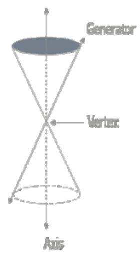
Syllabus Objectives: 2.1 – The student will graph relations or functions, including real-world applications. (Parabolas)

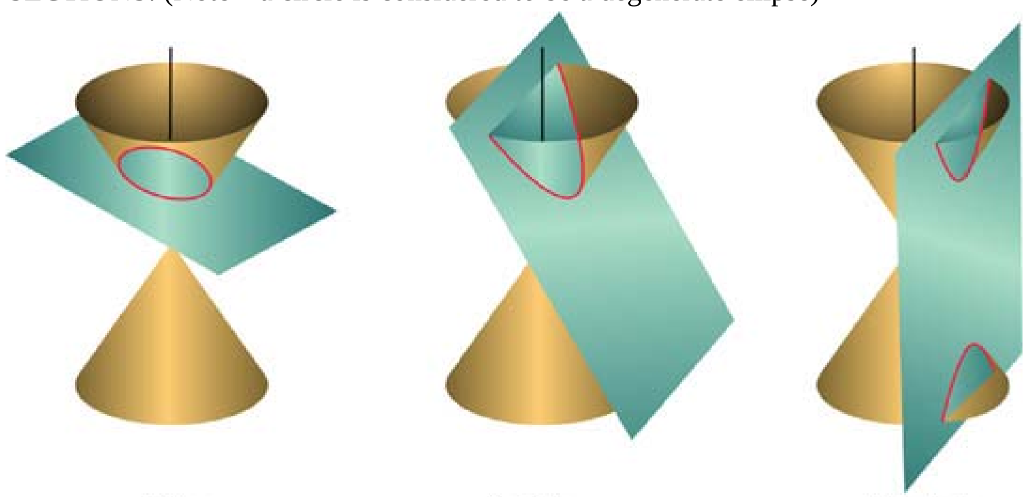
Conic Sections: the graphs of second degree (quadratic) equations in two variables

$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, where A, B, & C are not all zero

Conic Sections are all created from two cones:

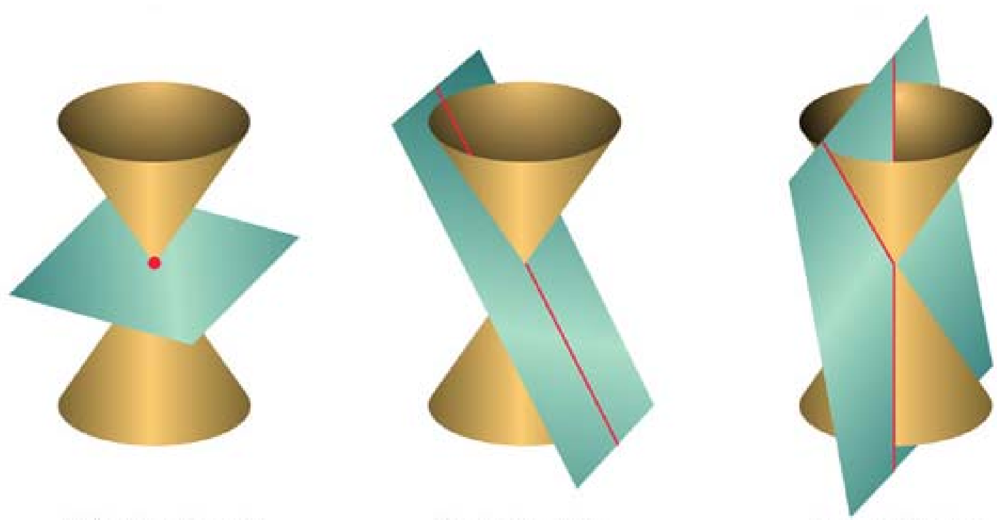


CONIC SECTIONS: (Note – a circle is considered to be a degenerate ellipse)



Ellipse   Parabola   Hyberbola

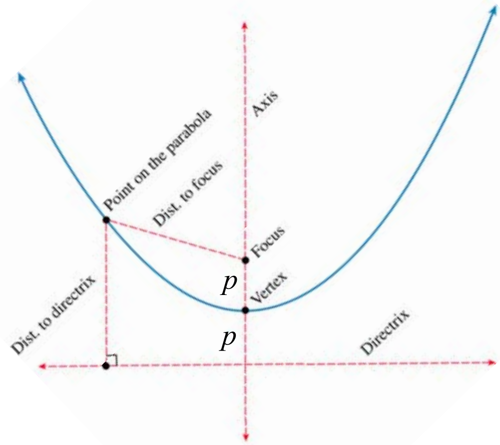
DEGENERATE CONICS:



Point: plane through cone's vertex only   Single line: plane tangent to cone   Intersecting lines

Parabola: a locus of points in a plane equidistant from a fixed point (**focus**) and a fixed line (**directrix**)

p = distance between focus to directrix and vertex
Focal Width = $4p$: the distance across the parabola at the focus



$y=a(x-h)^2+k$

$a=\frac{1}{4p}$

$y=\frac{1}{4p}(x-h)^2+k$

Parabolas with the Vertex (h,k):

Standard Equation	$(x-h)^2=4p(y-k)$	$(y-k)^2=4p(x-h)$
Opens	Upward or downward	To the right or left
Focus	$(h,k+p)$	$(h+p,k)$
Directrix	$y=k-p$	$x=h-p$
Axis	$x=h$	$y=k$
Focal Length	p	p
Focal Width	$\lvert 4p\rvert$	$\lvert 4p\rvert$

Note: Do NOT try to memorize this table. Determine first which way the parabola opens. Find the vertex, then recall that p is the distance from the vertex to the focus and the vertex to the directrix.

