Syllabus Objective: 11.1 – The student will solve real-world application problems involving combinatorics.

Discrete Math: math involving distinct values, not continuous values

For example, daily attendance at a school is discrete data; temperatures in a day are continuous data.

Simple Counting Problems

Ex: You have a bag with a penny, dime, nickel and quarter. How many ways can the sum be greater than $0.10 if…

1. With Replacement: you select a coin, write down the value and put the coin back in the bag. You then pull out a coin again and add the value to the first.

Possibilities:

1,1	5,1	*10,1	*25,1
1,5	5,5	*10,5	*25,5
*1,10	*5,10	*10,10	*25,10
*1,25	*5,25	*10,25	*25,25

There are **12 ways**.

2. Without Replacement: you select a coin and then select another coin (or select two coins at once).

Possibilities:

1,5	5,1	*10,1	*25,1
*1,10	*5,10	*10,5	*25,5
*1,25	*5,25	*10,25	*25,10

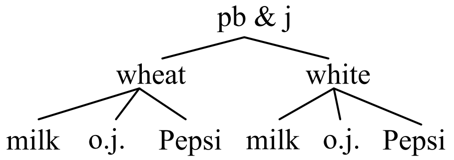
There are **10 ways**.

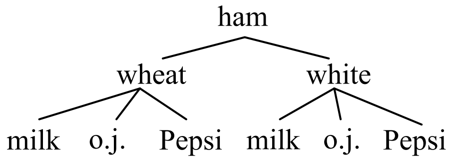
Fundamental Counting Principle: The number of ways that multiple events can occur is the product of the outcomes (number of possibilities) of each of those events.

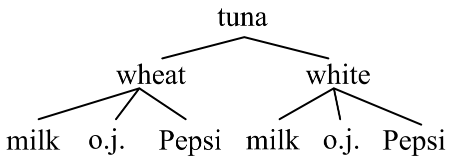
Ex: Your mom says you can have a pb & j, ham, or tuna sandwich on white or wheat bread with milk, orange juice, or Pepsi. How many different lunches could you have?

3 types of sandwiches, 2 types of bread, 3 types of drinks: $3\cdot 2\cdot 3=\boxed{\text{18 lunches}}$

or Tree Diagram:







There are 18 possible "paths" in the tree diagrams. $(3\cdot 2\cdot 3)$

Ex: How many different license plates can be made with 3 different letters and 4 digits?
There are 26 letters and 10 digits (0-9): $\underline{26}\cdot\underline{25}\cdot\underline{24}\cdot\underline{10}\cdot\underline{10}\cdot\underline{10}\cdot\underline{10}=\boxed{156{,}000{,}000}$

Permutation: an ordering (ranking) of outcomes in one event

Ex: How many ways can a family of five be arranged in a row for a photograph?
5 spaces: $\underline{5}\cdot\underline{4}\cdot\underline{3}\cdot\underline{2}\cdot\underline{1}=5!=\boxed{120}$

Factorial: $n!=n(n-1)(n-2)\ldots 3\cdot 2\cdot 1$

Definition: $0!=1$

Ex: Simplify $\dfrac{(2n+1)!}{(2n)!}$.

$$\frac{(2n+1)!}{(2n)!}=\frac{(2n+1)\cancel{(2n)(2n-1)\ldots 3\cdot 2\cdot 1}}{\cancel{(2n)(2n-1)\ldots 3\cdot 2\cdot 1}}=\boxed{2n+1}$$

Ex: How many ways can any two of the family members of the family of five be arranged?
2 spaces: $\underline{5}\cdot\underline{4}=\boxed{20}$ or $\dfrac{5!}{3!}$

Permutation Formula: permutation of *n* items arranged *r* at a time $\quad {}_nP_r=\dfrac{n!}{(n-r)!}$

Ex: How many ways can 8 runners be awarded gold, silver and bronze medals?

$${}_nP_r=\frac{n!}{(n-r)!}\Rightarrow {}_8P_3=\frac{8!}{(8-3)!}=\frac{8!}{5!}=\frac{8\cdot 7\cdot 6\cdot \cancel{5!}}{\cancel{5!}}=\boxed{336}$$

Distinguishable (distinct) Permutations: When an element is repeated, it must be divided out of the total number of arrangements. The number of permutations of *n* items with *r* representing elements that are repeated: $\dfrac{n!}{r_1!r_2!\ldots}$

Ex: How many ways can the letters in the word CLASSES be arranged?
There are 7 letters, so they can be arranged in 7! ways. However, because there are three letters that are the same (S), there will be arrangements repeated.
Number of ways the three S's can be arranged = 3!
Number of distinct arrangements: $\dfrac{7!}{3!}=\dfrac{7\cdot 6\cdot 5\cdot 4\cdot \cancel{3!}}{\cancel{3!}}=\boxed{840}$

Ex: How many distinguishable arrangements are there of the letters in the word TENNESSEE?
9 letters; repeats: 4-E's, 2-N's, 2-S's $\quad \dfrac{9!}{4!2!2!}=\dfrac{9\cdot 8\cdot 7\cdot 6\cdot 5\cdot \cancel{4!}}{\cancel{4!}2\cdot 2}=9\cdot 2\cdot 7\cdot 6\cdot 5=\boxed{3780}$

Combinations: a selection of outcomes (order is not relevant)

Combination Formula: ${}_nC_r = \frac{n!}{r!(n-r)!}$ ${}_nC_r$ is read "*n* choose *r*" or "*n* things taken *r* at a time"

▼ Note: This is the same as the permutation formula, with the same possibilities in different orders (*r*!) divided out.

Ex: How many ways can you select 4 of your 20 CD's for a road trip?

It doesn't matter what order you choose the CD's, so this is a combination problem.

$${}_{20}C_4 = \frac{20!}{4!(20-4)!} = \frac{20\cdot19\cdot18\cdot17\cdot\cancel{16!}}{4\cdot3\cdot2\cdot1\cdot\cancel{16!}} = 5\cdot19\cdot3\cdot17 = \boxed{4{,}845}$$

Ex: How many ways can you choose a committee of 5 members from a student government of 9 girls and 6 boys if there has to be 3 girls and 2 boys?

\# of ways to choose 3 girls · # of ways to choose 2 boys: ${}_9C_3 \cdot {}_6C_2 = \frac{9!}{3!6!}\cdot\frac{6!}{2!4!} = \frac{9\cdot8\cdot7}{3\cdot2}\cdot\frac{6\cdot5}{2} = \boxed{1{,}260}$

Card Problems:

Deck of Cards (52 cards): 4 suits (2 red, 2 black) – clubs, diamonds, hearts, spades

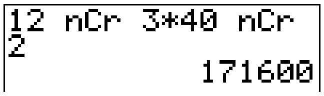
In each suit: 3 face cards (Jack, Queen, King); 9 numbers (2-10); 1 ace

Ex: How many 5-card hands have exactly 3 face cards?

