

Syllabus Objectives: 8.1 – The student will solve a given system of equations or system of inequalities.

Solution of a System of Linear Equations: the ordered pair (x, y) where the lines intersect

A solution can be substituted into both equations to make true statements.

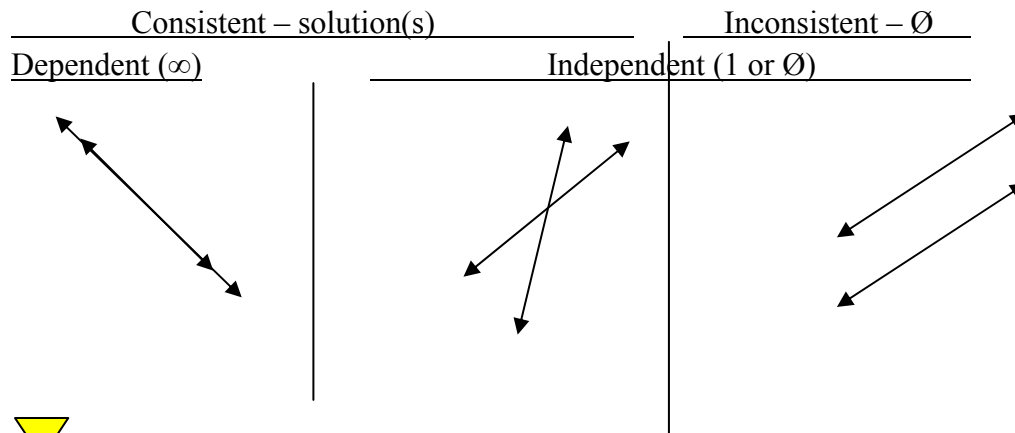
Special Cases:

- If the graphs of the equations in a system are parallel (do not intersect), then the system has NO SOLUTION. Systems with no solution are called **inconsistent**.

Systems with solutions are called **consistent**.

- If the graphs of the equations in a system are the same line (coincident), then the system has INFINITELY MANY SOLUTIONS. Systems with infinite solutions are called **dependent**.

Note: If the graphs are not coincident and intersect, then the system has EXACTLY ONE SOLUTION.



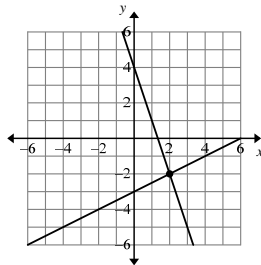
Systems of equations that are dependent graph lines are **coincident**.

Solving Linear Systems by Graphing

Ex: Graph and determine the point of intersection.

$$\begin{aligned} 3x + y &= 4 \\ x - 2y &= 6 \end{aligned}$$

Graph both lines on the same coordinate plane.



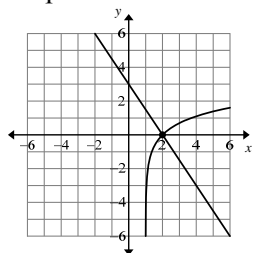
The lines intersect at $(2, -2)$.

Solving a System by Graphing (not linear)

Ex: Graph and find the solution of the system.

$$\begin{aligned} 3x + 2y &= 6 \\ y &= \ln(x - 1) \end{aligned}$$

Graph both functions on the same coordinate plane.



The solution of the system is the point at which the functions intersect: $\boxed{(2, 0)}$

Solving a System of Linear Equations by Substitution

Ex: Solve using substitution.

$$\begin{aligned} -3x + y &= 7 \\ -6x + 2y &= -8 \end{aligned}$$

Step One: Solve one of the equations for one of the variables (if necessary). Note: You may choose which variable to solve for.

$$\begin{aligned} -3x + y &= 7 \\ y &= 3x + 7 \end{aligned}$$

Step Two: Substitute the expression from Step One into the *other* equation of the system and solve for the other variable.

$$\begin{aligned} -6x + 2(3x + 7) &= -8 \\ -6x + 6x + 14 &= -8 \\ 14 &= -8 \end{aligned}$$

Note: Both variables were eliminated, and the resulting equation is an untrue statement. If the resulting equation was a true statement, then the lines are coincident and there are infinitely many solutions.

$\boxed{\text{No Solution}}$ (The lines are parallel.)

Solving Nonlinear Systems by **Substitution**

Ex: Solve the system. $x^2 + y^2 = 13$
 $y = x^2 - 1$

Step One: Solve one of the equations for one of the variables (if necessary).

$$y = x^2 - 1$$

$$x^2 = y + 1$$

Step Two: Substitute the expression from Step One into the *other* equation of the system and solve for the other variable.

$$x^2 + y^2 = 13$$

$$(y + 1) + y^2 = 13$$

$$y^2 + y - 12 = 0$$

$$(y + 4)(y - 3) = 0$$

$$y = -4, 3$$

Step Three: Substitute the value from Step Two into the equation from Step One and solve for the remaining variable.

$$x^2 = y + 1$$

$$x^2 = -4 + 1$$

$$x^2 = -3$$

No Solution

$$x^2 = 3 + 1$$

$$x^2 = 4$$

$$x = \pm 2$$

Step Four: Write your answer as an ordered pair and check in both of the original equations.

Solutions:

$$\boxed{(-2, 3), (2, 3)}$$



Note: We solved for x^2 instead of substituting the expression for y to avoid a high-degree polynomial equation to solve.

Solving Linear Systems by Linear Combinations (Elimination)

Ex: Solve the system. $-3y = -5x + 1$
 $4x - 6y = -10$

Step One: Write the two equations in standard form.

$$5x - 3y = 1$$

$$4x - 6y = -10$$

Step Two: Multiply one or both of the equations by a constant to obtain coefficients that are opposites for one of the variables.

$$5x - 3y = 1 \Rightarrow -10x + 6y = -2$$

$$4x - 6y = -10 \Rightarrow 4x - 6y = -10$$

Step Three: Add the two equations from Step Two. One of the variable terms should be eliminated. Solve for the remaining variable.

$$-6x + 0y = -12$$

$$-6x = -12$$

$$x = 2$$

$$5(2) - 3y = 1$$

Step Four: Substitute the value from Step Three into either one of the original equations to solve for the other variable.

$$10 - 3y = 1$$

$$-3y = -9$$

$$y = 3$$

Step Five: Write your answer as an ordered pair and check in the original system.