Graphing a Parabola

Ex: Find the vertex, focus, and directrix of $(x-3)^2=4(y+1)$. Then graph the parabola.

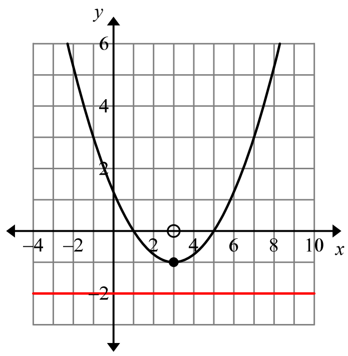
$(x-3)^2=4(y+1)$ is in the form $(x-h)^2=4p(y-k)$. Vertex: $(h,k)=(3,-1)$

Because x is being squared and $4p$ is positive, the parabola opens UP.

Focal Length: $4p = 4 \Rightarrow p = 1$ Focus: $(h, k+p) = (3, -1+1) = (3, 0)$ (1 unit above the vertex)

Directrix: $y = k - p \Rightarrow y = -1 - 1 \Rightarrow y = -2$ (1 unit below the vertex, horizontal line)

Graph:



Ex: Find the vertex, focus, and directrix of $y^2 + 6y + 8x + 25 = 0$. Then graph the parabola.

Write in standard form (complete the square):

$$y^2 + 6y + 8x + 25 = 0 \Rightarrow y^2 + 6y + 9 = -8x - 25 + 9 \Rightarrow (y+3)^2 = -8x - 16 \Rightarrow (y+3)^2 = -8(x+2)$$

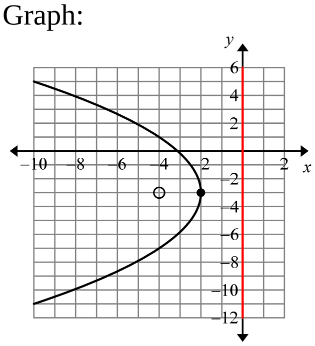
Vertex: $(-2, -3)$ Because y is being squared and $4p$ is negative, the parabola opens to the LEFT.

Focal Length: $4p = -8 \Rightarrow p = -2$

Focus: $(h+p, k) = (-2-2, -3) = (-4, -3)$ (2 units to the left of the vertex)

Directrix: $x = h - p \Rightarrow x = -2 + 2 \Rightarrow x = 0$ (2 units to the right of the vertex, vertical line)

Graph:



<u>Writing an Equation of a Parabola</u>

Ex: Write an equation in standard form of a parabola with vertex $(4, 0)$ and directrix $x = 5$.

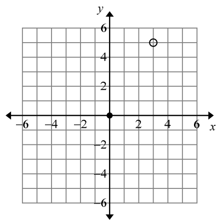
Because the directrix is a vertical line and the vertex is to the left of the directrix, the parabola opens left.

Use the equation $(y-k)^2 = 4p(x-h)$. Substitute $(4,0) = (h,k)$: $(y-0)^2 = 4p(x-4)$

The directrix is 1 unit from the vertex, so $p = -1$. ($p < 0$ because it opens left.)

Equation: $(y-0)^2 = 4(-1)(x-4) \Rightarrow \boxed{y^2 = -4(x-4)}$

Ex: Write an equation in standard form of a parabola with vertex $(0,0)$ and passes through the point $(3,5)$.



By the placement of the vertex and the point given, we can see that the parabola opens up.

Use the equation $(x-h)^2 = 4p(y-k)$. Substitute $(0,0)=(h,k)$: $(x-0)^2 = 4p(y-0) \Rightarrow x^2 = 4py$

Substitute the point $(3,5)=(x,y)$ and solve for *p*: $x^2 = 4py \Rightarrow 3^2 = 4p(5) \Rightarrow p = \frac{9}{20}$

$$x^2 = 4\left(\frac{9}{20}\right)y \Rightarrow \boxed{x^2 = \frac{9}{5}y}$$

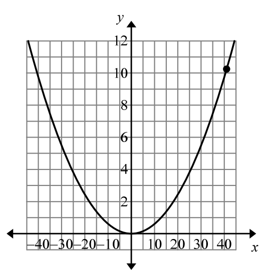
Paraboloid of Revolution: the result of rotating a parabola in three-dimensional space about its axis
Examples – flashlights, headlights, searchlights, satellite dishes, etc.

Application Problem – Paraboloid

Ex: A satellite dish is 82 feet in diameter. If it has a depth of 10.25 feet, how far from the vertex should the receiving antenna be placed? (Hint: The antenna should be placed at the focus.)

A parabola that represents the situation has a vertex of $(0,0)$ and passes through the point $(41,10.25)$.

Graph:



Equation of the parabola: $(x-h)^2 = 4p(y-k) \Rightarrow x^2 = 4py$

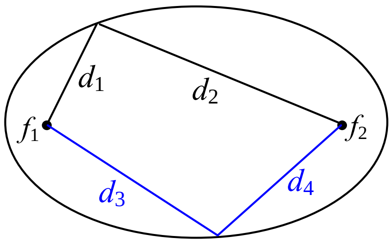
$(41,10.25)$: $41^2 = 4p(10.25) \Rightarrow p = 41$ **The focus is 41 ft from the vertex**.

You Try: Write an equation in standard form of the parabola with a focus of $(2,2)$ and directrix $x=-2$.
Hint: Remember that the vertex is halfway between the focus and the directrix.