\# of ways to have 3 face cards · # of ways to have 2 non-face cards = ${}_{12}C_3 \cdot {}_{40}C_2$

(There are 12 face cards, you are choosing 3. There are 40 non-face cards, you are choosing 2.)

${}_{12}C_3 \cdot {}_{40}C_2 = \boxed{171{,}600}$:



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12 nCr 3*40 nCr
2
            171600
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${}_nC_r$ is in Math Menu – Prob

You Try: 1. Simplify: $\frac{(3n+2)!}{(3n+4)!}$

2. How many ways can a group of 4 boys and 3 girls be chosen from your precalculus class?

QOD: It is illogical to call a numerical lock a "combination" lock. Explain why.

Syllabus Objective: 10.7 – The student will expand a given binomial.

Exploration: Expand the following.

$(x+1)^0 = 1$ $\quad$ $(x+1)^2 = 1x^2 + 2x + 1$

$(x+1)^1 = 1x + 1$ $\quad$ $(x+1)^3 = 1x^3 + 3x^2 + 3x + 1$

Pascal's Triangle

1 Row 0

1 1 Row 1

1 2 1 Row 2

1 3 3 1 Row 3

1 4 6 4 1 Row 4

1 5 10 10 5 1 Row 5

This pattern can be continued…

▽ The values in the triangle correspond to the coefficients of the binomial expansion of $(x+1)^n$.

They also correspond to ${}_nC_r$:

$${}_0C_0 = \frac{0!}{0!(0-0)!} = \frac{1}{1} = 1$$

$${}_1C_0 = \frac{1!}{0!(1-0)!} = \frac{1}{1} = 1 \qquad {}_1C_1 = \frac{1!}{1!(1-1)!} = \frac{1}{1} = 1$$

$${}_2C_0 = \frac{2!}{0!(2-0)!} = 1 \qquad {}_2C_1 = \frac{2!}{1!(2-1)!} = 2 \qquad {}_2C_2 = \frac{2!}{2!(2-2)!} = 1 \qquad \ldots$$

Notation: Combination (Binomial Coefficient) ${}_nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$

Expanding a Binomial

Ex: Expand. $(x-3)^4$

Use coefficients from Pascal's Triangle: 1 4 6 4 1

Exponent of first term (*x*) starts at 4 and goes down by 1 each time; exponent of second term (−3) starts at 0 and goes up by 1 each time.

$$(x-3)^4 = 1\left(x^4\right)(-3)^0 + 4\left(x^3\right)(-3)^1 + 6\left(x^2\right)(-3)^2 + 4\left(x^1\right)(-3)^3 + 1\left(x^0\right)(-3)^4 = \boxed{x^4 - 12x^3 + 54x^2 - 108x + 81}$$

▽ When expanding a binomial: The exponent of the first term starts at *n* and goes **down** by 1 each time. The exponent of the second term starts at 0 and goes **up** by 1 each time. The **sum** of the two exponents of each term should equal *n*.

If the second term is negative, the terms of the expansion alternate signs.

Finding a Particular Term of a Binomial Expansion: recall that the coefficients from Pascal's Triangle can be found individually by evaluating ${}_nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$.

Ex: Find the fifth term of the binomial expansion of $(2x+1)^{10}$.

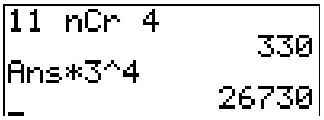
The first term of the expansion has $r=0$, so the fifth term has $r=4$.

Fifth term: $\binom{10}{4}(2x)^6(1)^4 = \frac{10!}{4!(10-4)!}\left(64x^6\right) = \frac{10\cdot 9\cdot 8\cdot 7}{4\cdot 3\cdot 2}\left(64x^6\right) = 13440x^6$

Ex: Find the coefficient of the term with a^7 in the expansion of $(a-3b)^{11}$.

Sum of exponents = 11, so the term has $(-3b)^4$; therefore, $r=4$.

$\binom{11}{4}(a)^7(-3b)^4 = 26730a^7b^4$



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11 nCr 4
                330
Ans*3^4
              26730
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You Try: Show that $\binom{n}{2} = \binom{n}{n-2}$.

QOD: Is the sum of EVERY row of Pascal's triangle an even number? Explain.

Syllabus Objectives: 11.2 – The student will calculate probabilities for a given sample space. 11.3 – The student will determine if two events are independent or dependent events. 11.4 – The student will solve problems using the addition rule, multiplication rule, and conditional probability. 11.9 – The student will solve problems using the binomial distribution.

Probability: $\dfrac{\text{\# of outcomes in event}}{\text{\# of outcomes in sample space}}$ Notation: $P(E)=\dfrac{n(E)}{n(S)}$

$0\le P(E)\le 1$ $P(E)=0$: Impossible Event $P(E)=1$: Certain Event

Ex: What is the probability of rolling a sum that is prime on a single roll of two fair number cubes?

Possibilities:

# on each die	**1**	**2**	**3**	**4**	**5**	**6**
1	$\underline{2}$	$\underline{3}$	4	$\underline{5}$	6	$\underline{7}$
2	$\underline{3}$	4	$\underline{5}$	6	$\underline{7}$	8
3	4	$\underline{5}$	6	$\underline{7}$	8	9
4	$\underline{5}$	6	$\underline{7}$	8	9	10
5	6	$\underline{7}$	8	9	10	$\underline{11}$
6	$\underline{7}$	8	9	10	$\underline{11}$	12

\# of prime numbers (underlined) = 15 total # of outcomes = 36

Probability of rolling a prime number: $\dfrac{15}{36}=\boxed{\dfrac{5}{12}}$

Ex: A card player chooses 2 cards from a full deck. What is the probability he will choose 2 red face cards?

\# of ways of choosing 2 red face cards: ${}_6C_2$ (There are 6 red face cards in the deck)

Total # of ways of choosing 2 cards: ${}_{52}C_2$

Probability of choosing 2 red face cards: $\dfrac{{}_6C_2}{{}_{52}C_2}=\boxed{\dfrac{15}{1326}}$ (use calculator)

Mutually Exclusive Events: two events that have no common outcomes

If $A\cap B=\varnothing$ (are mutually exclusive), then $P(A\text{ or }B)=P(A)+P(B)$.

Ex: What is the probability of drawing an even number or a king from a deck of cards?

These events are mutually exclusive because a king is not an even number.

Probability of drawing an even number: $P(E)=\dfrac{20}{52}$ Probability of drawing a king: $P(K)=\dfrac{4}{52}$

$$P(E\text{ or }K)=\frac{20}{52}+\frac{4}{52}=\frac{24}{52}=\boxed{\frac{6}{13}}$$

Not Mutually Exclusive Events: two events that have common outcomes

$$P(A \text{ or } B) = P(A) + P(B) - P(A \cap B)$$

Ex: What is the probability of drawing a red card or a six from a deck of cards?
These events are NOT mutually exclusive because there are red cards that are also sixes.

