simplify the function algebraically.

$$\lim_{x\to -3} \frac{9-x^2}{x+3} = \lim_{x\to -3} \frac{(3-x)\cancel{(3+x)}}{\cancel{(x+3)}} = \lim_{x\to -3}(3-x) = 3-(-3) = \boxed{6}$$

Nonexistent Limits

Ex: $\lim_{x\to -3} \frac{x^3-1}{x+3}$

We cannot use direct substitution, because of division by zero. Factoring the numerator, we have:

$\lim_{x\to -3} \frac{x^3-1}{x+3} = \lim_{x\to -3} \frac{(x-1)(x^2+x+1)}{x+3}$ This does not simplify to avoid division by zero.

So, $\lim_{x\to -3} \frac{x^3-1}{x+3}$ does not exist.

One-sided Limits

1. Right-Hand Limit: the limit of *f* as *x* approaches *c* from the right. $\lim_{x\to c^+} f(x)$

Ex: Find $\lim_{x\to 2^+} f(x)$, when $f(x) = \begin{cases} x^2 & x \le 2 \\ 3x-1 & x > 2 \end{cases}$

Because we are approaching from the right, we must substitute into the piece of the piecewise functions that represents values of *x* greater than 2.

$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} 3x-1 = 3(2)-1 = \boxed{5}$ Note: The limit value differs from the function value at 2.

2. Left-Hand Limit: the limit of *f* as *x* approaches *c* from the left. $\lim_{x\to c^-} f(x)$

Ex: $\lim_{x\to 2^-} \sqrt{x-2}$

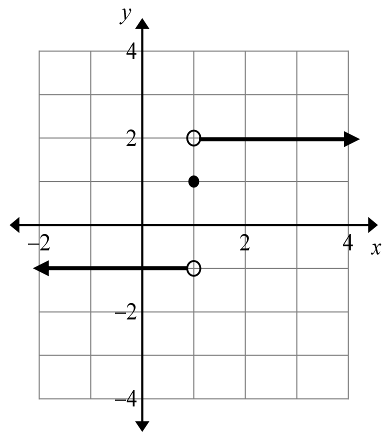
We cannot use direct substitution, because we cannot approach 2 from the left. The domain of the function $f(x) = \sqrt{x-2}$ is $[2,\infty)$. So $\lim_{x\to 2^-} \sqrt{x-2}$ does not exist.

Two-sided Limit

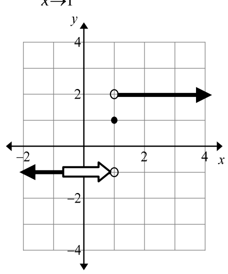
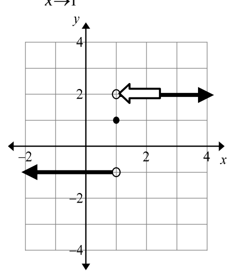
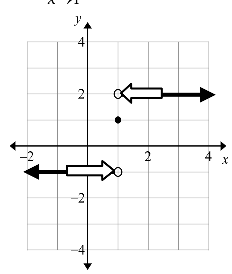
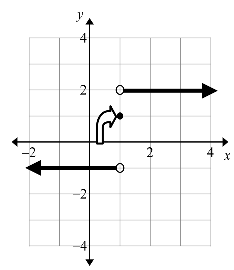
$$\lim_{x\to c} f(x)$$

** $f(x)$ has a limit as *x* approaches *c* if and only if the right and left hand limits at *c* exist and are equal**

Ex: Use the graph below of $f(x)$ to answer the following questions.



a. $\lim_{x \to 1^-} f(x)$ b. $\lim_{x \to 1^+} f(x)$ c. $\lim_{x \to 1} f(x)$ d. $f(1)$

$\lim_{x \to 1^-} f(x) = \boxed{-1}$ $\lim_{x \to 1^+} f(x) = \boxed{2}$ $\lim_{x \to 1} f(x) = \boxed{\text{does not exist}}$ $f(1) = \boxed{1}$

PROPERTIES OF LIMITS:

If $L, M, c,$ and k are real numbers and
$\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$, then

1. Sum and Difference Rules: $\lim_{x \to c}(f(x) \pm g(x)) = L \pm M$

The limit of the sum or difference of two functions is the sum or difference of their limits.