Solution: $\boxed{(2, 3)}$

Application Problems – Systems of Equations

Ex: Three gallons of a mixture is 60% water by volume. Determine the number of gallons of water that must be added to bring the mixture to 75% water.

Assign labels: w = gallons of water; t = total gallons of mixture

Write a system:

Amount (gallons)	$3 + w = t$
Value (percent water)	$3(0.60) + w(1.00) = t(0.75)$

Solve the system: Rewrite second equation. $1.8 + w = .75t \Rightarrow 180 + 100w = 75t$

Use substitution.	$180 + 100w = 75(3 + w)$	$25w = 45$
	$180 + 100w = 225 + 75w$	$w = \boxed{1.8 \text{ gallons}}$

Ex: A machine takes 3 minutes to form a bowl and 2 minutes to form a plate. The material costs \$0.25 for a bowl and \$0.20 for a plate. How many bowls and plates were made if the machine ran for 8 hours and \$44 was spent on materials?

Assign labels: $b = \#$ of bowls; $p = \#$ of plates

Write a system: Time $3b + 2p = 8(60)$ Multiply by 60 to change hours to minutes.
 Money $0.25b + 0.20p = 44$

Solve the system: Rewrite equations. $3b + 2p = 480$
 $25b + 20p = 4400$

Use elimination.	$30b + 20p = 4800$	$3(80) + 2p = 480$	Solution: 80 bowls, 120 plates
	$\underline{-25b - 20p = -4400}$	$2p = 240$	
	$5b = 400$	$p = 120$	
	$b = 80$		

You Try:

1. Solve the system:
$$\begin{cases} x + y = 6 \\ (x + y)^2 + \frac{x + y}{6} = x \end{cases}$$

(Hint: You do not need to substitute for a single variable. You can use an expression.)

2. A car radiator contains 10 quarts of a 30% antifreeze solution. How many quarts will have to be replaced with pure antifreeze if the resulting solution is to be 50% antifreeze?

QOD: Explain how you would choose the most efficient method for solving a system of equations.

Syllabus Objectives: 8.2 – The student will calculate the determinant of a square matrix. 8.3 – The student will find the inverse of a given square matrix.

Matrix: a rectangular arrangement of numbers (called **entries**) in rows and columns.

Plural of matrix: matrices

Dimensions (Order) of a Matrix: Number of Rows \times (“by”) Number of Columns

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 5 \\ 3 & -2 \end{bmatrix}$$

Entry/Element

$a_{32} = -2$ (row 3, column 2)

Matrix A is a 3x2 matrix.

Square Matrix: a matrix with the same number of rows and columns

Ex: Determine the order of the matrix. Is it a square matrix? Identify b_{24} .

$$B = \begin{bmatrix} 2 & 5 & 1 & 4 \\ 4 & 8 & 0 & 6 \\ -3 & 1 & -2 & -1 \end{bmatrix}$$

Order: Rows by Columns $\boxed{3 \times 4}$ No, it is not a square matrix.

b_{24} : Element in row 2, column 4 $\boxed{b_{24} = 6}$

Adding and Subtracting Matrices: matrices must have the same order

Add or subtract corresponding elements.

Multiplying by a Scalar: multiply all elements of the matrix by the scalar

Ex: Perform the operations. $\begin{bmatrix} 2 & 0 \\ 7 & -1 \end{bmatrix} - 2 \begin{bmatrix} 4 & -8 \\ 0 & 2 \end{bmatrix}$

Multiply by the scalar: $\begin{bmatrix} 2 & 0 \\ 7 & -1 \end{bmatrix} + \begin{bmatrix} -8 & 16 \\ 0 & -4 \end{bmatrix}$

Add the matrices: $\begin{bmatrix} 2+(-8) & 0+16 \\ 7+0 & -1+(-4) \end{bmatrix} = \boxed{\begin{bmatrix} -6 & 16 \\ 7 & -5 \end{bmatrix}}$

Matrix Multiplication: In order to multiply matrices, the number of columns in the first matrix must equal the number of rows in the second matrix. The product matrix will have the same number of rows of the first matrix and the same number of columns of the second matrix.



Matrix multiplication is NOT commutative. ($AB \neq BA$ for all matrices A and B.)

Let A be an $m \times n$ matrix, and B be an $n \times p$ matrix. The product $A \cdot B$ exists, and will have the dimensions $m \times p$.

Ex: Find the product AB (if possible). $A = \begin{bmatrix} 2 & 3 \\ -1 & 5 \\ 0 & -2 \end{bmatrix}$ $B = \begin{bmatrix} 4 & 1 \\ -3 & 6 \end{bmatrix}$

Step One: Determine if the product exists. If it does, find its dimensions.

Matrix A is a 3×2 matrix. Matrix B is a 2×2 matrix. The number of columns in matrix A equals the number of rows in matrix B . Therefore, the product exists and will be a 3×2 matrix.

Step Two: Multiply each entry in the *rows* of matrix A to each entry in the *columns* of matrix B . Then find the sum of these products.

$$\begin{bmatrix} 2(4)+3(-3) & 2(1)+3(6) \\ -1(4)+5(-3) & -1(1)+5(6) \\ 0(4)+-2(-3) & 0(1)+-2(6) \end{bmatrix} = \begin{bmatrix} -1 & 20 \\ -19 & 29 \\ 6 & -12 \end{bmatrix}$$

Ex: Find the product BA (if possible) of the matrices from the previous example.

Step One: Determine if the product exists. If it does, find its dimensions.

Matrix B is a 2×2 matrix. Matrix A is a 3×2 matrix. The number of columns in Matrix B does not equal the number of rows in matrix A . Therefore, the product BA does not exist.