Probability of drawing a red card: $P(R) = \frac{26}{52}$ Probability of drawing a six: $P(S) = \frac{4}{52}$

Probability of drawing a red six: $P(R \cap S) = \frac{2}{52}$

$$P(R \text{ or } 6) = \frac{26}{52} + \frac{4}{52} - \frac{2}{52} = \frac{28}{52} = \boxed{\frac{7}{13}}$$

Complementary Event: the complement of an event E is E' if $P(E) + P(E') = 1$, or $P(E') = 1 - P(E)$

Ex: What is the probability of NOT drawing a queen or a club?

Probability of drawing a queen or a club: $P(Q) + P(C) - P(Q \cap C) = \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}$

Probability of NOT drawing a queen or a club: $1 - P(Q \text{ or } C) = 1 - \frac{4}{13} = \boxed{\frac{9}{13}}$

Ex: At a school, 43% of the students are girls and half of the girls play sports. 45% of all the students in the school play sports.

a. What percent of the students who play sports are boys?

Half of the girls play sports: $\frac{43\%}{2} = 21.5\%$

Percent of students who play sports that are boys: $45\% - 21.5\% = \boxed{23.5\%}$

b. If a student is chosen at random, what is the probability that he is a boy who doesn't play sports?

Percent of students who are boys: 57%

Percent of students who are boys who do not play sports: $57\% - 23.5\% = \boxed{33.5\%}$

Conditional Probability: the probability of an event that depends on an earlier event

$$P(B|A) = \frac{P(A \text{ and } B)}{P(A)} \text{ or } \frac{P(A) \cdot P(B)}{P(A)} \text{ or } \frac{P(A \cap B)}{P(A)}$$ Read "the probability of B given A".

Ex: A shirt is drawn at random from one of two identical drawers. Drawer A has 3 t-shirts and 2 sweatshirts and Drawer B has 2 t-shirts. What is the probability that a t-shirt was drawn from drawer A?

$$\frac{\text{Probability of a t-shirt from drawer } A}{\text{Probability of a t-shirt}} = \frac{\frac{3}{5}}{\frac{5}{7}} = \frac{3}{5} \cdot \frac{7}{5} = \boxed{\frac{21}{25}}$$

Binomial Distribution: $\left[P(E)+P(E')\right]^n$, where $P(E)$: probability event happens, $P(E')$: probability event doesn't happen, *n*: number of trials

Ex: 10% of African-Americans are carriers of the genetic disease sickle-cell anemia.

a) Find the probability of having 5 carriers in a sample of 20 African-Americans.

$$(0.1+0.9)^{20}\text{:}\quad P(5)=\binom{20}{5}(0.1)^5(0.9)^{15}\approx\boxed{3.19\%}$$

b) Find the probability of having at most 2 carriers in a sample of 20 African-Americans.

$$(0.1+0.9)^{20}\text{:}$$

$$P(0)+P(1)+P(2)=\binom{20}{0}(0.1)^0(0.9)^{20}+\binom{20}{1}(0.1)^1(0.9)^{19}+\binom{20}{2}(0.1)^2(0.9)^{18}\approx\boxed{67.69\%}$$

You Try: Suppose Michael makes 80% of his free throws. If he shoots 20 free throws, what is the probability that he makes exactly 15? at least 17?

Note: Assume that his chance of making each one is independent of the other shots.

QOD: Explain how a tree diagram can help solve a problem involving conditional probability. Hint: Draw a tree diagram for the example given in the notes.

Syllabus Objective: 10.1 – The student will explore various types of sequences including arithmetic, geometric, recursive and product.

Sequence: an ordered list of numbers $a_1, a_2, a_3, \ldots$

Terms of a Sequence:

Number	1st	2nd	3rd	nth
Term	a_1	a_2	a_3	a_n
Said	"*a* sub 1"	"*a* sub 2"	"*a* sub 3"	"*a* sub *n*"

Finite Sequence: a sequence with *n* terms 1, 3, 5, 7

Infinite Sequence: a sequence that has infinitely many terms 2, 5, 8, 11, ...

Explicitly Defined Sequence: a formula that defines a sequence based upon *n*, which represents the term number

Ex: Find the first five terms of the sequence given by the formula. $a_n = 5 + 2n(-1)^n$

$a_1 = 5 + 2(1)(-1)^1 = 5 - 2 = 3$ $a_2 = 5 + 2(2)(-1)^2 = 5 + 4 = 9$

$a_3 = 5 + 2(3)(-1)^3 = 5 - 6 = -1$ $a_4 = 5 + 2(4)(-1)^4 = 5 + 8 = 13$

$a_5 = 5 + 2(5)(-1)^5 = 5 - 10 = -5$ $\boxed{3,\ 9,\ -1,\ 13,\ -5}$

Ex: Write an expression for the *n*th term of the sequence: $\frac{2}{1}, \frac{3}{2}, \frac{3}{3}, \frac{5}{4}, \ldots a_n$

Numerators = 1 + term number; Denominators = term number; Formula: $\boxed{a_n = \frac{n+1}{n}}$

Recursively Defined Sequence: given the initial term(s), subsequent terms are then defined using the previous term

Ex: Write the first five terms of the sequence. $a_1 = -11,\ a_n = a_{n-1} + 5$

▽ Note: a_{n-1} is the term before a_n.

$a_1 = -11 \Rightarrow a_2 = a_1 + 5 = -11 + 5 = -6$ $a_2 = -6 \Rightarrow a_3 = a_2 + 5 = -6 + 5 = -1$

$a_3 = -1 \Rightarrow a_4 = a_3 + 5 = -1 + 5 = 4$ $a_4 = 4 \Rightarrow a_5 = a_4 + 5 = 4 + 5 = 9$

First five terms: $\boxed{-11,\ -6,\ -1,\ 4,\ 9}$

Limits of Infinite Sequences: Let $\{a_n\}$ be a sequence of real numbers. If $\lim\limits_{n\to\infty} a_n =$ a finite number L, then the sequence **converges** and L is the limit of the sequence. Otherwise, the sequence **diverges**.

Ex: Determine if the sequence converges.

a) $\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, ..., \frac{1}{2n}, ...$ $\lim\limits_{n\to\infty} \frac{1}{2n} = 0$ converges

b) 0.1, 0.2, 0.3, 0.4, ... $a_n = \frac{n}{10}$; $\lim\limits_{n\to\infty} \frac{n}{10} = \infty$ no, diverges

c) $\frac{11}{1}, \frac{12}{2}, \frac{13}{3}, \frac{14}{4}, ...$ $a_n = \frac{n+10}{n}$; $\lim\limits_{n\to\infty} \frac{n+10}{n} = 1$ converges

d) $a_n = \frac{10n-2}{4-5n}$ $\lim\limits_{n\to\infty} \frac{10n-2}{4-5n} = -2$ converges

e) $a_n = 5^n$ $\lim\limits_{n\to\infty} 5^n = \infty$ no, diverges

f) $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, ...$ $a_n = \left(\frac{1}{2}\right)^n$; $\lim\limits_{n\to\infty} \left(\frac{1}{2}\right)^n = \lim\limits_{n\to\infty} \left(\frac{1}{2^n}\right) = 0$ converges

Arithmetic Sequence: each term is found by **adding** a constant (called the **common difference**) to the previous term

Common Difference: $d = a_n - a_{n-1}$

Ex: Write the recursive and explicit formulas for the sequence. 5, 11, 17, 23, ...