2. Product Rule: $\lim_{x \to c}(f(x) \cdot g(x)) = L \cdot M$

The limit of a product of two functions is the product of their limits.

3. Constant Multiple Rule: $\lim_{x \to c}(k \cdot f(x)) = k \cdot L$

The limit of a constant times a function is the constant times the limit of the function.

4. Quotient Rule: $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M}, M \neq 0$

The limit of a quotient of two functions is the quotient of their limits, provided the limit of the denominator is not zero.

5. Power Rule: If r and s are integers, $s \neq 0$, then

$\lim_{x \to c}(f(x))^{r/s} = L^{r/s}$ provided that $L^{r/s}$ is a real number.

Ex: Find $\lim_{x\to 0}\frac{\tan x}{x}$.

Using direct substitution, we divide by zero. So try algebraic manipulation. We can rewrite $\tan x$ as $\frac{\sin x}{\cos x}$.

$$\lim_{x\to 0}\frac{\tan x}{x}=\lim_{x\to 0}\frac{\frac{\sin x}{\cos x}}{x}=\lim_{x\to 0}\frac{\sin x}{x\cos x}=\lim_{x\to 0}\frac{\sin x}{x}\cdot\frac{1}{\cos x}$$

Using the product rule, $\lim_{x\to 0}\frac{\sin x}{x}\cdot\frac{1}{\cos x}=\lim_{x\to 0}\frac{\sin x}{x}\cdot\lim_{x\to 0}\frac{1}{\cos x}=1\cdot\frac{1}{1}=\boxed{1}$

<u>You Try:</u> Calculate the limit. $\lim_{x\to 2}\frac{t^2-3t+2}{t^2-4}$

<u>QOD:</u> Explain how left and right-hand limits relate to two-sided limits.

Syllabus Objectives: 12.3 – The student will understand, interpret and apply the concept of the difference quotient between two points on a curve as the average rate of change of the function over a given interval of the domain. 12.4 – The student will understand, interpret and apply the concept of the tangent line at a given point on a curve as the instantaneous rate of change of the function at that point.

Average Velocity: $\dfrac{\text{change in position}}{\text{change in time}}$ or $v_a = \dfrac{\Delta s}{\Delta t}$

Ex: What is the average velocity of a racecar that drives a quarter mile in 4.5 seconds?

Average velocity: $v_a = \dfrac{\Delta s}{\Delta t} = \dfrac{0.25 \text{ mi}}{4.5 \text{ sec}} = \dfrac{1}{18}\dfrac{\text{mi}}{\cancel{\text{sec}}} \cdot \dfrac{60}{1}\dfrac{\cancel{\text{sec}}}{\cancel{\text{min}}} \cdot \dfrac{60}{1}\dfrac{\cancel{\text{min}}}{\text{hr}} = \dfrac{3600}{18}\dfrac{\text{mi}}{\text{hr}} = \boxed{200 \text{ mph}}$

Discuss: What was the car's velocity at exactly 4 seconds?

More information would be needed to determine the answer. This is called **instantaneous velocity**.

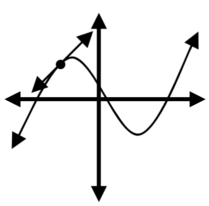
Slope: $\dfrac{\text{change in } y}{\text{change in } x} = \dfrac{\Delta y}{\Delta x}$

Discuss: Does a parabola have a slope? Graph $y = x^2$ and zoom in on the point $(2,4)$.

Yes, a parabola does have a slope. As we zoom in on the point, the graph appears to be a line.