Matrices on the Graphing Calculator

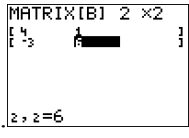
Ex: Evaluate the product from the previous example on the graphing calculator.

In the Matrix Menu, choose Edit, then Matrix A. Our matrix is a 3×2 , so enter in the dimensions. Then enter in the entries of the matrix. Keystrokes:

Keystrokes: 2^{nd} x^{-1} \rightarrow \rightarrow ENTER 3 ENTER 2 ENTER



Enter Matrix B the same way.



On the home screen, you can perform matrix operations.

Keystrokes: 2^{nd} x^{-1} 1 \times 2^{nd} x^{-1} 2 ENTER



Recall:

Identity of Addition: $a + \boxed{0} = a$

Identity of Multiplication: $a \cdot \boxed{1} = a$

Matrix Identity of Addition: $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Matrix Identity of Multiplication: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Ex: Find $\begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

$$\begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix}$$

Recall:

Additive Inverse: $a + \boxed{(-a)} = 0$

Multiplicative Inverse: $a \cdot \boxed{\frac{1}{a}} = 1$

Additive Inverse (matrices): $\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Determinant of a Matrix: If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $\det(A) = |A| = ad - bc$.

Nonsingular Matrix: a matrix that has a nonzero determinant

Inverse of a 2x2 Matrix: If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $A^{-1} = \frac{1}{|A|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, $|A| \neq 0$.

The product of $A \cdot A^{-1} = A^{-1} \cdot A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, the multiplicative identity.



Note: If $|A| = 0$, then A does not have an inverse. Be careful with notation. A^{-1} means the inverse of matrix A , NOT the reciprocal of matrix A .

Ex: Find the inverse of $B = \begin{bmatrix} -4 & 1 \\ 6 & -2 \end{bmatrix}$. Verify your answer.

Evaluate the determinant: $|B| = -4(-2) - 1(6) = 2$

Write the inverse: $B^{-1} = \frac{1}{2} \begin{bmatrix} -2 & -1 \\ -6 & -4 \end{bmatrix} \Rightarrow B^{-1} = \begin{bmatrix} -1 & -\frac{1}{2} \\ -3 & -2 \end{bmatrix}$

Verify by multiplying: $B \cdot B^{-1} = \begin{bmatrix} -4 & 1 \\ 6 & -2 \end{bmatrix} \cdot \begin{bmatrix} -1 & -\frac{1}{2} \\ -3 & -2 \end{bmatrix} = \begin{bmatrix} -4(-1) + 1(-3) & -4(-\frac{1}{2}) + 1(-2) \\ 6(-1) + (-2)(-3) & 6(-\frac{1}{2}) + (-2)(-2) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \odot$

Evaluating the Determinant of a 3×3 Matrix: Lattice Method

Ex: Evaluate the determinant. $\begin{vmatrix} 4 & 2 & -3 \\ 5 & 1 & 0 \\ -2 & 1 & 11 \end{vmatrix}$

Rewrite the first two columns to the right of the 3rd column: $\begin{vmatrix} 4 & 2 & -3 & 4 & 2 \\ 5 & 1 & 0 & 5 & 1 \\ -2 & 1 & 11 & -2 & 1 \end{vmatrix}$

Multiply along the diagonals. Find the sum of the products of the diagonals from left to right and the opposite of the products of the diagonals from right to left. (Blue – Red)

$$\begin{vmatrix} 4 & 2 & -3 & 4 & 2 \\ 5 & 1 & 0 & 5 & 1 \\ -2 & 1 & 11 & -2 & 1 \end{vmatrix}$$

$$\begin{vmatrix} 4 & 2 & -3 \\ 5 & 1 & 0 \\ -2 & 1 & 11 \end{vmatrix} = \boxed{-87}$$

$$-(6 + 0 + 110) + (44 + 0 + -15)$$

$$-116 + 29 = -87$$



Evaluating an Inverse on the Graphing Calculator

Ex: Calculate the determinant of the matrix on the calculator. Then find the inverse and verify.

$$A = \begin{bmatrix} 0 & 2 & -1 \\ 1 & -1 & 1 \\ 2 & 3 & 0 \end{bmatrix}$$



Enter the matrix into the calculator.

$$\boxed{\det([A])} \quad -1$$

On the home screen, find the determinant.

det(can be found in the Matrix Math menu.

Find the inverse: $\begin{bmatrix} [A]^{-1} \\ \begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -5 & -4 & 2 \end{bmatrix} \end{bmatrix}$

Verify: $\begin{bmatrix} [A] * [A]^{-1} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{bmatrix} \quad \begin{bmatrix} [A]^{-1} * [A] \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{bmatrix}$

You Try: Find the A^{-1} . Then verify that it is the inverse of A. $A = \begin{bmatrix} -2 & 6 \\ -1 & 5 \end{bmatrix}$.

QOD: Which matrix operation(s) are commutative? Explain.

Syllabus Objective: 8.4 – The student will solve application problems involving matrix operations with and without technology.

Triangular Form of a System: an equivalent form of a system from which the solution is easy to read; all lead coefficients are equal to 1

Ex: Solve the system in triangular form.

$$\begin{cases} x - 2y - 2z = 7 \\ y - 2z = -7 \\ z = 3 \end{cases}$$

Use back substitution: $y - 2(3) = -7 \Rightarrow y = -1$
 $x - 2(-1) - 2(3) = 7 \Rightarrow x = 11$

Write solution as an ordered triple: $\boxed{(11, -1, 3)}$

Gaussian Elimination: transforming a system to triangular form by

- interchanging any two equations of the system
- multiplying (or dividing) one of the equations by any nonzero real number
- adding a multiple of one equation to any other equation in the system

Ex: Solve the system using Gaussian Elimination.

$$\begin{array}{rcl} 2x + y - z = 0 & \text{Eq. I} \\ x - y + 2z = -3 & \text{Eq. II} \\ -x + 2y - 3z = 7 & \text{Eq. III} \end{array}$$