Common Difference: $d = 11 - 5 = 6$

Recursive: $a_1 = 5, a_n = a_{n-1} + 6$

n	1	2	3	4	5	n
a_n	5	$5+6$	$5+2(6)$	$5+3(6)$	$5+4(6)$	$5+(n-1)6$

Explicit: $a_n = 5 + (n-1)6 \Rightarrow \boxed{a_n = 6n - 1}$

Explicit Formula for an Arithmetic Sequence: $\boxed{a_n = a_1 + (n-1)d}$

Ex: Find a_{17} for the sequence $-3, 4, 11, 18, ...$.

This is an arithmetic sequence with $d = 7$.

$a_n = a_1 + (n-1)d \Rightarrow a_n = -3 + (n-1)(7) \Rightarrow a_n = 7n - 10$ $a_{17} = 7(17) - 10 = \boxed{109}$

Ex: In an arithmetic sequence, $a_3 = 14$ and $a_8 = 44$, write the first five terms and a formula.

To find *d*, write a system of equations: $\begin{array}{l} 14 = a_1 + 2d \\ 44 = a_1 + 7d \end{array}$ Solve the system: $\begin{array}{r} -14 = -a_1 - 2d \\ \underline{44 = a_1 + 7d} \\ 30 = 5d \;\Rightarrow\; d = 6 \\ 14 = a_1 + 12 \;\Rightarrow\; a_1 = 2 \end{array}$

First five terms: $\boxed{2, 8, 14, 20, 26}$ Formula: $a_n = 2 + (n-1)(6) \Rightarrow \boxed{a_n = 6n - 4}$

OR – find the common difference using a rate of change (slope): $d = \frac{44-14}{8-3} = \frac{30}{5} = 6$

<u>Geometric Sequence</u>: each term is found by **multiplying** a constant (called the **common ratio**) to the previous term

Common Ratio: $r = \frac{a_n}{a_{n-1}}$

Ex: Write the recursive and explicit formulas for the sequence. 3, 6, 12, 24, 48, ...

Common Ratio: $r = \frac{6}{3} = 2$

Recursive: $a_1 = 3, a_n = 2 \cdot a_{n-1}$

n	1	2	3	4	5	n
a_n	$3 \cdot 1$	$3 \cdot 2$	$3 \cdot 4$	$3 \cdot 8$	$3 \cdot 16$	$3 \cdot 2 \cdot 2 \cdot 2 ...$
a_n	$3 \cdot 2^0$	$3 \cdot 2^1$	$3 \cdot 2^2$	$3 \cdot 2^3$	$3 \cdot 2^4$	$3 \cdot 2^{n-1}$

Explicit: $\boxed{a_n = 3 \cdot 2^{n-1}}$

<u>Explicit Formula for an Geometric Sequence</u>: $\boxed{a_n = a_1 \cdot r^{n-1}}$

Ex: Find a formula for a_n and find a_{10} for the sequence $16, -8, 4, -2, ...$

This is a geometric sequence with common ratio $r = -\frac{8}{16} = -\frac{1}{2}$.

Formula: $\boxed{a_n = 16 \cdot \left(-\frac{1}{2}\right)^{n-1}}$ $a_{10} = 16 \cdot \left(-\frac{1}{2}\right)^9 = 2^4 \frac{1}{(-2)^9} = -\frac{1}{2^5} = \boxed{-\frac{1}{32}}$

<u>You Try:</u> What comes next in the sequence? 1, 1, 2, 3, 5, 8, … Write a recursive rule for the sequence.
Note: This is called the "Fibonacci Sequence".

<u>QOD</u>: Can every sequence be defined as arithmetic or geometric? Explain.

Syllabus Objectives: 10.2 – The student will rewrite a given series using summation notation. 10.3 – The student will find a given partial sum of a series. 10.4 – The student will find the sum of an infinite series. 10.6 – The student will solve application problems involving series.

Series: the sum of a list of terms $2+5+8+11+14$

Summation (Sigma) Notation: $\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \ldots + a_n$ (*i*: index of summation)

Ex: Find the sum of the given series.

a) $\sum_{i=1}^{4}(2+3i)$ $\sum_{i=1}^{4}(2+3i) = (2+3\cdot 1)+(2+3\cdot 2)+(2+3\cdot 3)+(2+3\cdot 4) = \boxed{38}$

b) $\sum_{k=2}^{5}(-1)^k\cdot(2k)$

$$\sum_{k=2}^{5}(-1)^k\cdot(2k) = (-1)^2\cdot(2\cdot 2)+(-1)^3\cdot(2\cdot 3)+(-1)^4\cdot(2\cdot 4)+(-1)^5\cdot(2\cdot 5) = 4-6+8-10 = \boxed{-4}$$

c) $\sum_{n=0}^{6}\cos(n\pi)$ $\sum_{n=0}^{4}\cos(n\pi) = \cos 0 + \cos\pi + \cos 2\pi + \cos 3\pi + \cos 4\pi = 1-1+1-1+1 = \boxed{1}$

d) $\sum_{i=0}^{\infty}\sin(n\pi)$ $\sum_{i=0}^{\infty}\sin(n\pi) = \sin 0 + \sin\pi + \sin 2\pi + \ldots = 0+0+0+\ldots = \boxed{0}$

e) $\sum_{i=1}^{\infty}5\left(\frac{1}{10}\right)^i$ $\sum_{i=1}^{\infty}5\left(\frac{1}{10}\right)^i = 0.5+0.05+0.005+\ldots = 0.55\overline{5} = \boxed{\frac{5}{9}}$

Arithmetic Series: each term is found by **adding** a constant (called the **common difference**) to the previous term

Story of Karl F. Gauss (1777 – 1855): Find the sum of the first 100 natural numbers.

$$S_n = 1+2+3+\ldots+98+99+100 \qquad \begin{array}{ll} 1\ \ +2\ \ +3+\ldots+98+99+100 & S_{100} \\ \underline{100+99+98+\ldots\ \ +3\ \ +2\ \ \ +1} & \underline{S_{100}} \\ 101+101+101+\ldots+101+101+101 & 2S_{100} \end{array}$$

$$2S_{100} = 101(100) \Rightarrow 2S_{100} = 10100 \Rightarrow \boxed{S_{100} = 5050}$$

Formula – Sum of an Arithmetic Series: $\boxed{S_n = \frac{n(a_1+a_n)}{2}}$

Recall: Formula for the n^{th} Term of an Arithmetic Sequence/Series: $\boxed{a_n = a_1 + (n-1)d}$

Ex: Find S_{100} and write the series in sigma notation. 5, 8, 11, 14, …

$a_n = a_1 + (n-1)d$: $a_{100} = 5 + (100-1)(3) = 302$ $\quad S_n = \dfrac{n(a_1 + a_n)}{2}$: $S_{100} = \dfrac{100(5+302)}{2} = \boxed{15{,}350}$

Sigma Notation: $\sum_{n=1}^{100}(5 + (n-1)\cdot 3) = \boxed{\sum_{n=1}^{100} 3n+2}$

Ex: A theater has 30 seats in the 1st row and 2 more in each subsequent row. How many seats are there if there are 78 seats in the last row?