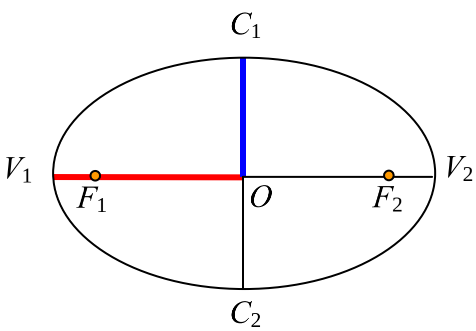
QOD: Explain the relationship between the vertex, focus, and directrix and how these help you determine the orientation of the parabola.

Syllabus Objectives: 2.1 – The student will graph relations or functions, including real-world applications. (Ellipses)

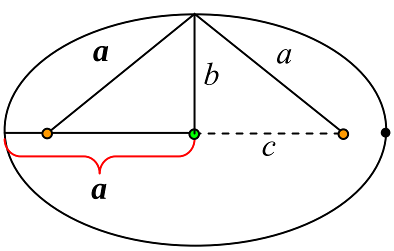
Ellipse: the locus of points where the sum of the distances to two fixed points (**foci**) is constant
Plural of focus: foci



$$d_1 + d_2 = d_3 + d_4$$



O: center $(0,0)$
V: vertices $(-a,0)$ & $(a,0)$
F: foci $(-c,0)$ & $(c,0)$
C: covertices $(-b,0)$ & $(b,0)$
Major Axis: $2a$ Minor Axis: $2b$
Length of Semimajor Axis: a
Length of Semiminor Axis: b
Distance from Center to Focus: c
$c^2 = a^2 - b^2$



The distance from the center to a vertex is b, and the distance from the center to a covertex is a. If the major axis is vertical, all of this is reversed. In an ellipse, a is associated with the major axis.

Eccentricity: a value that describes the "roundness" of the ellipse $\boldsymbol{e = \frac{c}{a}}$

In an ellipse, $0 < e < 1$. If $e = 0$, the graph is a circle. If $e = 1$, the graph is a line.

Don't confuse this e with the number e used with exponential and logarithmic equations!

Equation of an Ellipse: $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ or $\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$

a is the larger number (represents the major axis)

<u>Graphing an Ellipse</u>

Ex: Graph the ellipse and state its eccentricity. Find the foci. $\frac{(x-2)^2}{9}+\frac{(y+1)^2}{25}=1$

Center (h,k): $(2,-1)$ $\quad a^2=25 \quad b^2=9$

$a=5 \quad b=3$

Major axis is vertical because a is under the y-term.

Vertices: $(2,-1+a)\Rightarrow(2,-1+5)\Rightarrow(2,4)$ and $(2,-1-a)\Rightarrow(2,-1-5)\Rightarrow(2,-6)$

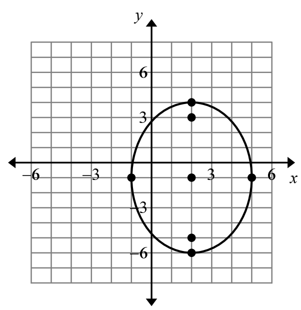
Covertices: $(2+b,-1)\Rightarrow(2+3,-1)\Rightarrow(5,-1)$ and $(2-b,-1)\Rightarrow(2-3,-1)\Rightarrow(-1,-1)$

Length of Major Axis: $2a=10$ Length of Minor Axis: $2b=6$

Foci: $c^2=a^2-b^2 \Rightarrow c^2=5^2-3^2 \Rightarrow c^2=16 \Rightarrow c=4$

$(2,-1+c)\Rightarrow(2,-1+4)\Rightarrow(2,3)$ and $(2,-1-c)\Rightarrow(2,-1-4)\Rightarrow(2,-5)$

Eccentricity: $e=\frac{c}{a}=\frac{4}{5}$ Graph:



<u>Transform General Form to Standard Form</u>

Ex: Write the equation in standard form. $9x^2+5y^2+36x-30y+36=0$

Complete the square:

$9(x^2+4x)+5(y^2-6y)=-36 \Rightarrow 9(x^2+4x+4)+5(y^2-6y+9)=-36+36+45 \Rightarrow 9(x+2)^2+5(y-3)^2=45$

Set equal to 1: $9(x+2)^2+5(y-3)^2=45 \Rightarrow \frac{9(x+2)^2}{45}+\frac{5(y-3)^2}{45}=\frac{45}{45} \Rightarrow \frac{(x+2)^2}{5}+\frac{(y-3)^2}{9}=1$

<u>Writing an Equation of an Ellipse</u>

Ex: Write an equation of an ellipse if the foci are $(0,-3)$ and $(0,3)$ and the major axis has a length of $2\sqrt{13}$.

Center – halfway between the foci: $(0,0)$ Major Axis: $2a=2\sqrt{13} \Rightarrow a=\sqrt{13}$

Distance from center to a focus: $c=3$ $\quad c^2=a^2-b^2 \Rightarrow 9=13-b^2 \Rightarrow b=2$

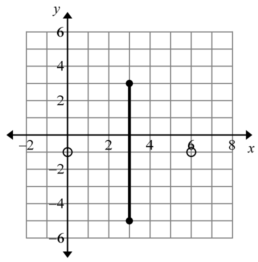
The foci are on the major axis, so the major axis is vertical (a under y-term).

$\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$: $\boxed{\frac{x^2}{4}+\frac{y^2}{13}=1}$

Ex: Write an equation of an ellipse if a focus is $(0,-1)$ and a covertex is $(3,3)$.

Covertex – endpoint of the minor axis. Reflect over the major axis to find the other covertex, $(3,-5)$.

Focus – reflect over the minor axis to find the other focus, $(6,-1)$.