Switch Eq. I & Eq. II (leading coefficient of 1 on x-term):

$$\begin{array}{rcl} x - y + 2z = -3 & \text{Eq. I} \\ 2x + y - z = 0 & \text{Eq. II} \\ -x + 2y - 3z = 7 & \text{Eq. III} \end{array}$$

Add Eq. I + Eq. III:

$$\begin{array}{rcl} x - y + 2z = -3 & & \\ -x + 2y - 3z = 7 & & \\ \hline y - z = 4 & & \end{array}$$

We now have:

$$\begin{array}{rcl} x - y + 2z = -3 & \text{Eq. I} \\ 2x + y - z = 0 & \text{Eq. II} \\ y - z = 4 & \text{Eq. III} \end{array}$$

Add $-2(\text{Eq. I}) + \text{Eq. II}$:

$$\begin{array}{rcl} -2x + 2y - 4z = 6 & & \\ 2x + y - z = 0 & & \\ \hline 3y - 5z = 6 & & \end{array}$$

We now have:

$$\begin{array}{rcl} x - y + 2z = -3 & \text{Eq. I} \\ 3y - 5z = 6 & \text{Eq. II} \\ y - z = 4 & \text{Eq. III} \end{array}$$

Switch Eq. II & Eq. III (leading coefficient of 1 on y-term):

$$\begin{array}{rcl} x - y + 2z = -3 & \text{Eq. I} \\ y - z = 4 & \text{Eq. II} \\ 3y - 5z = 6 & \text{Eq. III} \end{array}$$

Add $-3(\text{Eq. II}) + \text{Eq. III}$:

$$\begin{array}{rcl} -3y + 3z = -12 & & \\ 3y - 5z = 6 & & \\ \hline -2z = -6 \Rightarrow z = 3 & & \end{array}$$

Triangular Form:

$$\begin{array}{rcl} x - y + 2z = -3 \\ y - z = 4 \\ z = 3 \end{array}$$

Now that we have the system in the triangular form, solve by back substitution. Solution: $\boxed{(-2, 7, 3)}$

Special Systems: A linear system can have exactly one solution, infinitely many solutions, or no solution.

$$x - 2y + z = 4$$

Ex: Solve the system. $3x - 6y + 3z = 7$

$$2x + y + 4z = 2$$

$$-3x + 6y - 3z = -12$$

Add $-3(\text{Eq. I}) + \text{Eq. II}$:
$$\frac{3x - 6y + 3z = 7}{0 = -5}$$

All variables were eliminated, and the result is a false statement. Therefore, this system has NO SOLUTION (the lines do not intersect).

Coefficient Matrix of a System: The coefficient matrix of the system

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2$$

$$a_3x + b_3y + c_3z = d_3$$

is
$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}.$$

Augmented Matrix of a System: The augmented matrix of the system

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2$$

$$a_3x + b_3y + c_3z = d_3$$

is
$$\begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{bmatrix}.$$

Row Echelon Form of a Matrix: the augmented matrix that corresponds to the triangular form of the original system of equations

- rows consisting entirely of zeros (if any) occur at the bottom of the matrix
- the first entry in any row with nonzero entries is 1
- the column subscript of the leading 1 entries increases as the row subscript increases (the leading 1's move to the right as we move down the rows)

For Example, the triangular system
$$\begin{array}{r} x - y + 2z = -3 \\ y - z = 4 \\ z = 3 \end{array}$$
 as an augmented matrix:
$$\begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

We will write a system as an augmented matrix, then use **Elementary Row Operations** to put the matrix in **Row Echelon Form** to solve the system. Elementary Row Operations:

- interchange rows
- multiply all elements of a row by a nonzero number
- add a multiple of one row to any other row



Note: These are the same strategies we used in Gaussian Elimination! Tip: Start in the lower left corner of the matrix.

Ex: Solve the system using an augmented matrix in row echelon form.

$$\begin{aligned} 3x + 2y + z &= 5 \\ x - y + z &= 4 \\ x + 3y - 2z &= -3 \end{aligned}$$

Write as an augmented matrix:
$$\begin{bmatrix} 3 & 2 & 1 & 5 \\ 1 & -1 & 1 & 4 \\ 1 & 3 & -2 & -3 \end{bmatrix}$$

Perform the elementary row operations:
$$\begin{bmatrix} 3 & 2 & 1 & 5 \\ 1 & -1 & 1 & 4 \\ 1 & 3 & -2 & -3 \end{bmatrix} \Rightarrow R1 \leftrightarrow R2 \Rightarrow \begin{bmatrix} 1 & -1 & 1 & 4 \\ 3 & 2 & 1 & 5 \\ 1 & 3 & -2 & -3 \end{bmatrix} \Rightarrow -3R1 + R2 \Rightarrow$$

$$\begin{bmatrix} 1 & -1 & 1 & 4 \\ 0 & 5 & -2 & -7 \\ 1 & 3 & -2 & -3 \end{bmatrix} \Rightarrow -R1 + R3 \Rightarrow \begin{bmatrix} 1 & -1 & 1 & 4 \\ 0 & 5 & -2 & -7 \\ 0 & 4 & -3 & -7 \end{bmatrix} \Rightarrow -R3 + R2 \Rightarrow \begin{bmatrix} 1 & -1 & 1 & 4 \\ 0 & 1 & 1 & 0 \\ 0 & 4 & -3 & -7 \end{bmatrix} \Rightarrow -4R2 + R3 \Rightarrow \begin{bmatrix} 1 & -1 & 1 & 4 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -7 & -7 \end{bmatrix}$$

$$\Rightarrow -\frac{1}{7}R3 \Rightarrow \begin{bmatrix} 1 & -1 & 1 & 4 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \text{ Row Echelon Form!} \quad \text{Triangular Form of System: } \begin{aligned} x - y + z &= 4 \\ y + z &= 0 \\ z &= 1 \end{aligned}$$