This is an arithmetic series: $a_1 = 30,\ d = 2,\ a_n = 78$

$a_n = a_1 + (n-1)d$: $78 = 30 + (n-1)\cdot 2 \Rightarrow 2n + 28 = 78 \Rightarrow n = 25$

Sigma Notation: $\sum_{n=1}^{25} 2n+28$

$S_n = \dfrac{n(a_1 + a_n)}{2}$: $S_{25} = \dfrac{25(30+78)}{2} = \boxed{1{,}350 \text{ seats}}$

<u>Geometric Series</u>: each term is found by **multiplying** a constant (called the **common ratio**) to the previous term

<u>Formula – Sum of a Geometric Series:</u> $\boxed{S_n = \dfrac{a_1(1-r^n)}{1-r}}$

<u>Recall: Formula for the n^{th} Term of a Geometric Sequence/Series:</u> $\boxed{a_n = a_1 \cdot r^{n-1}}$

Ex: Find S_{10} for the series 3 + 6 + 12 + 24 + ….

This is a geometric series with $r = 2$. Use $S_n = \dfrac{a_1(1-r^n)}{1-r}$: $S_{10} = \dfrac{3(1-2^{10})}{1-2} = -3(1-2^{10}) = \boxed{3{,}069}$

Ex: Suppose $a_4 = 54$ and $a_7 = 1458$ in a geometric series. Find S_7.

$a_n = a_1 \cdot r^{n-1}$: $\begin{array}{l} 54 = a_1 \cdot r^3 \\ 1458 = a_1 \cdot r^6 \end{array}$ We can eliminate a_1 by dividing the two equations.

$\dfrac{1458 = \cancel{a_1} \cdot r^6}{54 = \cancel{a_1} \cdot r^3} \Rightarrow 27 = r^3 \Rightarrow r = 3$ $\quad 54 = a_1 \cdot r^3 \Rightarrow 54 = a_1 \cdot 3^3 \Rightarrow a_1 = 2$

$S_n = \dfrac{a_1(1-r^n)}{1-r}$: $S_7 = \dfrac{2(1-3^7)}{1-3} = -1(1-3^7) = \boxed{2{,}186}$

<u>Infinite Geometric Series</u>: If $|r| < 1$, then the series **converges**.

Let $|r| < 1$. Then $\lim_{n\to\infty} \dfrac{a_1(1-r^n)}{1-r} = \dfrac{a_1}{1-r}$. $\quad$ So $\boxed{S_\infty = \dfrac{a_1}{1-r},\ |r| < 1}$.

Ex: Find S_∞ for the series. $4+2+1+\dots$

$r=\frac{1}{2}<1$, so the series converges. Use $S_\infty=\frac{a_1}{1-r}$: $S_\infty=\frac{4}{1-\frac{1}{2}}=\boxed{8}$

Repeating Decimals

Ex: Convert the repeating decimal to fraction form. $3.252525\dots$

$3.252525\dots=3+\left(0.25+0.0025+0.000025+\dots\right)$

$\left(0.25+0.0025+0.000025+\dots\right)$ is an infinite geometric series with $a_1=0.25$ and $r=0.01$.

$S_\infty=\frac{a_1}{1-r}$: $S_\infty=\frac{0.25}{1-0.01}=\frac{0.25}{0.99}=\frac{25}{99}$ So $3.252525\dots=\boxed{3\frac{25}{99}}$

You Try: Find the sum. $\sum_{n=0}^{\infty}27\cdot\left(\frac{1}{3}\right)^n$

QOD: Can an infinite geometric series with $r=-1$ converge? Explain.

Syllabus Objective: 10.5 – The student will prove rules for sums of series using the theory of mathematical induction.

Proof by Mathematical Induction

Step 1: Show that P_1 is true. (Plug in a 1.)

Step 2: Show that for any positive integer *k*, if P_k is true, then P_{k+1} is also true.

(For sums: $S_{k+1} = S_k + a_{k+1}$)

Prove the following by mathematical induction:

Ex: Prove that $S_n = 5+7+9+11+...+(3+2n) = n(n+4)$.

Step 1: $S_1 = 1(1+4) = 5$ TRUE

Step 2: If S_k is true, then

$$\begin{array}{lcl} S_k + a_{k+1} = k(k+4) + (3+2(k+1)) & & S_{k+1} = (k+1)(k+1+4) \\ = k^2+4k+3+2k+2 & \text{and} & = (k+1)(k+5) \\ = k^2+6k+5 & \longleftrightarrow & = k^2+6k+5 \end{array}$$

Ex: Prove that $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+...+\frac{1}{2^n} = \frac{2^n-1}{2^n}$.

Step 1: $S_1 = \frac{2^1-1}{2^1} = \frac{1}{2}$ TRUE

Step 2: If S_k is true, then

$$\begin{array}{lcl} S_k + a_{k+1} = \frac{2^k-1}{2^k} + \frac{1}{2^{k+1}} & \text{and} & S_{k+1} = \frac{2^{k+1}-1}{2^{k+1}} \\ = \frac{2^{k+1}-2+1}{2^{k+1}} & & \\ = \frac{2^{k+1}-1}{2^{k+1}} & \longleftrightarrow & \end{array}$$

Ex: Prove that 5 is a factor of $(4^{2n}-1)$. (Or that $(4^{2n}-1)$ is divisible by 5.)

Step 1: P_1: $4^{2(1)}-1 = 16-1 = 15$; $5(3) = 15$, 5 is a factor. TRUE

Step 2: If $P_k = 4^{2k}-1 = 5r$ is true, then

$$\begin{aligned} P_{k+1} &= 4^{2(k+1)}-1 \\ &= 4^{2k+2}-1 \\ &= 16 \cdot 4^{2k}-1 \\ &= 16(4^{2k}-1)+15 \end{aligned}$$

$P_k = 4^{2k} - 1$ is divisible by 5, so $16\left(4^{2k} - 1\right)$ is also divisible by 5. 15 is divisible by 5. So $16\left(4^{2k} - 1\right) + 15$ is a sum of multiples of 5, therefore it is divisible by 5.

You Try: Prove by mathematical induction that $n^2 > 2n$ for $n \geq 3$.

QOD: In mathematical induction, what is assumed and what is to be shown?

Syllabus Objectives: 11.5 – The student will summarize distributions of data using histograms, frequency tables, stemplots, and boxplots. 11.6 – The student will fit a curve to a scatterplot of bivariate data.