Rate of Change

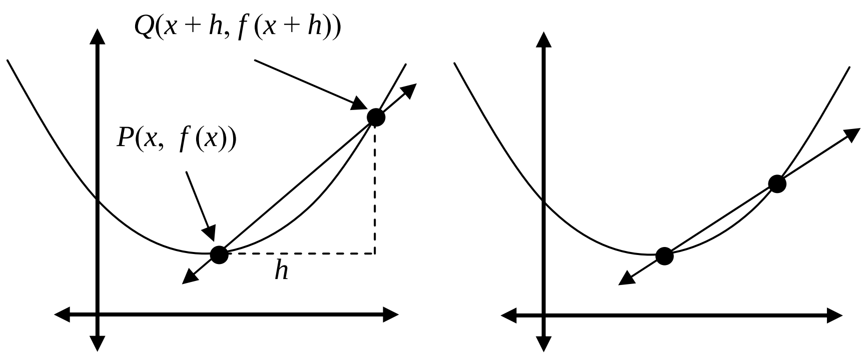
- In linear equations, the rate of change is constant (the slope)
- The slope of the tangent line at a point gives us the rate of change at that instant, or the slope of the curve at that point.



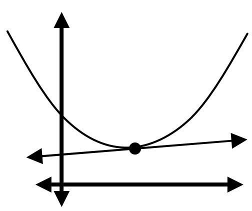
The line shown is a tangent line.

Pierre Fermat (1629)

1. Start with the slope of a secant through $P(x, f(x))$ and a point $Q(x+h, f(x+h))$ nearby.
2. Find the limiting value of the secant slope as *Q* approaches *P*.
3. This is the slope of the curve at *P* and the slope of the tangent line to the curve at *P*.



<u>Slope of the Secant Line</u>: $m_{\text{sec}} = \dfrac{\Delta y}{\Delta x} = \dfrac{f(x+h) - f(x)}{(x+h) - x} = \dfrac{f(x+h) - f(x)}{h}$



Let $P \to Q$, then $h \to 0$.

<u>Slope of the Tangent Line</u>: $\boxed{m_{\tan} = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}}$

<u>Tangent to a Curve</u> – the line through *P* with the slope as calculated above

<u>Slope of the Curve:</u> The slope of $y = f(x)$ at the point $P(a, f(a))$ is $m = \lim\limits_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$

Note: As *h* approaches 0, the two points approach one point. The slope of the curve at point *P* is the same as the slope of the tangent line at point *P*.

Ex: Find the slope of $f(x) = 10x - 2x^2$ at $(3, 12)$.

$f(x) = 10x - 2x^2 \qquad f(x+h) = 10(x+h) - 2(x+h)^2$

$$m = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{10(x+h) - 2(x+h)^2 - (10x - 2x^2)}{h} = \lim_{h \to 0} \frac{10x + 10h - 2x^2 - 4x - 2h^2 - 10x + 2x^2}{h}$$

$$= \lim_{h \to 0} \frac{10h - 4xh - 2h^2}{h} = \lim_{h \to 0} \frac{h(10 - 4x - 2h)}{h} = \lim_{h \to 0} (10 - 4x - 2h) = 10 - 4x$$

At $(3, 12)$: $m = 10 - 4x = 10 - 4(3) = \boxed{-2}$

Alternate Method:

Use $m = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$ with $a=3$: $f(3)=10(3)-2(3)^2=12$ $\quad f(3+h)=10(3+h)-2(3+h)^2$

$$\lim_{h\to 0}\frac{10(3+h)-2(3+h)^2-12}{h}=\lim_{h\to 0}\frac{30+10h-18-12h-2h^2-12}{h}=\lim_{h\to 0}\frac{-2h-2h^2}{h}=\lim_{h\to 0}\frac{h(-2-2h)}{h}$$
$$=\lim_{h\to 0}(-2-2h)=\boxed{-2}$$

<u>Derivative</u>: The **derivative** of a function *f* with respect to *x*: $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

Note: $f'(x)$ is a function. (Read "*f* prime of *x*".) Derivative is SLOPE.