Center: Intersection of the major and minor axes. $(3,-1)$

Distance between center and covertex: $b=4$ Distance between center and focus: $c=3$

$$c^2=a^2-b^2 \Rightarrow 9=a^2-16 \Rightarrow a=5$$

Equation (major axis is horizontal): $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1 \Rightarrow \boxed{\frac{(x-3)^2}{25}+\frac{(y+1)^2}{16}=1}$

Application Problem – Orbits: planets, asteroids, and comets all follow elliptical paths as they orbit the Sun.

Ex: The orbit of Halley's comet is 36.18 AU long and 9.12 AU wide. Write an equation that represents its elliptical orbit.

Note: An AU (astronomical unit) is a unit of length equal to approx. 150 million km (93 million miles). It is the mean distance between the Earth and the Sun over one Earth orbit.

Major axis: $2a=36.18$, $a=18.09$ Minor Axis: $2b=9.12$, $b=4.56$ Center: $(0,0)$

Equation: $\frac{x^2}{(18.09)^2}+\frac{y^2}{(4.56)^2}=1 \Rightarrow \boxed{\frac{x^2}{327.25}+\frac{y^2}{20.79}=1}$

You Try: Graph the ellipse. State the eccentricity and foci. $\frac{x^2}{16}+\frac{(y-2)^2}{4}=1$

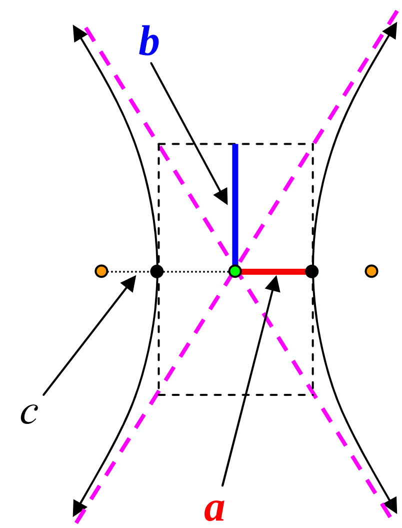
QOD: A "whispering gallery" is shaped like an ellipse. If a person stands at one focus and whispers, he can be heard clearly by a person standing at the other focus. Why?

Syllabus Objectives: 2.1 – The student will graph relations or functions, including real-world applications. (Hyperbolas)

<u>Hyperbola</u>: the set of all points in a plane whose distances from two fixed points (**foci**) in the plane have a constant difference (**2*a***)

Standard Form of the Equation of a Hyperbola: $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$ or $\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$

Note: In the equation of a hyperbola, *a* is always associated with the positive (first) term.



a: the distance from the center to the vertex
c: the distance from the center to a focus
$c^2=a^2+b^2$
Transverse Axis: $2a$
Semitransverse Axis: a
Conjugate Axis: $2b$
Semiconjugate Axis: b

The hyperbola shown has an equation of the form $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$ (opens left and right).

Vertices: $(h\pm a,k)$ Foci: $(h\pm c,k)$

The hyperbola with the equation $\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$ opens up and down.

Vertices: $(h,k\pm a)$ Foci: $(h,k\pm c)$

<u>Asymptotes</u>: a hyperbola has two slant asymptotes

$y-k=\pm\frac{b}{a}(x-h)$ (If hyperbola opens left and right, b = rise, a = run.)

or $y-k=\pm\frac{a}{b}(x-h)$ (If hyperbola opens up and down, a = rise, b = run.)

<u>Eccentricity</u>: $e=\frac{c}{a}$ For a hyperbola, $e>1$.

Discuss: What happens as *c* approaches 1?

Graphing a Hyperbola

Ex: Sketch the graph of the hyperbola. State the center, vertices, foci, asymptotes, and eccentricity. $\frac{(x-2)^2}{16}-\frac{(y+1)^2}{9}=1$

Center: $(h,k)=(2,-1)$ Hyperbola opens right and left.

Vertices are to the right and left of the center.

$a^2=16 \Rightarrow a=4$: $(2+4,-1)=(6,-1)$ & $(2-4,-1)=(-2,-1)$

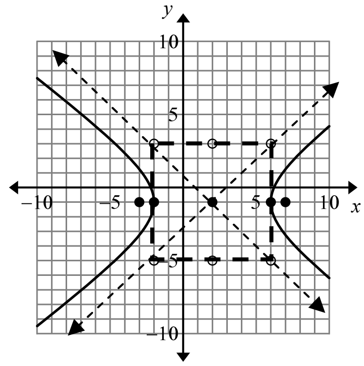
$c^2=a^2+b^2 \Rightarrow c^2=16+9 \Rightarrow c=5$

Foci are c units to the right and left of the center.

$(2+5,-1)=(7,-1)$ & $(2-5,-1)=(-3,-1)$

Asymptotes: $y-k=\pm\frac{b}{a}(x-h) \Rightarrow y+1=\pm\frac{3}{4}(x-2)$ Eccentricity: $e=\frac{c}{a}=\frac{5}{4}$

Graph: (Drawing a rectangle with the asymptotes as diagonals helps to sketch the graph.)



Ex: Sketch the graph of the hyperbola. State the center, vertices, foci, asymptotes, and eccentricity. $\frac{(y-2)^2}{16}-\frac{x^2}{4}=1$

Center: $(h,k)=(0,2)$ Hyperbola opens up and down.

Vertices are above and below center.