Statistics: the science of data (discrete or continuous)

Individual: usually people, objects, transactions, or events that form a population (set) we wish to study

Variable: a characteristic or property of an individual

Categorical (qualitative) Variable: variables that cannot be measured on a natural numerical scale; they can only be put into groups

Quantitative Variable: measurements that are recorded on a naturally occurring numerical scale

Ex: Classify the following variables as categorical or quantitative.

Unemployment rates of the 50 states	QUANTITATIVE
Name for the size of a car (compact, mid-size, etc.)	CATEGORICAL
Political party	CATEGORICAL
Temperature	QUANTITATIVE
A taste-tester's ranking	QUANTITATIVE

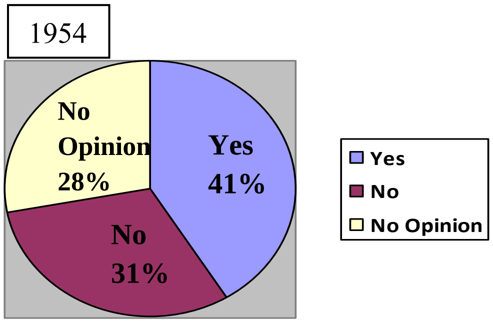
Displaying Categorical Data

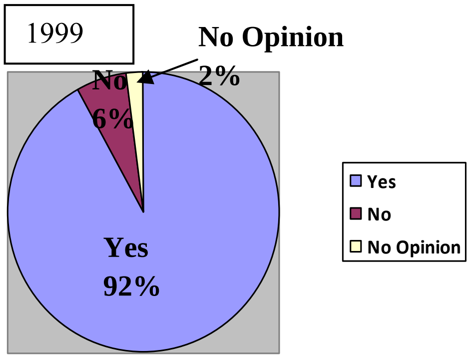
- Bar Chart
- Pie Chart
- Circle Graph
- Picture Graph

Ex: Construct a pie chart and bar graph for each year given the table.

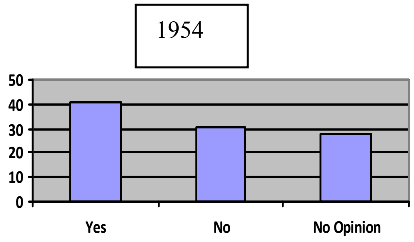
Is smoking a cause of lung cancer?	**1954**	**1999**
Yes	41%	92%
No	31%	6%
No Opinion	28%	2%

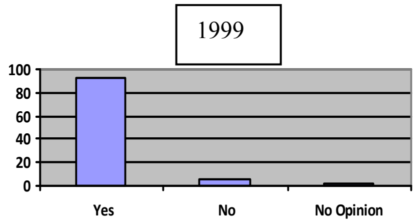
Pie Chart:





Bar Graph:


1954


1999

<u>Displaying Quantitative Data</u>

<u>Stemplot (Stem-and-Leaf)</u>: arranges data by separating the digits into stems (the beginning digit(s)) and leaves (the ending digit(s))

Ex: Make a stemplot for the percent of student loans in default in 12 states.

AZ	CA	CO	HI	ID	MT	NV	NM	OK	OR	UT	WA
12.1	11.4	9.5	12.8	7.1	6.4	10.1	7.5	11.2	7.9	6.0	8.4

Stemplot – Use the whole number as the stems and the tenths digits as the leaves.

6	0 4
7	1 5 9
8	4
9	5
10	1
11	2 4
12	1

Key: $6|0 = 6.0\%$

<u>Split Stem Stemplot</u>: stems are split to spread out the data

Ex: Make a split stemplot for the SAT scores from 2000.

CA	CT	HI	ME	MD	MA	NV	NH	NJ	NY	RI	VT	VA	WV
1015	1017	1007	1004	1016	1024	1027	1039	1011	1000	1005	1021	1009	1037

Stemplot – Use the first three digits for the stems. Split the stems 100, 101, and 103 in two to spread out the data.

100	0	4	5
100	7	9	
101	1	5	
101	6	7	
102	1	4	
102	7		
103			
103	7	9	

Key: $100|0 = 1000$

Back-to-Back Stemplot: stemplot that plots two sets of data on separate sides

Ex: The table lists the percent of graduates taking the SAT and their average math score. Create a back-to-back stemplot for the states with less than 25% on the left.

Arizona	22%	520
California	44%	484
Colorado	29%	511
Idaho	16%	501
Nevada	24%	486
New Mexico	12%	524
Oregon	50%	486
Texas	45%	462
Utah	6%	536
Washington	37%	494

Less than 25%

	46	2
	47	
6	48	4 6
	49	4
1	50	
	51	1
4 0	52	
6	53	

Greater than 25%

Key: $46|2 = 462; 6|48 = 486$

Frequency Table: lists the number of occurrences per interval

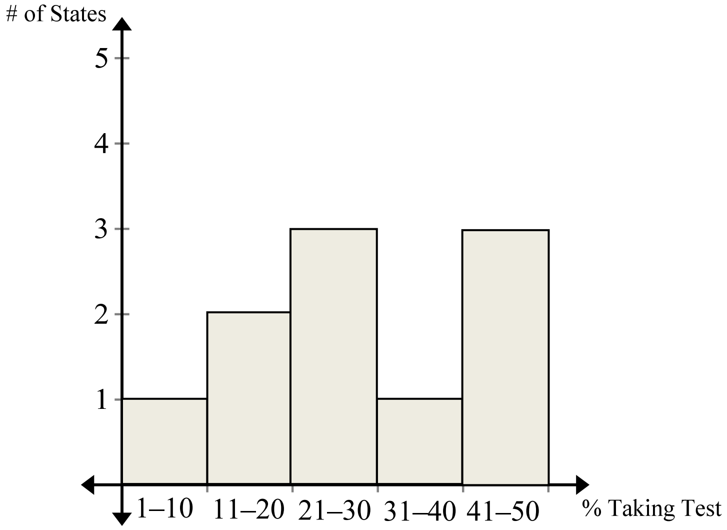
Ex: Create a frequency table for the percent of students taking the SAT from the table above.

Use tally marks:

Percent	# of States
1 – 10	|
11 – 20	| |
21 – 30	| | |
31 – 40	|
41 – 50	| | |

Histogram: displays the information from a frequency table using intervals

Ex: Create a histogram using the frequency table above.