<u>Notation</u>: Derivative of *f* at *x* = $f'(x),\ y',\ \frac{dy}{dx},\ D_x[y],\ \frac{d}{dx}[f(x)]$

Ex: Find the derivative of $f(x)=2x^2$.

$$f'(x)=\lim_{h\to 0}\frac{2(x+h)^2-2x^2}{h}=\lim_{h\to 0}\frac{2x^2+4xh+2h^2-2x^2}{h}=\lim_{h\to 0}\frac{4xh+2h^2}{h}=\lim_{h\to 0}4x+2h=\boxed{4x}$$

<u>Calculating a Derivative at *a*</u>: $f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$

Ex: Find $f'(3)$ if $f(x)=2x+4$.

$$f'(3)=\lim_{h\to 0}\frac{f(3+h)-f(3)}{h}=\lim_{h\to 0}\frac{2(3+h)+4-(2(3)+4)}{h}=\lim_{h\to 0}\frac{6+2h+4-10}{h}=\lim_{h\to 0}\frac{2h}{h}=\boxed{2}$$

<u>Differentiability</u>

When $f'(a)$ does not exist:

a. **Corner** – one-sided derivatives differ

Ex: $y=|x|$ at $x=0$

The left-hand derivative at $x=0$ is −1, and the right-hand derivative at $x=0$ is 1.

b. **Cusp** – the slopes of the secant lines approach ∞ from one side and $-\infty$ from the other

Ex: $y=x^{\frac{2}{3}}$ at $x=0$

The slopes of the secant lines approach $-\infty$ from the left of $x=0$, and they approach ∞ from the right of $x=0$.

c. **Vertical Tangent** – the slopes of the secant lines approach either ∞ or $-\infty$ from both sides

Ex: $y = \sqrt[3]{x}$ at $x = 0$

The slopes of the secant lines approach ∞ from both sides.

d. **Discontinuity** – the function has a discontinuity

Ex: $f(x) = \begin{cases} 1 & x \le 0 \\ -1 & x > 0 \end{cases}$ at $x = 0$

The function has a jump discontinuity at $x = 0$.

<u>Teacher Note:</u> Have students graph the functions above to get a visual of each type.

<u>You Try</u>: Find the formula for the slope of $f(x) = \dfrac{1}{x+4}$ and then use it to find the slopes at $(0, 0.25)$ and $(-2, 0.5)$.

<u>QOD</u>: A differentiable function is always continuous. Is the converse to this statement true? Explain.

Syllabus Objectives: 12.6 – The student will understand, interpret, and apply the concept of area under a curve as a cumulative total of some quantity. 12.7 – The student will model application problems involving areas under curves. 12.8 – The student will explore topics connecting higher-level mathematics to real-world applications.

Recall: Distance = rate x time, or $d = rt$, where r is the average rate (speed)

s is used to represent position, so Δs would be the distance traveled: $r = \dfrac{\Delta s}{\Delta t}$

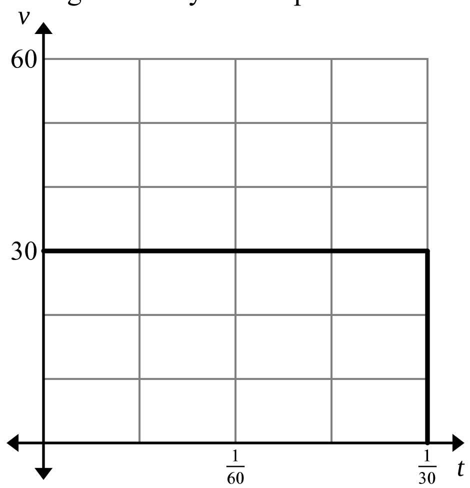
Ex: If it takes you 2 minutes to travel 1 mile from your house, how fast were you traveling?

$r = \dfrac{\Delta s}{\Delta t}$: $r = \dfrac{1 \text{ mi}}{2 \text{ min}} \cdot \dfrac{60 \text{ min}}{1 \text{ hr}} = \boxed{30 \text{ mph}}$

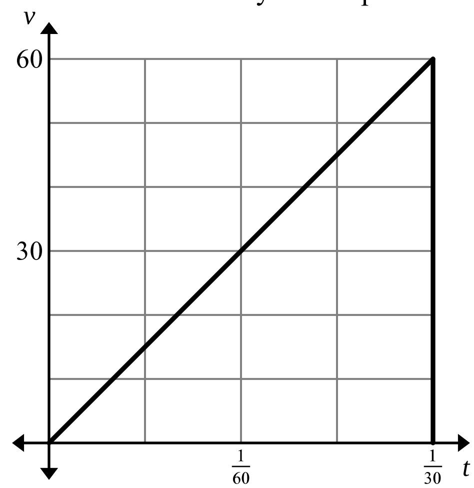
The Connection to Areas

Ex: Graph the time (in hours) as a function of the velocity.

