TAKE IT TO THE MAT

A NEWSLETTER ADDRESSING THE FINER POINTS OF MATHEMATICS INSTRUCTION

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This month's *Take It to the MAT* topic is courtesy of a Clark County School District high school teacher. The teacher asks, "I know that I can show the divergence of the harmonic series by various tests, like the integral test, but is there a way to show it diverges without those tests?" The answer to the question is, "Yes." In the following paragraphs is a fairly conceptual "proof" that the harmonic series diverges.

First, let's review what the harmonic series is.

The *harmonic series* is defined as $\sum_{k=1}^{\infty}\frac{1}{k}$. That is, $\sum_{k=1}^{\infty}\frac{1}{k}=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$.

It seems odd that this infinite sum would not converge on some value, that it diverges to infinity. Other infinite series, like $\sum_{k=1}^{\infty}\frac{1}{2^k}=\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots$, converge to a fixed value. In the case of $\sum_{k=1}^{\infty}\frac{1}{2^k}=\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots$, the infinite sum can be shown to be 1. But that's not the case with $\sum_{k=1}^{\infty}\frac{1}{k}=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$. As the teacher who posed the question put it, "I'm adding a smaller and smaller amount with each term, and those amounts approach zero. Wouldn't we get to the point where we're adding such a small amount, that the sum should converge to some value?" Agreed. By continually adding what amounts to zero, it's counterintuitive that the sum would diverge to infinity.

Consider this series: $1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots$. Does it converge or diverge? If we consider that we are adding $\frac{1}{2}$ ad infinitum, it's not hard to see that we'll never converge on a fixed value; it diverges.

Now consider the following groups of terms:

The sum $\frac{1}{3}+\frac{1}{4}$: Since $\frac{1}{3}>\frac{1}{4}$, and $\frac{1}{4}+\frac{1}{4}=\frac{1}{2}$, then $\frac{1}{3}+\frac{1}{4}>\frac{1}{2}$.

The sum $\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}$: Since $\frac{1}{5}>\frac{1}{6}>\frac{1}{7}>\frac{1}{8}$, and $\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}=\frac{1}{2}$, then $\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}>\frac{1}{2}$.

We can similarly show that the sum of the next eight terms, $\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}+\frac{1}{14}+\frac{1}{15}+\frac{1}{16}>\frac{1}{2}$.

It's not hard to see that the pattern will continue for the next sixteen, thirty-two, etc. terms.

If we look at the last term in each group, it's a power of one-half, and there are half as many terms in the group as the value of the denominator. If each of the terms were equal to the last, then the sum would be exactly one-half. That is, if the last fraction in the group were called $\frac{1}{n}$, then the sum would be $\underbrace{\frac{1}{n}+\frac{1}{n}+\ldots+\frac{1}{n}+\frac{1}{n}}_{\frac{n}{2}\text{ terms}}=\left(\frac{n}{2}\right)\left(\frac{1}{n}\right)=\frac{1}{2}$.

The actual situation is different. Each of the terms in the group is greater than the next. Again, if we call the last term $\frac{1}{n}$, then $\frac{1}{n-\left(\frac{n}{2}-1\right)}>\frac{1}{n-\left(\frac{n}{2}-2\right)}>\cdots>\frac{1}{n-1}>\frac{1}{n}$, and

$$\frac{1}{n-\left(\frac{n}{2}-1\right)}+\frac{1}{n-\left(\frac{n}{2}-2\right)}+\cdots+\frac{1}{n-1}+\frac{1}{n}>\frac{1}{2}.$$

Time to cut to the chase. If we take groups of terms in the harmonic series, such that the last term in the group is a power of one-half, and the number of terms in the group is half of the value of the last denominator, then the sum of those terms is greater than one-half.

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{1}{k}&=1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\left(\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}+\frac{1}{14}+\frac{1}{15}+\frac{1}{16}\right)+\cdots\\&=1+\frac{1}{2}+\left(>\frac{1}{2}\right)+\left(>\frac{1}{2}\right)+\left(>\frac{1}{2}\right)+\cdots\end{aligned}$$

Since we're continually adding values larger than one-half, the series will diverge. The harmonic series diverges—no tests needed.

One last thing: the sequence of sums $\frac{1}{3}+\frac{1}{4}$, $\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}$, $\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}+\frac{1}{14}+\frac{1}{15}+\frac{1}{16}$, etc. increases and seems to converge on a particular value. Determining that value is an exercise left to the reader.