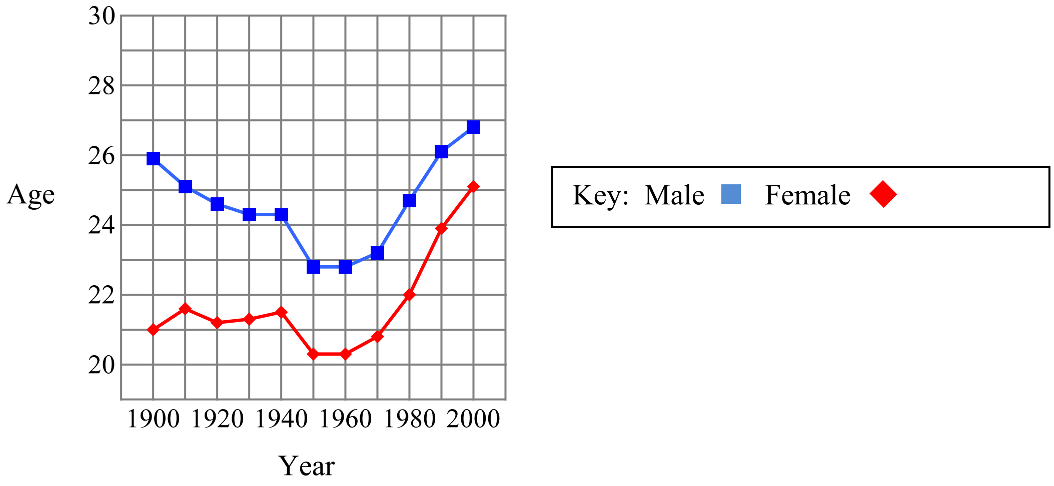


Time Plot: a line graph (not necessarily linear) where discrete points are connected to reveal trends in data over time

Ex: Make time plots for the data below on the same set of axes. Which decade did the largest decrease occur for each? Give a possible explanation.

Year	Male	Female
1900	25.9	21.9
1910	25.1	21.6
1920	24.6	21.2
1930	24.3	21.3
1940	24.3	21.5
1950	22.8	20.3
1960	22.8	20.3
1970	23.2	20.8
1980	24.7	22.0
1990	26.1	23.9
2000	26.8	25.1

Largest decrease occurred in 1940 – 1950.



<u>You Try</u>: Create a histogram for the SAT scores from 2000 table.

<u>QOD</u>: How is a histogram different from a bar graph?

Syllabus Objectives: 11.7 – The student will describe quantitative data using measures of center and measures of variability. 11.8 – The student will solve problems using normal distributions.

Measures of Central Tendency (Measures of Center)

Given a set of numbers $\{x_1, x_2, x_3, \ldots x_n\}$:

Mean (average): $\overline{X} = \dfrac{x_1 + x_2 + x_3 + \ldots + x_n}{n}$

Median: when the data are written *in order*, the middle number if *n* is odd or the mean of the two middle numbers if *n* is even

Mode: the number(s) that occur the most

Ex: Find the measures of center for the test scores of a precalculus class (out of 70).
47, 38, 14, 26, 33, 39, 67, 39, 16, 44, 44, 40, 33, 45, 36, 55, 28, 53, 43, 43, 35

Write the values in order: 14, 16, 26, 28, 33, 33, 35, 36, 38, 39, 39, 40, 43, 43, 44, 45, 47, 53, 55, 56, 67

Mean: $\overline{X} = \dfrac{14+16+\ldots+56+67}{21} = \dfrac{818}{21} = \boxed{38.95}$

Median: $\boxed{39}$

Modes: $\boxed{33, 39, 43, 44}$

Weighted Mean: assigning weights to each number before the mean is computed (same as frequencies)

Measure of Variability

Range: the difference between the maximum and minimum values of a data set

▼ Note: Do not confuse this with the range of a function!

Ex: Find the mean, median, mode and range:

Score	10	9	8	7	6	5	4	3	2	1	0
Frequency	4	4	4	4	2	2	5	3	0	2	2

Mean: $\overline{X} = \dfrac{10(4)+9(4)+8(4)+7(4)+6(2)+5(2)+4(5)+3(3)+2(0)+1(2)+0(2)}{4+4+4+4+2+2+5+3+0+2+2} = \dfrac{189}{32} \approx \boxed{5.9}$

Median: $\dfrac{6+7}{2} = \boxed{6.5}$ Mode: $\boxed{4}$ Range: $10 - 0 = \boxed{10}$

Quartiles: values the separate the data into fourths

First Quartile (Q_1): the median of the lower half of the data

Second Quartile (Q_2): the median of the data

Third Quartile (Q_3): the median of the upper half of the data

Interquartile Range: the difference between the third and first quartiles; comprising the middle half of the data $IQR = Q_3 - Q_1$

Five-Number Summary: $\{\text{minimum}, Q_1, \text{ median}, Q_2, \text{ maximum}\}$

Boxplot (Box-and-Whisker Plot): a plot that displays the five-number summary in the form of whiskers (minimum, maximum) and a box (quartiles 1, 2, and 3)

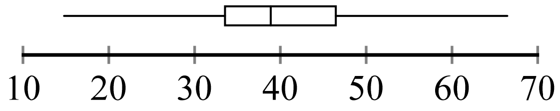
Ex: Draw a boxplot for the test data.

47, 38, 14, 26, 33, 39, 67, 39, 16, 44, 44, 40, 33, 45, 36, 55, 28, 53, 43, 43, 35

Write the data in order: 14, 16, 26, 28, 33, 33, 35, 36, 38, 39, 39, 40, 43, 43, 44, 45, 47, 53, 55, 56, 67

14, 16, 26, 28, $\boxed{33, 33}$, 35, 36, 38, 39, $\boxed{39}$, 40, 43,43, 44, $\boxed{45, 47}$, 53, 55, 56, 67

$Q_1 = \frac{33+33}{2} = 33$ median = 39 $Q_3 = \frac{45+47}{2} = 46$



Outlier: a number in a data set that is more than 1.5(IQR) below Q_1 or above Q_3

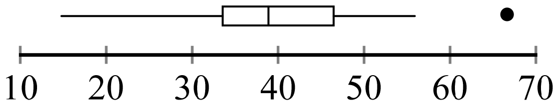
Modified Boxplot: separates outliers as isolated points

Ex: Identify any outliers in the example above and draw a modified boxplot.

IQR = $Q_3 - Q_1 = 46 - 33 = 13$ $1.5(IQR) = 1.5(13) = 19.5$

Outliers would have to be less than $33 - 19.5 = 13.5$ or more than $46 + 19.5 = 65.5$. Outlier = $\boxed{67}$

Modified boxplot:

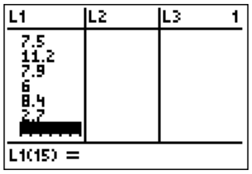


Boxplots on the Graphing Calculator

Ex: Make a boxplot for the percent of student loans in default.

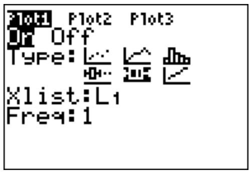
AK	AZ	CA	CO	HI	ID	MT	NV	NM	OK	OR	UT	WA	WY
19.7	12.1	11.4	9.5	12.8	7.1	6.4	10.1	7.5	11.2	7.9	6.0	8.4	2.7

Enter the data into a List:



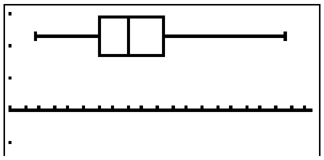
```
L1     L2     L3    1
 7.5
 11.2
 7.9
 6
 8.4
 2.7

L1(15) =
```

Turn on boxplot in StatPlot:



```
Plot1  Plot2  Plot3
On Off
Type:
Xlist:L1
Freq:1
```

Zoom Stat:



Standard Deviation: Typically, two-thirds of the data values are within one standard deviation of the mean.

s: standard deviation when calculated from a sample;
σ: standard deviation when calculated from the entire population

Note: σ is the lower case Greek letter sigma.

