Syllabus Objective: 2.15 – The student will interpret differential equations via slope fields.

Differential Equation: an equation containing a derivative, for example, $\frac{dy}{dx} = y \ln x$

Initial Value Problem: the problem of finding a function *y* of *x* when we are given its derivative and its value at a particular point

Initial Condition: the value of f for one value of *x*

Ex1: Find a particular solution to the differential equation $\frac{dy}{dx} = \cos x - 2x$, for which $f(0) = 1$.

Find the antiderivative of $\cos x - 2x$: $y = \sin x - x^2 + C$

Use the initial value to find *C*: $f(0) = 1;\ 1 = \sin(0) - 0^2 + C \Rightarrow C = 1$ $\boxed{y = \sin x - x^2 + 1}$

Slope Field (Direction Field): for the first order differential equation $\frac{dy}{dx} = f(x, y)$, a plot of short line segments with slopes $f(x, y)$ at the lattice points (x, y) in the plane

First order differential equations can be expressed in the form $\frac{dy}{dx} = f(x, y)$. The solutions of the differential equations are certain functions. The differential equation defines the slope at the point (x, y) of the certain curve of the function that passes through this point. For each point (x, y), the differential equation defines a line segment with slope $f(x, y)$. We say that the differential equation defines the slope (or direction) field of the differential equation.

Drawing a Slope Field

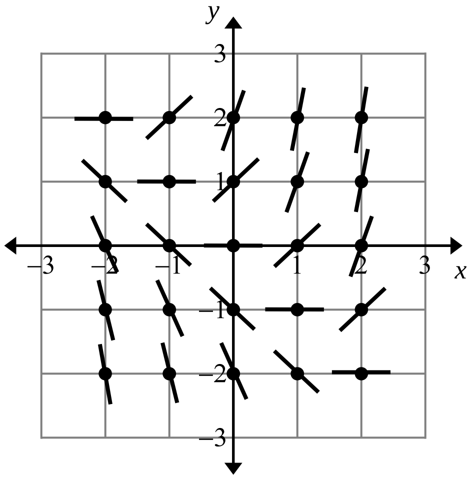
- Evaluate the differential equation at various points, (x, y).
- At each of these points, (x, y), sketch a line segment with the slope found by evaluating the differential equation.

Ex2: Draw a slope field for the differential equation. $\frac{dy}{dx} = x + y$

Make a table and choose values for (x, y).

x	−2	−2	−2	−2	−2	−1	−1	−1	−1	−1	0	0	0	0	0	1	1	1	1	1	2	2	2	2	2
y	−2	−1	0	1	2	−2	−1	0	1	2	−2	−1	0	1	2	−2	−1	0	1	2	−2	−1	0	1	2
$\frac{dy}{dx} = x + y$	−4	−3	−2	−1	0	−3	−2	−1	0	1	−2	−1	0	1	2	−1	0	1	2	3	0	1	2	3	4

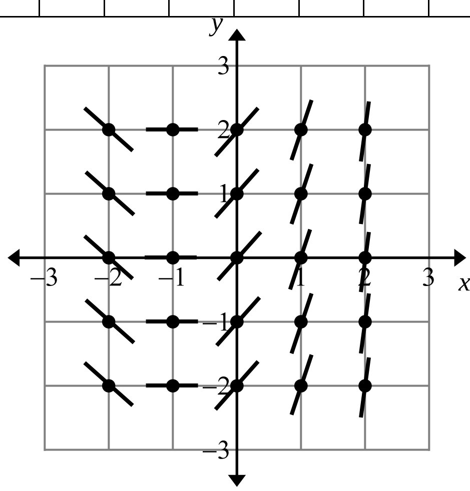
Draw the line segments with slopes found in the table.



Slope Field for $\frac{dy}{dx} = x + y$:

Ex3: Draw a slope field for $\frac{dy}{dx} = x + 1$.

x	−2	−2	−2	−2	−2	−1	−1	−1	−1	−1	0	0	0	0	0	1	1	1	1	1	2	2	2	2	2
y	−2	−1	0	1	2	−2	−1	0	1	2	−2	−1	0	1	2	−2	−1	0	1	2	−2	−1	0	1	2
$\frac{dy}{dx} = x + 1$	−1	−1	−1	−1	−1	0	0	0	0	0	1	1	1	1	1	2	2	2	2	2	3	3	3	3	3

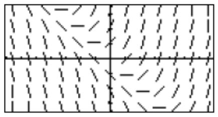


This differential equation, $\frac{dy}{dx} = x + 1$, is **<u>autonomous</u>**. The slopes of the tangent lines in the field only depend upon one variable.