Average Velocity = 30 mph



Instantaneous Velocity = 30 mph at t = 1 min



Calculate the areas of each region.

Average Velocity: $A = 30\left(\dfrac{1}{30}\right) = 1$

Instantaneous Velocity: $A = \dfrac{1}{2}\left(\dfrac{1}{30}\right)60 = 1$

What do the areas represent? Explain.

The areas represent the total distance traveled, 1 mile. To calculate the area, we multiplied velocity (mph) times time (hours). $\dfrac{\text{mi}}{\text{hr}} \cdot \text{hr} = \text{mi}$

Approximating Areas Under Curves

Riemann Sum: the sum of the areas of rectangles that lie under a curve

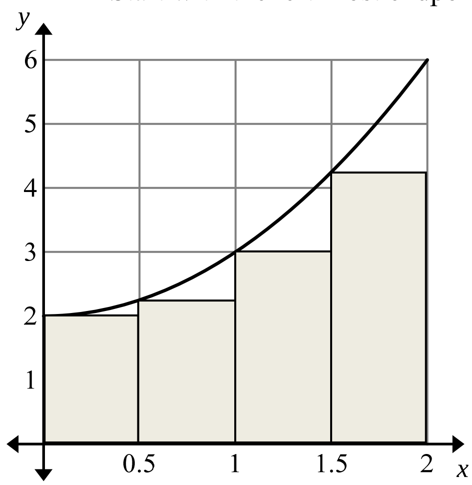
LRAM: Left Rectangular Approximation Method

RRAM: Right Rectangular Approximation Method

Ex: Use Riemann sums to approximate the area of the region bounded by the graph of $f(x) = x^2 + 2$, the x-axis, $x = 0$, and $x = 2$, using 4 subintervals.

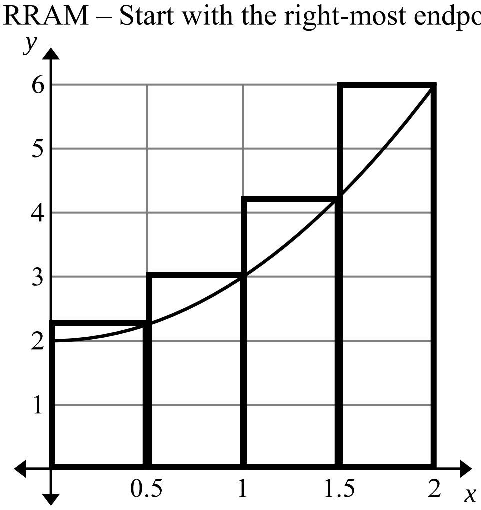
The height of each rectangle is the corresponding function value of the endpoint.
The width of each rectangle is the length of each subinterval = 0.5.

LRAM – Start with the left-most endpoint.



LRAM: $A \approx (0.5) \cdot f(0) + (0.5) \cdot f(0.5) + (0.5) \cdot f(1) + (0.5) \cdot f(1.5)$
$= (0.5) \cdot 2 + (0.5) \cdot 2.25 + (0.5) \cdot 3 + (0.5) \cdot 4.25 = \boxed{5.75}$

RRAM – Start with the right-most endpoint.



RRAM: $A \approx (0.5) \cdot f(0.5) + (0.5) \cdot f(1) + (0.5) \cdot f(1.5) + (0.5) \cdot f(2)$
$= (0.5) \cdot 2.25 + (0.5) \cdot 3 + (0.5) \cdot 4.25 + (0.5) \cdot 6 = \boxed{7.75}$

Discuss: Which approximation is better? Describe two ways we could find a better approximation.