$a^2=16 \Rightarrow a=4$: $(0,2+4)=(0,6)$ & $(0,2-4)=(0,-2)$

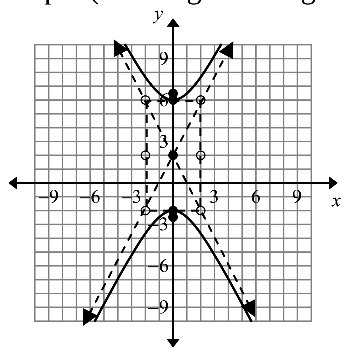
$c^2=a^2+b^2 \Rightarrow c^2=16+4 \Rightarrow c=\sqrt{20}=2\sqrt{5}$

Foci are c units above and below the center. $(0,2+2\sqrt{5})$ & $(0,-2\sqrt{5})$

Asymptotes: $y-k=\pm\frac{a}{b}(x-h) \Rightarrow y-2=\pm\frac{4}{2}(x-0) \Rightarrow y-2=\pm 2x$

Eccentricity: $e=\frac{c}{a}=\frac{2\sqrt{5}}{4}=\frac{\sqrt{5}}{2}$

Graph: (Drawing a rectangle with the asymptotes as diagonals helps to sketch the graph.)



Transform General Form to Standard Form

Ex: Write the standard equation of $x^2-4y^2-2x-16y-11=0$.

Complete the square:

$$x^2-2x-4y^2-16y=11 \Rightarrow \left(x^2-2x+1\right)-4\left(y^2+4y+4\right)=11+1-16 \Rightarrow (x-1)^2-4(y+2)^2=-4$$

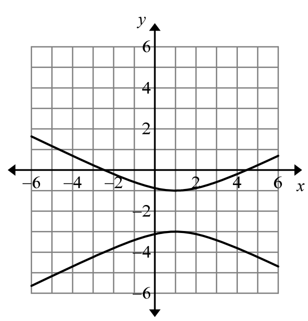
Set equal to 1: $\dfrac{(x-1)^2}{-4}-\dfrac{4(y+2)^2}{-4}=\dfrac{-4}{-4} \Rightarrow \boxed{\dfrac{(y+2)^2}{1}-\dfrac{(x-1)^2}{4}=1}$

If students need more practice:

Center: $(1,-2)$ Vertices: $(1,-1)$, $(1,-3)$ Foci: $\left(1,-2\pm\sqrt{5}\right)$

Asymptotes: $y+2=\pm\dfrac{1}{2}(x-1)$ Eccentricity: $e=\sqrt{5}$

Graph:



Writing an Equation of a Hyperbola

Ex: Write an equation of a hyperbola if the foci are $(4,0)$ and $(4,10)$ and the vertices are $(4,1)$ and $(4,9)$.

Center is halfway between the foci (or the vertices): $(4,5)$

a = distance between center and vertices: $a=4$ *c* = distance between center and foci: $c=5$

$c^2=a^2+b^2 \Rightarrow 25=16+b^2 \Rightarrow b=3$ Opens up and down

$$\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1 \Rightarrow \boxed{\frac{(y-5)^2}{16}-\frac{(x-4)^2}{9}=1}$$

Ex: Write an equation of a hyperbola if the vertices are $(-2,1)$ and $(2,1)$ and it passes through the point $(5,4)$.

Center is halfway between the vertices: $(0,1)$ a = distance between center and vertices: $a=2$

Opens left and right: $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1 \Rightarrow \frac{(x-0)^2}{4}-\frac{(y-1)^2}{b^2}=1$

Substitute $(x,y)=(5,4)$ and solve for b^2:

$$\frac{x^2}{4}-\frac{(y-1)^2}{b^2}=1 \Rightarrow \frac{25}{4}-\frac{(4-1)^2}{b^2}=1 \Rightarrow -\frac{9}{b^2}=-\frac{21}{4} \Rightarrow 21b^2=36 \Rightarrow b^2=\frac{12}{7}$$

Equation: $\boxed{\frac{x^2}{4}-\frac{(y-1)^2}{\frac{12}{7}}=1}$

<u>You Try</u>: Write an equation of a hyperbola if the foci are $\left(0,\pm\sqrt{13}\right)$ and the equation of an asymptote is $2y=3x$.

<u>QOD</u>: Explain why the eccentricity of a hyperbola must be greater than 1.

Syllabus Objective: 2.8 – The student will construct the graph of a function under a given <u>translation</u>, dilation, or reflection (or <u>rotation</u>).

<u>Review</u>: Which conic is it? $4x^2+9y^2-16x-18y-11=0$

Complete the square: $4\left(x^2-4x+4\right)+9\left(y^2-2y+1\right)=11+16+9 \Rightarrow 4\left(x-2\right)^2+9\left(y-1\right)^2=36$

Set equal to 1: $\frac{\left(x-2\right)^2}{9}+\frac{\left(y-1\right)^2}{4}=1$ $\boxed{\text{ellipse}}$

<u>Graphing on the Calculator</u>

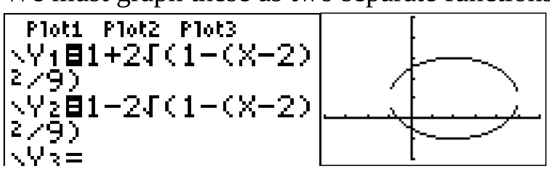
Ex: Graph the ellipse from the example above on the calculator.

To graph on the calculator, we must solve for y: Always use standard form to solve for y.

$$\frac{\left(x-2\right)^2}{9}+\frac{\left(y-1\right)^2}{4}=1 \Rightarrow \frac{\left(y-1\right)^2}{4}=1-\frac{\left(x-2\right)^2}{9} \Rightarrow \left(y-1\right)^2=4-\frac{4}{9}\left(x-2\right)^2$$

$$y-1=\pm\sqrt{4-\frac{4}{9}\left(x-2\right)^2} \Rightarrow y=1\pm 2\sqrt{1-\frac{\left(x-2\right)^2}{9}}$$

We must graph these as two separate functions on the calculator:



Note: It is calculator error that prevents it from enclosing the ellipse.