Matching Slope Fields to Differential Equations

Ex4: Match the slope fields with their differential equations.

(A)



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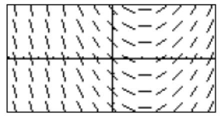

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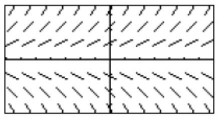

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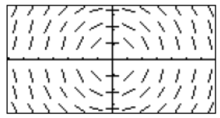
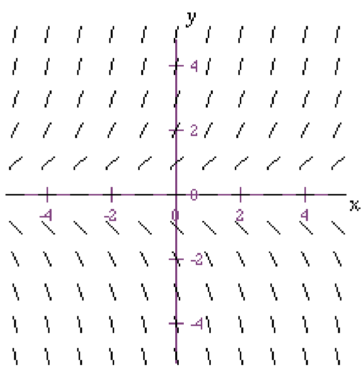


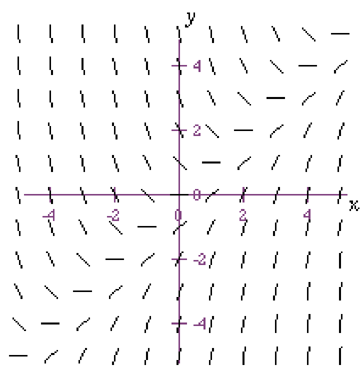
1. $\frac{dy}{dx} = 0.5x - 1$ (B) 2. $\frac{dy}{dx} = \frac{1}{2}y$ (C) 3. $\frac{dy}{dx} = -\frac{x}{y}$ (D) 4. $\frac{dy}{dx} = x + y$ (A)

Sketching a Solution Curve in a Slope Field

Ex5: Consider the slope fields below. Sketch at least 5 solution curves for each differential equation.

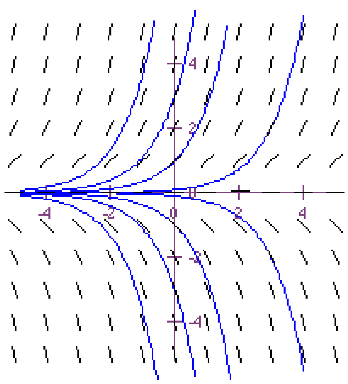


Slope Field for $\frac{dy}{dx} = y$

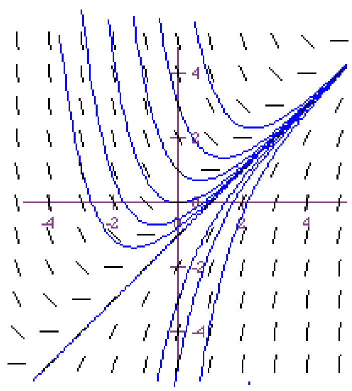


Slope Field for $\frac{dy}{dx} = x - y$

Possible solution curves:



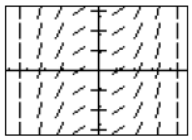
Some solutions for $\frac{dy}{dx} = y$



Some solutions for $\frac{dy}{dx} = x - y$

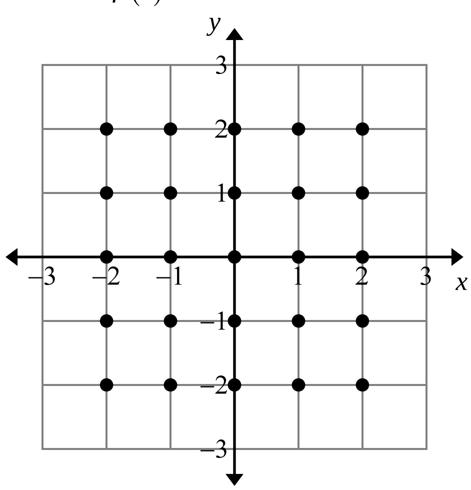
Note that the solution curves thread their way through the fields much like a leaf in a stream following the streamlines created by the current.