One approximation is too small (LRAM) and the other is too big (RRAM). A better approximation could be found by using more rectangles (smaller subintervals). Or we could find the average of the two.

Definite Integral: the value of the area under the nonnegative curve $f(x)$ from on the interval $[a,b]$

Notation: $\int_a^b f(x)\,dx$ Read: "The integral of *f* of *x* *dx* from *a* to *b*."

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

n = number of subintervals is approaching ∞ $f(x_i)$ = height of rectangles at each interval endpoint

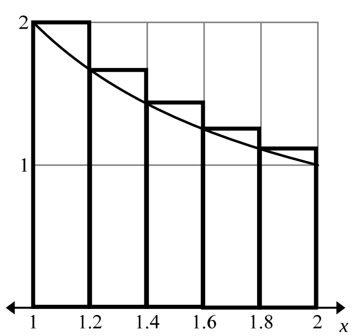
Δx = width of each subinterval

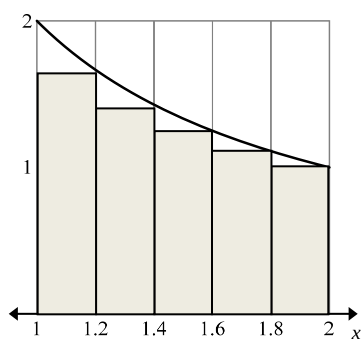
If this limit exists, then *f* is **integrable** on $[a,b]$.

Ex: Estimate the area bounded by $y = \frac{2}{x}$ and the *x*-axis from $x = 1$ to $x = 2$, using 5 subintervals.

LRAM: $\int_1^2 \frac{2}{x}dx \approx 0.2\left(f(1)+f(1.2)+f(1.4)+f(1.6)+f(1.8)\right) = 0.2\left(2+\frac{5}{3}+\frac{10}{7}+\frac{5}{4}+\frac{10}{9}\right) \approx \boxed{1.49}$

RRAM: $\int_1^2 \frac{2}{x}dx \approx 0.2\left(f(1.2)+f(1.4)+f(1.6)+f(1.8)+f(2)\right) = 0.2\left(\frac{5}{3}+\frac{10}{7}+\frac{5}{4}+\frac{10}{9}+1\right) \approx \boxed{1.29}$

LRAM: 

RRAM: 

Because the LRAM is too big and the RRAM is too small, we can make our estimate better by finding their average: $\frac{1.49+1.29}{2} = 1.39$

<u>Numerical Integral</u>: The calculator can estimate a definite integral using fnInt (TI-84).

Ex: Check the estimation from the example above for $\int_1^2 \frac{2}{x}dx$.

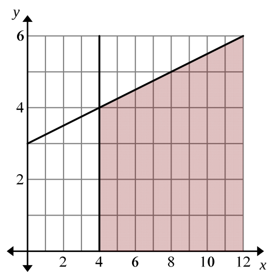
fnInt can be found in the MATH menu. Type in fnInt(function, variable of integration, lower limit, upper limit)

```
fnInt(2/X,X,1,2)
      1.386294361
```

Evaluating a Definite Integral Using Areas

Ex: Evaluate $\int_4^{12}\left(\frac{1}{4}x+3\right)dx$ by computing the area.

Graph: Find the area of the shaded region in the interval $[4,12]$.



The region is a trapezoid with bases of length 4 and 6 and a height of $12-4=8$.

$$A_{\text{trap}}=\frac{1}{2}h\left(b_1+b_2\right)=\frac{1}{2}(8)(4+6)=40 \text{ sq units}; \text{ So } \int_4^{12}\left(\frac{1}{4}x+3\right)dx=\boxed{40}$$

You Try: Estimate the area bounded by $y=4-x^2$ and the x-axis from $x=0$ to $x=2$ using 4 subintervals. Compare your answer with the estimate from the calculator.

QOD: Another rectangular approximation method is called the MRAM (midpoint rectangular approximation method). Would this method give the same result as finding the average of the LRAM and RRAM? Explain.