<u>Discriminant Test</u>: If an equation is written in the form $Ax^2+Bxy+Cy^2+Dx+Ey+F=0$, then the disriminant is B^2-4AC.

$B^2-4AC>0$ Hyperbola

$B^2-4AC=0$ Parabola

$B^2-4AC<0$ Ellipse

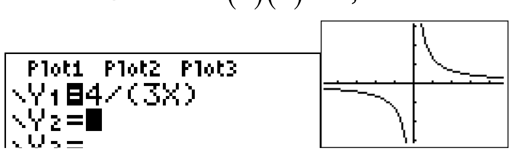
Ex: Verify that the conic in the review is an ellipse using the discriminant.

$4x^2+9y^2-16x-18y-11=0$ $A=4,\ B=0,\ C=9$ $B^2-4AC=0^2-4\left(4\right)\left(9\right)<0$, so ELLIPSE!

Ex: Identify the conic using the discriminant. Then solve for y and graph. $3xy-4=0$

$A=0,\ B=3,\ C=0$ $B^2-4AC=9^2-4\left(0\right)\left(0\right)>0$, so HYPERBOLA

$3xy-4=0 \Rightarrow y=\frac{4}{3x}$



<u>Translation</u>: a slide

<u>Translation of Axes</u>: produces a new set of translated axes parallel to the original axes

<u>Translation Formulas</u>: $x = x' + h$ and $y = y' + k$ or $x' = x - h$ and $y' = y - k$

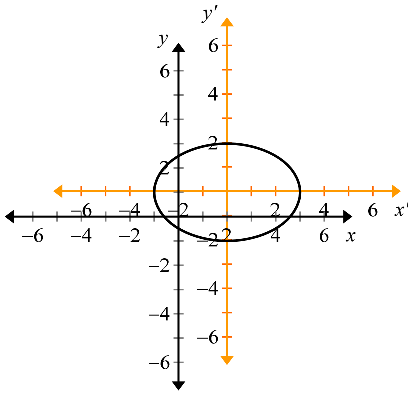
Ex: Sketch the graph of the ellipse. Translate the coordinate axes so that the origin is at the center of the ellipse. $4x^2 + 9y^2 - 16x - 18y - 11 = 0$

Standard form:

$$4x^2 + 9y^2 - 16x - 18y - 11 = 0 \Rightarrow 4\left(x^2 - 4x + 4\right) + 9\left(y^2 - 2y + 1\right) = 11 + 16 + 9$$

$$4(x-2)^2 + 9(y-1)^2 = 36 \Rightarrow \frac{(x-2)^2}{9} + \frac{(y-1)^2}{4} = 1$$

Let $x' = x - 2$ and $y' = y - 1$: $\frac{(x')^2}{9} + \frac{(y')^2}{4} = 1$ Graph with original *xy*-axes overlaid:



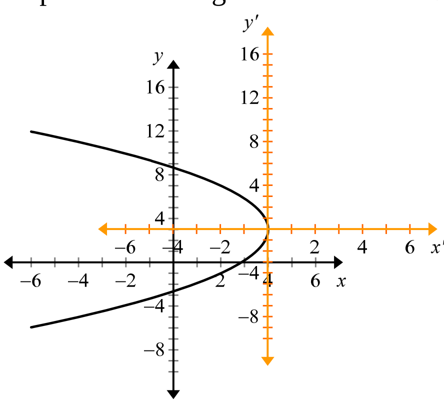
Ex: Identify the type of conic, write the equation in standard form, translate the conic to the origin, and sketch it in the transformed system. $-y^2 - 8x + 6y + 23 = 0$

Discriminant: $A = 0,\ B = 0,\ C = -1$ $B^2 - 4AC = 0^2 - 4(0)(-1) = 0$, so PARABOLA

Standard form: $-y^2 - 8x + 6y + 23 = 0 \Rightarrow -\left(y^2 - 6y + 9\right) = 8x - 23 - 9 \Rightarrow (y-3)^2 = -8(x-4)$

Let $x' = x - 4$ and $y' = y - 3$: $(y')^2 = -8(x')$

Graph with the original axes overlaid:



<u>Rotation of Axes:</u> the origin stays fixed, and the x- and y-axes are rotated through an angle α

<u>Rotation Formulas</u>: $0 < \alpha < 2\pi$, α is the angle of rotation

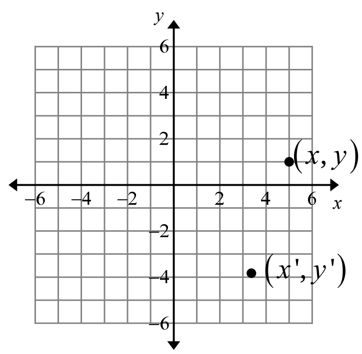
$$x = x'\cos\alpha - y'\sin\alpha \qquad y = x'\sin\alpha + y'\cos\alpha \qquad \cot 2\alpha = \frac{A-C}{B},\ 0 < \alpha < \frac{\pi}{2}$$

$$x' = x\cos\alpha + y\sin\alpha \qquad y' = -x\sin\alpha + y\cos\alpha$$

Ex: Using the point $P(x, y)$ and the rotation information, find the coordinates of P in the rotated $x'y'$ system. $P(5,1)$, $\alpha = 60^\circ$

$$P(x,y) = P(5,1): \quad x' = x\cos\alpha + y\sin\alpha \Rightarrow x' = 5\cos 60^\circ + 1\sin 60^\circ \Rightarrow x' = \frac{5}{2} + \frac{\sqrt{3}}{2}$$

$$y' = -x\sin\alpha + y\cos\alpha \Rightarrow y' = -5\sin 60^\circ + 1\cos 60^\circ \Rightarrow y' = -\frac{5\sqrt{3}}{2} + \frac{1}{2}$$



$$(x', y') = \left(\frac{5}{2} + \frac{\sqrt{3}}{2}, -\frac{5\sqrt{3}}{2} + \frac{1}{2}\right)$$

<u>You Try:</u> Identify the type of conic, write the equation in standard form, translate the conic to the origin, and sketch it in the translated system. $-y^2 - 8x + 6y + 23 = 0$

<u>QOD</u>: How is the discriminant of a conic section in general form different from the discriminant used with quadratic equations (parabolas)?