Ex6: The slope field for a certain differential equation is shown below. Which of the following could be a specific solution to that differential equation?



(A) $y = \sin x$ (B) $y = \cos x$ (C) $y = x^2$ **(D) $y = \frac{1}{6}x^3$ (E) $y = \ln x$

You Try: Draw a slope field for the differential equation $\frac{dy}{dx} = -\frac{y}{x}$. Then sketch a solution curve with initial condition $f(1) = 2$.



QOD: Explain how the line segments in a slope field relate to the tangent lines of the solution curve(s).

Syllabus Objective: 3.1 – The student will determine the antiderivative of an elementary function. 3.2 – The student will solve antiderivatives using initial conditions. 3.14 – The student will solve integration problems using substitution of variables.

Indefinite Integral: The set of all antiderivatives of a function $f(x)$ is the **indefinite integral** of f with respect to *x* and is denoted by $\int f(x)dx = F(x)+C$, where *C* is the constant of integration. Read "The indefinite integral of *f* with respect to *x* is $F(x)+C$."

Ex1: Evaluate $\int 3x^2 dx$.

The antiderivative of $f(x)=3x^2$ is $F(x)=x^3$. So $\int 3x^2 dx = \boxed{x^3+C}$.

Integral Rules

1. $\int x^n dx = \frac{x^{n+1}}{n+1}+C,\ n \neq -1$
2. $\int \frac{dx}{x} = \ln|x|+C$
3. $\int e^{kx}dx = \frac{e^{kx}}{k}+C$
4. $\int a^x dx = \frac{a^x}{\ln a}+C$
5. $\int \sin kx dx = -\frac{\cos kx}{k}+C$
6. $\int \cos kx dx = \frac{\sin kx}{k}+C$
7. $\int \sec^2 x dx = \tan x + C$
8. $\int \csc^2 x dx = -\cot x + C$
9. $\int \sec x \tan x dx = \sec x + C$
10. $\int \csc x \cot x dx = -\csc x + C$

Properties of Indefinite Integrals: Let *k* be a real number.

1. Constant Multiple Rule: $\int kf(x)dx = k\int f(x)dx$
2. Sum and Difference Rule: $\int f(x) \pm g(x)dx = \int f(x)dx \pm \int g(x)dx$

Ex2: Find each indefinite integral.

1. $\int x^7 dx$ Rule #1: $\int x^7 dx = \boxed{\frac{x^8}{8} + C}$

2. $\int e^{-3x} dx$ Rule #3: $\int e^{-3x} dx = \boxed{\frac{e^{-3x}}{-3} + C}$

3. $\int \frac{1}{\sqrt{x}} dx$ Rule #1: $\int \frac{1}{\sqrt{x}} dx = \int x^{-\frac{1}{2}} dx = \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + C = \boxed{2\sqrt{x} + C}$

4. $\int \cos\frac{x}{2} dx$ Rule #6: $\int \cos\frac{x}{2} dx = \frac{\sin\frac{x}{2}}{\frac{1}{2}} + C = \boxed{2\sin\frac{x}{2} + C}$

5. $\int \left(x^2 - 2x + 5\right) dx$ Sum/Difference Property: $\int \left(x^2 - 2x + 5\right) dx = \boxed{\frac{x^3}{3} - x^2 + 5x + C}$

<u>Integration by Substitution</u>: When evaluating an integral of a function that is not "simple" (i.e. exactly one of the rules above), try defining a new function, *u*, and substitute *u* and *du* back into the integral.

Examples Involving Composite Functions

Ex3: Evaluate the following indefinite integrals.

1. $\int \cos(7x + 5) dx$

Define *u*. Use the argument of the trig function. $u = 7x + 5$

Find *du* by differentiating. $du = 7dx$

Solve for *dx*. $dx = \frac{du}{7}$

Substitute into the original integral. $\int \cos u \frac{du}{7}$

Note: The integral should only be in terms of *u*.