Solution of System (use back substitution): $\boxed{(2, -1, 1)}$

Reduced Row Echelon Form: every column that has a leading 1 has zeros elsewhere

Ex: Write the augmented matrix above in reduced row echelon form.

$$\begin{bmatrix} 1 & -1 & 1 & 4 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \Rightarrow -R3 + R2 \Rightarrow \begin{bmatrix} 1 & -1 & 1 & 4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \Rightarrow -R3 + R1 \Rightarrow \begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \Rightarrow R2 + R1 \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Reduced Row Echelon Form:
$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$
 Solution of system is the last column! $\boxed{(2, -1, 1)}$

Special Case: Infinite Solutions

$$x + y - 3z = -1$$

Ex: Solve the system.

$$y - z = 0$$

$$-x + 2y = 1$$

$$\begin{bmatrix} 1 & 1 & -3 & -1 \\ 0 & 1 & -1 & 0 \\ -1 & 2 & 0 & 1 \end{bmatrix} \Rightarrow R1 + R3 \Rightarrow \begin{bmatrix} 1 & 1 & -3 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 3 & -3 & 0 \end{bmatrix} \Rightarrow \frac{1}{3}R3 \Rightarrow \begin{bmatrix} 1 & 1 & -3 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix} \Rightarrow -R2 + R3 \Rightarrow \begin{bmatrix} 1 & 1 & -3 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Notice that the last row in the matrix represents the equation $0 = 0$, so we know there are infinitely many solutions. We can represent these solutions as an ordered triple in terms of z .

$$y - z = 0 \Rightarrow y = z$$

$$x + y - 3z = -1 \Rightarrow x = -y + 3z - 1 \Rightarrow x = -z + 3z - 1 \Rightarrow x = 2z - 1$$

Solutions: $\boxed{(2z - 1, z, z)}$

Recall: Inverses can be used to solve the equation $a \cdot x = b$. $x = \frac{1}{a} \cdot b$

Solving Systems Using Inverses: The system $a_1x + b_1y = c_1$
 $a_2x + b_2y = c_2$ can be written as $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$. So the

solution of the system is $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}^{-1} \cdot \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$.

Note: The matrix $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$ is called the **coefficient matrix**.

Ex: Solve the system $-4x + y = 4$
 $6x - 2y = 3$ using the inverse matrix.

Write as a matrix equation: $\begin{bmatrix} -4 & 1 \\ 6 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$

Find the inverse of the coefficient matrix: $\begin{bmatrix} -4 & 1 \\ 6 & -2 \end{bmatrix}^{-1} = \frac{1}{(-4)(-2) - (1)(6)} \begin{bmatrix} -2 & -1 \\ -6 & -4 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -2 & -1 \\ -6 & -4 \end{bmatrix}$

Multiply both sides by the inverse: $\frac{1}{2} \begin{bmatrix} -2 & -1 \\ -6 & -4 \end{bmatrix} \cdot \begin{bmatrix} -4 & 1 \\ 6 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -2 & -1 \\ -6 & -4 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 3 \end{bmatrix}$

Note: Multiplying the coefficient matrix by its inverse results in the identity matrix.

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -2 & -1 \\ -6 & -4 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -2(4) + (-1)(3) \\ -6(4) + (-4)(3) \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -11 \\ -36 \end{bmatrix} = \begin{bmatrix} -\frac{11}{2} \\ -18 \end{bmatrix} \quad \text{Solution: } \boxed{\left(-\frac{11}{2}, -18\right)}$$



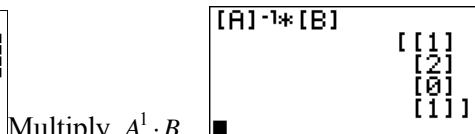
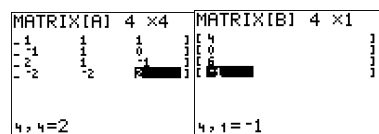
Solving a System Using Inverse Matrices

Ex: Solve the system.

$$\begin{aligned} x + y + z + w &= 4 \\ 2x - y + z &= 0 \\ 3x + 2y + z - w &= 6 \\ x - 2y - 2z + 2w &= -1 \end{aligned}$$

Write as a matrix equation: $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & -1 & 1 & 0 \\ 3 & 2 & 1 & -1 \\ 1 & -2 & -2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 6 \\ -1 \end{bmatrix}$

Enter the coefficient matrix as Matrix A and the constant matrix as Matrix B.



Multiply $A^{-1} \cdot B$. Solution: $\boxed{(1, 2, 0, 1)}$



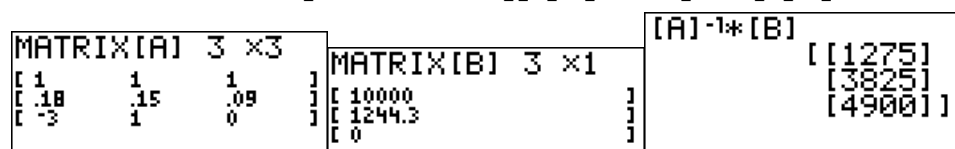
Application Problems

Ex: A man borrows a total of \$10,000; some at 18%, 15%, and 9%. He borrows 3 times as much at 15% as he borrows at 18%, and the total interest on all of the loans is \$1244.25. How much did he borrow at each percentage rate?

Assign labels: x = amount borrowed at 18%; y = amount at 15%; z = amount at 9%

Write a system: Total amount borrowed: $x + y + z = 10000$
 Total interest: $0.18x + 0.15y + 0.09z = 1244.25$
 Other information given: $3x = y \Rightarrow -3x + y = 0$

Solve by inverses: $\begin{bmatrix} 1 & 1 & 1 \\ 0.18 & 0.15 & 0.09 \\ -3 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10000 \\ 1244.25 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0.18 & 0.15 & 0.09 \\ -3 & 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 10000 \\ 1244.25 \\ 0 \end{bmatrix}$



He borrowed \$1275 at 18%, \$3825 at 15%, and \$4900 at 9%.