Syllabus Objective: 6.2 – The student will transform functions between Cartesian and polar form.

Conic Section: the locus of points (P) in a plane such that the ratio of the distances from P to F (focus – pole) to the distances from P to D (directrix) equals a constant: e (**eccentricity**)

Recall: Ellipse: $0 < e < 1$ Parabola: $e = 1$ Hyperbola: $e > 1$

Polar Form:

Vertical Directrix $x = \pm k$: $r = \dfrac{ek}{1 \pm e\cos\theta}$ Horizontal Directrix $y = \pm k$: $r = \dfrac{ek}{1 \pm e\sin\theta}$

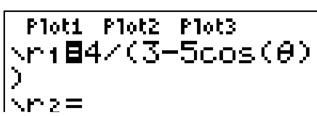
$k = 2p$: the distance from the focus to the directrix

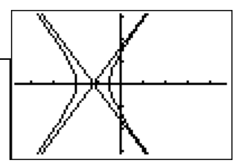
Ex: Determine the eccentricity, type of conic, and directrix. Then graph. $r = \dfrac{4}{3 - 5\cos\theta}$

Divide every term by 3 to write in polar form: $r = \dfrac{\frac{4}{3}}{\frac{3}{3} - \frac{5}{3}\cos\theta} \Rightarrow r = \dfrac{\frac{4}{3}}{1 - \frac{5}{3}\cos\theta}$

$r = \dfrac{ek}{1 - e\cos\theta}$ Eccentricity: $\boxed{e = \frac{5}{3}} > 1$, so HYPERBOLA

Directrix: $ek = \frac{4}{3} \Rightarrow \frac{5}{3}k = \frac{4}{3} \Rightarrow k = \frac{4}{5}$ Vertical Directrix $x = -k$: $\boxed{x = -\frac{4}{5}}$

Graph in Polar Mode: 



Note: Because the calculator connects the pixels, it appears as if it is drawing the asymptotes. It really is not!

Ex: Determine the eccentricity, type of conic, and directrix. $r = \dfrac{12}{6 - 2\sin\theta}$

Divide every term by 6 to write in polar form: $r = \dfrac{\frac{12}{6}}{\frac{6}{6} - \frac{2}{6}\sin\theta} \Rightarrow r = \dfrac{2}{1 - \frac{1}{3}\sin\theta}$

$r = \dfrac{ek}{1 - e\sin\theta}$ Eccentricity: $0 < \boxed{e = \frac{1}{3}} < 1$, so ELLIPSE

Directrix: $ek = 2 \Rightarrow \frac{1}{3}k = 2 \Rightarrow k = 6$ Horizontal Directrix $y = -k$: $\boxed{y = -6}$

Writing a Conic Section in Polar Form

Ex: Find a polar equation of the parabola with its focus at the pole and directrix $x = 2$.

Parabola eccentricity: $e = 1$ Vertical Directrix: Use $r = \dfrac{ek}{1 + e\cos\theta}$ $k = 2$

$$r = \frac{(1)(2)}{1 + (1)\cos\theta} \Rightarrow \boxed{r = \frac{2}{1 + \cos\theta}}$$

Ex: Find the polar equation of a parabola with the focus at the pole and vertex $(2,0)$.

Parabola eccentricity: $e = 1$

Vertex is $(2,0)$, so $p = 2$ (the focus/pole is 2 units to the left of the vertex)

Directrix is 2 units to the right of the vertex, and is vertical: $x = 4$, so $k = 4$

$$r = \frac{(1)(4)}{1 + (1)\cos\theta} \Rightarrow \boxed{r = \frac{4}{1 + \cos\theta}}$$

Ex: Find the polar equation of a hyperbola with the focus at the pole and vertices $(2,0)$ and $(6,0)$.

Center (halfway between vertices): $(4,0)$ Distance from center to pole/focus: $c = 4$

Distance from center to a vertex: $a = 2$ Eccentricity: $e = \dfrac{c}{a} = \dfrac{4}{2} = 2$

Solve for *k*, using $r = 2$ when $\theta = 0^\circ$: $2 = \dfrac{2k}{1 + 2\cos 0^\circ} \Rightarrow 2 = \dfrac{2}{3}k \Rightarrow k = 3$

Equation: $\boxed{r = \dfrac{6}{1 + 2\cos\theta}}$

Ex: Find the polar equation of a conic section with the focus at the pole, $e = 6$ and directrix $r = -2\csc\theta$.