Evaluate the definite integral. $\int \cos u \frac{du}{7} = \frac{1}{7}\int \cos u du = \frac{1}{7}\sin u + C$

Rewrite the answer in terms of *x*. $= \boxed{\frac{1}{7}\sin(7x + 5) + C}$

2. $\int \sqrt{4x-1}\,dx$

Define *u*. Use the radicand. $u = 4x - 1$

Find *du* by differentiating. $du = 4dx$

Solve for *dx*. $dx = \frac{du}{4}$

Substitute into the original integral. $\int \sqrt{u}\,\frac{du}{4}$

Note: The integral should only be in terms of *u*.

Evaluate the definite integral. $\int \sqrt{u}\,\frac{du}{4} = \frac{1}{4}\int u^{\frac{1}{2}}du = \frac{1}{4}\cdot\frac{2}{3}u^{\frac{3}{2}} + C$

Rewrite the answer in terms of *x*. $= \boxed{\frac{1}{6}(4x-1)^{\frac{3}{2}} + C}$

Examples Involving the Product/Quotient of Functions

Ex4: Evaluate the following indefinite integrals.

1. $\int (x^2 + 2x - 3)(x+1)\,dx$

Define *u*. Use the more complicated function. $u = x^2 + 2x - 3$

Find *du* by differentiating. $du = (2x+2)dx$

Solve for *dx*. $dx = \frac{du}{2x+2}$

Substitute into the original integral. $\int u(x+1)\frac{du}{2x+2} = \int u\,\cancel{(x+1)}\,\frac{du}{2\cancel{(x+1)}} = \frac{1}{2}\int u\,du$

Note: The integral should only be in terms of *u*. If you picked the correct function to use as *u*, all expressions containing *x* should cancel.

Evaluate the indefinite integral. $\frac{1}{2}\int u\,du = \frac{1}{2}\cdot\frac{u^2}{2} + C$

Rewrite the answer in terms of *x*. $= \boxed{\frac{1}{4}(x^2 + 2x - 3)^2 + C}$

2. $\int \tan x dx$

Rewrite using the trig identity $\tan x = \dfrac{\sin x}{\cos x}$. $\int \dfrac{\sin x}{\cos x} dx$

Define *u*. $u = \cos x$

Teacher Note: Allow students to choose $u = \sin x$. Trial and error is an important part of *u*-substitution.

Find *du* by differentiating. $du = -\sin x dx$

Solve for *dx*. $dx = \dfrac{du}{-\sin x}$

Substitute into the original integral. $\int \dfrac{\cancel{\sin x}}{u} \dfrac{du}{-\cancel{\sin x}} = \int -\dfrac{du}{u}$

Note: The integral should only be in terms of *u*. If you picked the correct function to use as *u*, all expressions containing *x* should cancel.

Evaluate the indefinite integral. $\int -\dfrac{du}{u} = -\ln|u| + C$

Rewrite the answer in terms of *x*. $= \boxed{-\ln|\cos x| + C}$ or $\boxed{\ln|\sec x| + C}$

Evaluating Definite Integrals Using *u*-Substitution

Method 1: Leaving the limits of integration as-is; converting back to a function in terms of *x*

Ex: Evaluate the definite integral. $\int_0^{\frac{\pi}{4}} \tan x \sec^2 x dx$

Define *u*. $u = \tan x$

Find *du* by differentiating. $du = \sec^2 x dx$

Solve for *dx*. $dx = \dfrac{du}{\sec^2 x}$

Substitute into the original integral. $\int u \cancel{\sec^2 x} \dfrac{du}{\cancel{\sec^2 x}} = \int u du$

Note: The limits of integration do not apply to *u*, so leave them off of this integral.

Evaluate the indefinite integral. $\int u du = \dfrac{u^2}{2}$

Rewrite the answer in terms of *x*. $= \dfrac{\tan^2 x}{2}$

Evaluate the definite integral using the original limits of integration.

$$= \frac{\tan^2 x}{2}\Bigg|_0^{\frac{\pi}{4}} = \frac{1}{2}\left(\tan^2 \frac{\pi}{4} - \tan^2 0\right) = \frac{1}{2}(1-0) = \boxed{\frac{1}{2}}$$

Method 2: Convert the limits of integration to values in terms of *u*

Ex: Evaluate the definite integral. $\int_{-1}^{1} 3x^2\sqrt{x^3+1}dx$

Define *u*. $u = x^3 + 1$

Find *du* by differentiating. $du = 3x^2 dx$

Solve for *dx*. $dx = \dfrac{du}{3x^2}$

Substitute into the original integral. $\int \cancel{3x^2} \sqrt{u} \dfrac{du}{\cancel{3x^2}} = \int \sqrt{u} du$

Note: The limits of integration do not apply to *u*, so leave them off of this integral.