Ex: Determine the quadratic function that contains the points (1,3), (2,10), & (-2,-6).

Quadratic function: $y = ax^2 + bx + c$ We must find a , b , and c .

(1,3): $3 = a + b + c$

Substitute each ordered pair into $y = ax^2 + bx + c$.

(2,10): $10 = 4a + 2b + c$

(-2,-6): $-6 = 4a - 2b + c$

$\begin{bmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 4 & -2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 3 \\ 10 \\ -6 \end{bmatrix}$ $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 4 & -2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ 10 \\ -6 \end{bmatrix}$ Solve on graphing calculator.

$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix}$ Solution: $y = x^2 + 4x - 2$

You Try: 1. Solve the system from the second application problem using reduced row echelon form.

2. A piggy bank contains quarters, dimes, and nickels. The number of dimes is 1 less than the number of nickels. The number of nickels is 2 more than twice the number of quarters. The coins are worth \$5.70. How many nickels are there?

QOD: When solving systems using inverse matrices, does it matter which order you multiply by the inverse matrix? Explain.

Syllabus Objective: 8.5 – The student will use partial fractions to decompose rational expressions with distinct linear factors.

Recall: To add the rational expressions $\frac{3}{x-4} + \frac{2}{x+3}$, we must find a common denominator.

$$\frac{3}{x-4} + \frac{2}{x+3} = \frac{3(x+3) + 2(x-4)}{(x-4)(x+3)} = \frac{3x+9+2x-8}{(x-4)(x+3)} = \frac{5x+1}{x^2-x-12} \quad \text{or} \quad \frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$$

What if we wanted to go from the sum, $\frac{5x+1}{x^2-x-12}$, back to the addends, $\frac{3}{x-4} + \frac{2}{x+3}$?

Partial Fraction Decomposition: writing rational expressions as the sum of simpler ones

Steps – Case I: Distinct (no repeated) Linear Factors in the Denominator

1. Factor the denominator.
2. Write the factors as a sum.
3. Clear the fractions (multiply by the LCD).
4. Write and solve the resulting system of equations.
5. Write the resulting partial fractions.

Ex: Find the partial fraction decomposition of $\frac{5x-10}{2x^2+3x-2}$.

1. Factor the denominator: $\frac{5x-10}{2x^2+3x-2} = \frac{5x-10}{(2x-1)(x+2)}$

2. Write the factors as a sum. $\frac{5x-10}{(2x-1)(x+2)} = \frac{A}{2x-1} + \frac{B}{x+2}$

3. Clear the fractions. $5x-10 = A(x+2) + B(2x-1)$
 $5x-10 = Ax + 2A + 2Bx - B$

4. Write and solve the system. $5x-10 = Ax + 2Bx - B + 2A$ $A + 2B = 5$
 $5x-10 = (A+2B)x + (2A-B)$ $2A - B = -10$

$$\begin{array}{rcl} A + 2B = 5 & \Rightarrow \times(-2) \Rightarrow & -2A - 4B = -10 & A + 2B = 5 \\ 2A - B = -10 & \Rightarrow & \underline{2A - B = -10} & A + 2(4) = 5 \\ & & -5B = -20 \Rightarrow & B = 4 & A = -3 \end{array}$$

5. Write the resulting partial fractions. $\frac{5x-10}{(2x-1)(x+2)} = \frac{A}{2x-1} + \frac{B}{x+2} = \frac{-3}{2x-1} + \frac{4}{x+2}$

Solution: $\boxed{\frac{4}{x+2} - \frac{3}{2x-1}}$

Steps – Case II: Repeated Linear Factors in the Denominator

1. Factor the denominator.
2. Write the factors as a sum. Any repeated factors, f^n will be written as n factors in the fractions.

$$\frac{A}{f^n} = \frac{A_1}{f} + \frac{A_2}{f^2} + \dots + \frac{A_n}{f^n}$$

3. Clear the fractions (multiply by the LCD).
4. Write and solve the resulting system of equations.
5. Write the resulting partial fractions.

Ex: Write $\frac{x+2}{x^2-2x+1}$ as a decomposition of partial fractions.

1. Factor the denominator. $\frac{x+2}{x^2-2x+1} = \frac{x+2}{(x-1)^2}$
2. Write the factors as a sum. Any repeated factors, f^n will be written as n factors in the fractions.

$$\frac{x+2}{(x-1)^2} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2}$$

3. Clear the fractions (multiply by the LCD).
4. Write and solve the resulting system of equations.

$$\begin{aligned} x+2 &= A(x-1)+B \Rightarrow x+2 = Ax - A + B \\ A &= 1 & -1+B &= 2 \\ -A+B &= 2 & B &= 3 \end{aligned}$$

5. Write the resulting partial fractions.

$$\frac{x+2}{x^2-2x+1} = \boxed{\frac{1}{x-1} + \frac{3}{(x-1)^2}}$$

Steps – Case III: Irreducible Distinct Quadratic Factors in the Denominator

1. Factor the denominator.
2. Write the factors as a sum. Any irreducible quadratic factor in the denominator must have a linear expression in the numerator, i.e. $(ax+b)$.
3. Clear the fractions (multiply by the LCD).
4. Write and solve the resulting system of equations.
5. Write the resulting partial fractions.

Ex: Find the partial fraction decomposition of $\frac{8x^2+5x+18}{(x^3-8)}$.