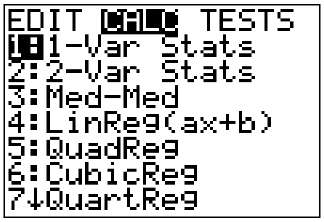
$$s=\sqrt{\frac{1}{n-1}\cdot\sum_{i=1}^{n}\left(X_i-\bar{X}\right)^2} \qquad \sigma=\sqrt{\frac{1}{n}\cdot\sum_{i=1}^{n}\left(X_i-\bar{X}\right)^2}$$

Variance: σ^2

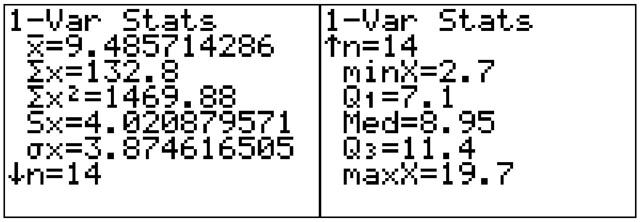
Standard deviation is strongly affected by outliers.

Ex: Find the standard deviation and the variance for the percent of student loans in default.
For data entered in a list, we can compute one-variable statistics.

STAT-CALC Menu:



```
EDIT CALC TESTS
1:1-Var Stats
2:2-Var Stats
3:Med-Med
4:LinReg(ax+b)
5:QuadReg
6:CubicReg
7↓QuartReg
```



```
1-Var Stats
 x̄=9.485714286
 Σx=132.8
 Σx²=1469.88
 Sx=4.020879571
 σx=3.874616505
↓n=14
```

```
1-Var Stats
↑n=14
 minX=2.7
 Q₁=7.1
 Med=8.95
 Q₃=11.4
 maxX=19.7
```

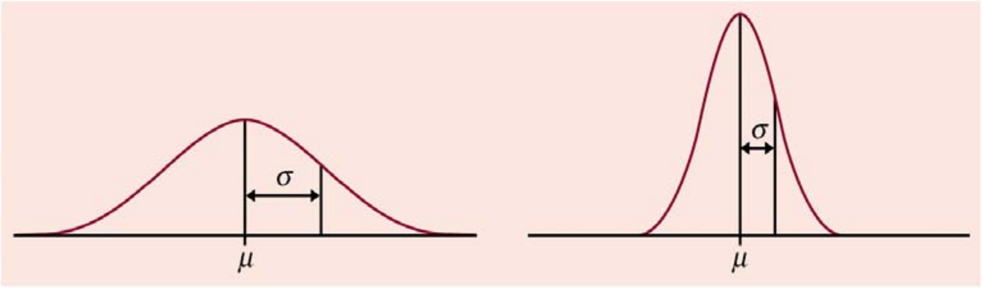
Standard deviation: $S_x=\boxed{4.02}$ (choose this one because this is a sample of states, not all)

Variance: $\approx\boxed{15.01}$

Normal Distributions: continuous distributions with the following properties:

- The curve is single-peaked (unimodal).
- The shape is symmetric.
- More specifically, the distribution is bell-shaped.
- The curve is described by two things: its mean, μ, and its standard deviation, σ.

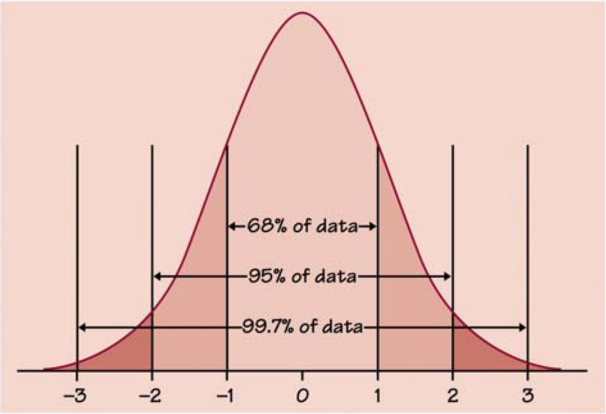
Examples of Normal distributions:



This is more spread out. The distr. has a larger σ.

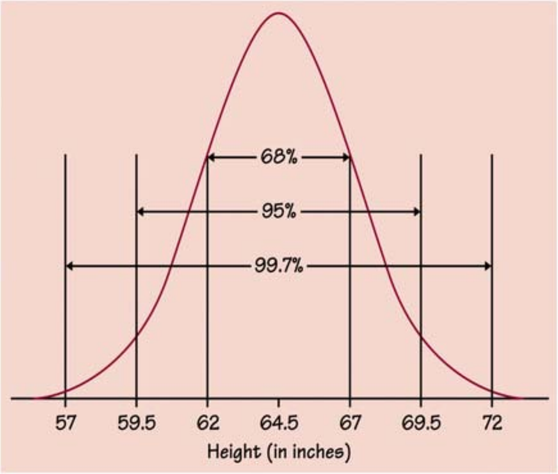
This is less spread out. The distr. has a smaller σ.

The 68-95-99.7 Rule, also known as the Empirical Rule

- In a Normal distribution, 68% of the observations fall within one σ of μ.
- 95% of the observations fall within two σ of μ.
- 99.7% of the observations fall within three σ of μ.
- 

- Notation: $N(\mu,\sigma)$
- If N(64.5, 2.5), then we have a Normal distribution with a mean of 64.5 and a standard deviation of 2.5.

Here is the Normal distribution of heights with a mean of 64.5 inches and a S.D. of 2.5 inches. The 68-95-99.7 Rule shows that 68% of the heights will be found between 62 and 67 inches, 95% of the heights will be found between 59.5 and 69.5 inches and 99.7% of the heights will be between 57 and 72 inches.



Ex: Suppose that we know that SAT I Math scores follow an approximately Normal distribution with mean 500 and standard deviation 100. $N(500,100)$

Using the Empirical Rule, approximately 68% of students scored between 400 and 600 (within 1 S.D.), 95% scored between 300 and 700 (2 S.D.) and 99.7% scored between 200 and 800 (3 S.D.).

<u>You Try</u>: Create a modified boxplot for the percent of student loans in default. Is it different from the boxplot? Explain why or why not.

AK	AZ	CA	CO	HI	ID	MT	NV	NM	OK	OR	UT	WA	WY
19.7	12.1	11.4	9.5	12.8	7.1	6.4	10.1	7.5	11.2	7.9	6.0	8.4	2.7

<u>QOD</u>: What does it mean for a teacher to grade a set of tests “on a curve”?