$r = -2\csc\theta \Rightarrow r = \dfrac{-2}{\sin\theta} \Rightarrow r\sin\theta = -2 \Rightarrow y = -2$ $e = 6$

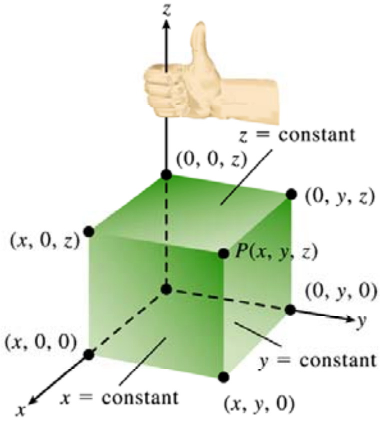
Equation: $r = \dfrac{(6)(2)}{1 - 6\sin\theta} \Rightarrow \boxed{r = \dfrac{12}{1 - 6\sin\theta}}$

You Try: Find the polar equation of a conic section with the focus at the pole, $e = 6$, and directrix $r = -3\sec\theta$.

QOD: What is the advantage to writing equations of conic sections in polar form?

Objective: The student will graph three-dimensional figures and analyze properties and equations in space. (Note: This is NOT in the course syllabus, but should be taught if time allows.)

Three-Dimensional Cartesian Coordinate System: consists of 3 axes, x, y, and z; consists of 3 coordinate planes, the xy-plane, the xz-plane, and the yz-plane; divided into 8 **octants**



Note: Octant I is the octant in which all three coordinates are positive.

Distance Formula: the distance d between the points $P(x_1, y_1, z_1)$ & $Q(x_2, y_2, z_2)$

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

Ex: Find the distance between the points $(-1,2,3)$ & $(4,5,-6)$.

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2} \Rightarrow d = \sqrt{(-1-4)^2 + (2-5)^2 + (3-(-6))^2} \Rightarrow d = \sqrt{25+9+81} = \boxed{\sqrt{115}}$$

Midpoint Formula: the midpoint of the line segment PQ with endpoints $P(x_1, y_1, z_1)$ & $Q(x_2, y_2, z_2)$

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$

Ex: Find the midpoint of the line segment with endpoints $(-1,2,3)$ & $(4,5,-6)$.

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right) \Rightarrow M = \left(\frac{-1+4}{2}, \frac{2+5}{2}, \frac{3+(-6)}{2}\right) = \boxed{\left(\frac{3}{2}, \frac{7}{2}, -\frac{3}{2}\right)}$$

Sphere: the set of points in space that lie a fixed distance (called the **radius**) from a fixed point called the **center**

Equation of a Sphere: with center (h,k,l) and radius r $\quad (x-h)^2 + (y-k)^2 + (z-l)^2 = r^2$

Ex: Write an equation of the sphere with radius 4 and center $(3,-5,1)$.

$$(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2 \Rightarrow \boxed{(x-3)^2 + (y+5)^2 + (z-1)^2 = 16}$$

Equation of a Plane: $Ax + By + Cz + D = 0$, where A, B, and C are not all zero

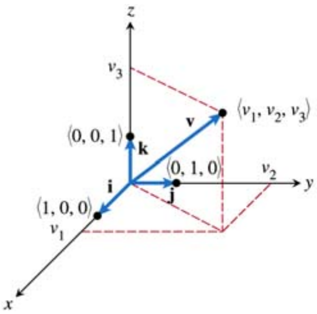
Ex: Sketch the plane $15x+12y+30z=60$.

Find the intercepts: $(4,0,0)$, $(0,5,0)$, $(0,0,2)$

Plot the points, draw the triangle they form, and shade to show the plane.

<u>Vectors in Space</u>: represent forces, displacements, and velocities in three dimensions $\mathbf{v}=\langle v_1,v_2,v_3\rangle$

Operations with vectors in 3-dimensions are the same as operations in 2-dimensions.



Ex: Simplify. $\langle 1,6,-3\rangle-2\langle -4,0,3\rangle$

$\langle 1,6,-3\rangle-2\langle -4,0,3\rangle=\langle 1,6,-3\rangle+\langle 8,0,-6\rangle=\boxed{\langle 9,6,-9\rangle}$

<u>Magnitude of a Vector</u>

Ex: Find $\|\mathbf{v}\|$ if $\mathbf{v}=\langle 5,1,-2\rangle$.

$\|\mathbf{v}\|=\sqrt{5^2+1^2+(-2)^2}=\sqrt{25+1+4}=\boxed{\sqrt{30}}$

<u>Dot Product</u>

Ex: Find the dot product $\langle 3,-7,1\rangle\cdot\langle -5,2,0\rangle$.

$\langle 3,-7,1\rangle\cdot\langle -5,2,0\rangle=3(-5)+(-7)(2)+1(0)=-15-14+0=\boxed{-29}$

<u>Equation of a Line in Space – Vector and Parametric Form</u>: a line through a point $P_0(x_0,y_0,z_0)$ in the direction of vector $\mathbf{v}=\langle a,b,c\rangle$

Vector Form: $\mathbf{r}=\mathbf{r}_0+t\mathbf{v}$, $\mathbf{r}=\langle x,y,z\rangle$ and $\mathbf{r}_0=\langle x_0,y_0,z_0\rangle$

Parametric Form: $x=x_0+at,\ y=y_0+bt,\ z=z_0+ct$, t is a real number

Ex: Write the equations of a line that passes through the points $(1,-2,5)$ & $(3,1,-4)$ in vector and parametric forms.

Direction Vector: $\mathbf{v}=\langle 3-1,1-(-2),-4-5\rangle=\langle 2,3,-9\rangle$

$\mathbf{r}=\mathbf{r}_0+t\mathbf{v}:\mathbf{r}=\langle 1,-2,5\rangle+t\langle 2,3,-9\rangle \Rightarrow \langle x,y,z\rangle=\langle 1+2t,-2+3t,5-9t\rangle$ or

$(1+2t)\mathbf{i}+(-2+3t)\mathbf{j}+(5-9t)\mathbf{k}$

Parametric Equations: $x=1+2t,\ y=-2+3t,\ z=5-9t$