1. Factor the denominator (difference of two cubes). $\frac{8x^2+5x+18}{(x^3-8)} = \frac{8x^2+5x+18}{(x-2)(x^2+2x+4)}$
2. Write the factors as a sum. Any irreducible quadratic factor in the denominator must have a linear expression in the numerator, i.e. $(ax+b)$.

$$\frac{8x^2+5x+18}{(x-2)(x^2+2x+4)} = \frac{A}{(x-2)} + \frac{Bx+C}{(x^2+2x+4)}$$

3. Clear the fractions (multiply by the LCD). $8x^2+5x+18 = A(x^2+2x+4) + (Bx+C)(x-2)$
 $8x^2+5x+18 = Ax^2+2Ax+4A+Bx^2-2Bx+Cx-2C$

4. Write and solve the resulting system of equations.

$$\begin{aligned}
 8x^2 + 5x + 18 &= (A+B)x^2 + (2A-2B+C)x + (4A-2C) & A+B &= 8 \\
 & & 2A-2B+C &= 5 \\
 & & 4A-2C &= 18
 \end{aligned}$$

$$\begin{aligned}
 \begin{bmatrix} 1 & 1 & 0 & 8 \\ 2 & -2 & 1 & 5 \\ 4 & 0 & -2 & 18 \end{bmatrix} & \xrightarrow{-2R_1+R_3} \begin{bmatrix} 1 & 1 & 0 & 8 \\ 2 & -2 & 1 & 5 \\ 0 & 4 & -4 & 8 \end{bmatrix} & \xrightarrow{-2R_1+R_2} \begin{bmatrix} 1 & 1 & 0 & 8 \\ 0 & -4 & 1 & -11 \\ 0 & 4 & -4 & 8 \end{bmatrix} & \xrightarrow{R_2+R_3} \begin{bmatrix} 1 & 1 & 0 & 8 \\ 0 & -4 & 1 & -11 \\ 0 & 0 & -3 & -3 \end{bmatrix} \\
 \xrightarrow{\frac{R_3}{-3}} \begin{bmatrix} 1 & 1 & 0 & 8 \\ 0 & -4 & 1 & -11 \\ 0 & 0 & 1 & 1 \end{bmatrix} & \xrightarrow{-R_3+R_2} \begin{bmatrix} 1 & 1 & 0 & 8 \\ 0 & -4 & 0 & -12 \\ 0 & 0 & 1 & 1 \end{bmatrix} & \xrightarrow{\frac{R_2}{-4}} \begin{bmatrix} 1 & 1 & 0 & 8 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{bmatrix} & \xrightarrow{-R_2+R_1} \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{bmatrix} & A=5, B=3, C=1
 \end{aligned}$$

5. Write the resulting partial fractions.

$$\frac{8x^2 + 5x + 18}{x^3 - 8} = \frac{5}{x-2} + \frac{3x+1}{x^2+2x+4}$$

Irreducible Quadratic Factors

Ex: Find the partial fraction decomposition. $\frac{x^3 - x^2 + 5x - 2}{(x^2 + 3)^2}$

$$\frac{x^3 - x^2 + 5x - 2}{(x^2 + 3)^2} = \frac{Ax + B}{x^2 + 3} + \frac{Cx + D}{(x^2 + 3)^2}$$

$$x^3 - x^2 + 5x - 2 = (Ax + B)(x^2 + 3) + Cx + D$$

$$x^3 - x^2 + 5x - 2 = Ax^3 + 3Ax + Bx^2 + 3B + Cx + D$$

$$A = 1$$

$$B = -1$$

$$3(1) + C = 5$$

$$3(-1) + D = -2$$

$$3A + C = 5$$

$$C = 2$$

$$D = 1$$

$$3B + D = -2$$

$$x^3 - x^2 + 5x - 2 = Ax^3 + Bx^2 + (3A + C)x + (3B + D)$$

$$\frac{x^3 - x^2 + 5x - 2}{(x^2 + 3)^2} = \frac{x-1}{x^2+3} + \frac{2x+1}{(x^2+3)^2}$$

Improper Fractions: use long division to write in the form $q(x) + \frac{r(x)}{d(x)}$

Ex: Find the partial fraction decomposition. $\frac{4x^3 + 9x^2 - 2x + 2}{x(x+2)}$

Long Division: $x^2 + 2x \overline{) 4x^3 + 9x^2 - 2x + 2}$

$$4x + 1 + \frac{-4x + 6}{x^2 + 2x}$$

Decompose: $\frac{-4x + 6}{x(x+2)} = \frac{A}{x} + \frac{B}{x+2}$

$$-4x + 6 = A(x+2) + Bx$$

$$A + B = -4$$

$$A = 3$$

$$-4x + 6 = Ax + 2A + Bx$$

$$2A = 6$$

$$3 + B = -4$$

$$\frac{-4x + 6}{x(x+2)} = \frac{3}{x} - \frac{7}{x+2}$$

$$-4x + 6 = (A+B)x + 2A$$

$$B = -7$$

$$\frac{4x^3 + 9x^2 - 2x + 2}{x(x+2)} = 4x + 1 + \frac{3}{x} - \frac{7}{x+2}$$

You Try: Find the partial fraction decomposition of $\frac{3x^3 - 6x^2 + 7x - 2}{(x^2 - 2x + 2)^2}$.

QOD: Are partial fraction decompositions always unique? Explain.

Syllabus Objective: 2.1 – The student will graph relations or functions, including real-world applications.

Half-Plane: the graph of a linear inequality

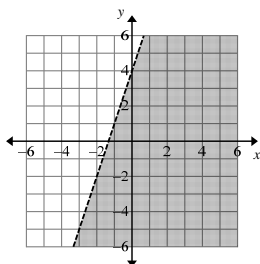
	Number Line	Coordinate Plane
$< \& >$	Open dot \circ	Dotted (dashed) line $\leftarrow \text{-----} \rightarrow$
$\leq \& \geq$	Closed (solid) dot \bullet	Solid Line $\leftarrow \text{=====} \rightarrow$

Note: You can use a test point to determine which half-plane to shade.

Ex: Draw the graph of the inequality. $y - 3x < 4$

Graph the line $y - 3x = 4$, or $y = 3x + 4$. The line will be “dashed” because it is a $<$ sign.

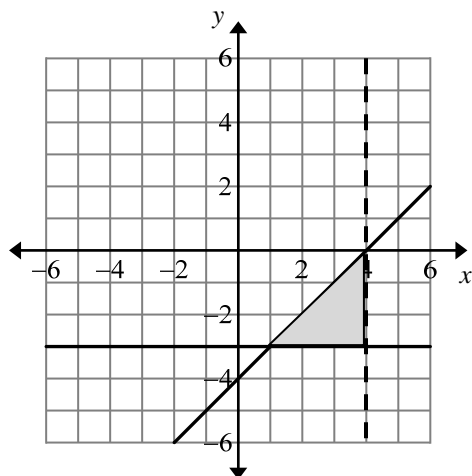
Pick a test point: $(0,0)$ $y - 3x < 4$
 $0 - 0 < 4$ true Shade the half-plane that contains the test point.



Solution of a System of Inequalities: the intersection of the solutions of all of the inequalities in the system; common solutions to all

Ex: Graph and determine if the solution is bounded or unbounded.

$$\begin{aligned} x &< 4 \\ y &\geq -3 \\ x - y &\geq 4 \end{aligned}$$



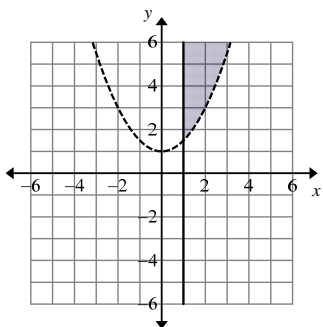
To be sure we shaded the correct region, use a test point.

Test Point $(3,-2)$: $x < 4$ $3 < 4$
 $y \geq -3$ $-2 \geq -3$ ☺ The region is **bounded**.
 $x - y \geq 4$ $3 - (-2) \geq 4$

Ex: Graph the system of inequalities and state if the solution is bounded or unbounded.

$$y > \frac{1}{2}x^2 + 1$$

$$x \geq 1$$



The region is unbounded.

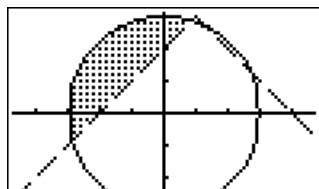
Ex: Solve the system of inequalities.

$$y > -|x - 1| + 3$$

$$x^2 + y^2 \leq 9$$

$y > -|x - 1| + 3$: vertex at (1, 3); opens down; shade above

$x^2 + y^2 \leq 9$: circle with center (0, 0); radius = 3; shade inside



Solution is the shaded region they have in common.

Linear Programming: decision making by finding a minimum or maximum value of a linear function given a set of constraints

Constraints: a system of linear inequalities

Feasible Region: the solution set found using the constraints

Objective Function: an equation applied to a feasible region that we want to minimize or maximize

Vertex Points: the corner points of the feasible region



The maximum or minimum of a linear programming problem always occurs at a vertex point.

Solving a Linear Programming Problem

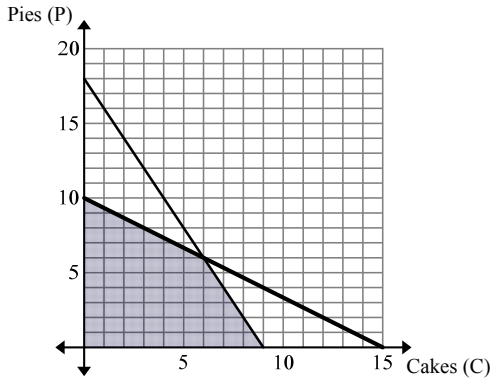
- Write the system of inequalities that represent the constraints.
- Graph the feasible region.
- Write the objective function.
- Test the vertex points in the objective function and answer the question.

Ex: Mrs. Arquette is baking as many cakes and pies as possible using the 18 eggs and 15 cups of milk she has on hand. Her cake recipe requires 2 eggs and 1 cup of milk. Her pie recipe requires 1 egg and 1.5 cups of milk. How many cakes and pies should she make?

Precalculus Notes: Unit 7 – Systems of Equations and Matrices

Constraints: $C \geq 0$ $2C + P \leq 18$ (eggs)
 $P \geq 0$ $C + 1.5P \leq 15$ (milk) C : number of cakes; P : number of pies

Graph:



Objective Function: $T = C + P$ Vertex Points: $(0, 10)$, $(0, 0)$, $(9, 0)$, $(6, 6)$

$(0, 10): T = 0 + 10 = 10$

$(0, 0): T = 0 + 0 = 0$

$(9, 0): T = 9 + 0 = 9$

$(6, 6): T = 6 + 6 = 12$

Maximum: 6 Cakes & 6 Pies

Ex: It takes 2 hours to manufacture a pair of skis and 1 hour for a snowboard. The finishing time for both is 1 hour. The maximum time available for manufacturing is 40 hours and for finishing is 32 hours each week. The profit for a pair of skis is \$70 and the profit for a snowboard is \$50. The manufacturer must produce at least 8 snowboards every week because of a contract with a sporting goods store. Find the maximum profit.

Constraints: $k \geq 0$ $2k + w \leq 40$ (manufacturing)
 $w \geq 8$ $k + w \leq 32$ (finishing) k : number of pairs of skis; w : number of snowboards

Graph:



Objective Function: $P = 70k + 50w$ Vertex Points: $(0, 8)$, $(0, 0)$, $(0, 32)$, $(8, 24)$, $(16, 8)$

$(0, 8): P = 70(0) + 50(8) = \400

$(0, 0): P = 70(0) + 50(0) = \0

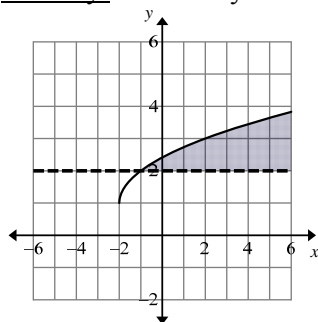
$(0, 32): P = 70(0) + 50(32) = \1600

$(8, 24): P = 70(8) + 50(24) = \1760

$(16, 8): P = 70(16) + 50(8) = \1520

Maximum Profit: \$1760 (8 pairs of skis, 24 snowboards)

You Try: Write a system of inequalities that represents the graph.



QOD: What does the feasible region represent in a linear programming problem? How are the corner points special?