Find new limits of integration in terms of *u*. $u = x^3 + 1$: When $x = -1,\ u = (-1)^3 + 1 = 0$.

When $x = 1,\ u = (1)^3 + 1 = 2$.

Evaluate the new definite integral.

$$\int_0^2 \sqrt{u} du = \int_0^2 u^{\frac{1}{2}} du = \frac{2}{3}\left[u^{\frac{3}{2}}\right]_0^2 = \frac{2}{3}\left[2^{\frac{3}{2}} - 0^{\frac{3}{2}}\right] = \frac{2}{3}\sqrt{8} = \boxed{\frac{4\sqrt{2}}{3}}$$

You Try: Integrate $\int \frac{1}{\cos^2 2x} dx$. (Hint: Use a trig identity.)

QOD: Describe some strategies you can use when choosing a *u* to integrate by substitution.

Sample AP Calculus AB Exam Question(s):

1. $\int_0^1 e^{-4x} dx =$

   (A) $\frac{-e^{-4}}{4}$

   (B) $-4e^{-4}$

   (C) $e^{-4} - 1$

   (D) $\frac{1}{4} - \frac{e^{-4}}{4}$

   (E) $4 - 4e^{-4}$

2. $\int x^2 \cos\left(x^3\right) dx =$

   (A) $-\frac{1}{3}\sin\left(x^3\right) + C$

   (B) $\frac{1}{3}\sin\left(x^3\right) + C$

   (C) $-\frac{x^3}{3}\sin\left(x^3\right) + C$

   (D) $\frac{x^3}{3}\sin\left(x^3\right) + C$

   (E) $\frac{x^3}{3}\sin\left(\frac{x^4}{4}\right) + C$

3. Using the substitution $u = 2x + 1$, $\int_0^2 \sqrt{2x+1}\, dx$ is equivalent to

(A) $\frac{1}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} \sqrt{u}\, du$

(B) $\frac{1}{2}\int_0^2 \sqrt{u}\, du$

(C) $\frac{1}{2}\int_1^5 \sqrt{u}\, du$

(D) $\int_0^2 \sqrt{u}\, du$

(E) $\int_1^5 \sqrt{u}\, du$

Syllabus Objective: 3.3 – The student will find solutions to separable differential equations, including *y*' = *ky* as it relates to growth and decay.

Solving a Separable Differential Equation

- Separate the variables. Get all *x* and *dx* expressions on one side, *y* and *dy* expressions on the other. Teacher Note: On the AP Exam Free Response, a student will receive 0 points if the variables are not separated!
- Integrate both sides. Don't forget the constant!
  - Note: The constant is only necessary on one side, because a constant minus another constant is still a constant.
- Rewrite the solution as a function of *y* in terms of *x* (if possible).
- Use the initial condition to solve for *C* (if necessary).

Ex1: Solve the differential equation . $\frac{1}{x}\frac{dy}{dx}=2y$

Separate the variables. $\frac{1}{x}\frac{dy}{dx}=2y \Rightarrow dy=2y\cdot xdx \Rightarrow \frac{dy}{2y}=xdx$

Integrate both sides. Don't forget the constant! $\int\frac{dy}{2y}=\int xdx \Rightarrow \frac{1}{2}\ln|y|=\frac{x^2}{2}+C$

Rewrite the solution as a function of *y* in terms of *x*.

$$\frac{1}{2}\ln|y|=\frac{x^2}{2}+C \Rightarrow \ln|y|=x^2+C \Rightarrow |y|=e^{x^2+C}$$

We can remove the absolute value bars because e^{x^2+C} is always positive. Also, we will rewrite e^{x^2+C} using the rules of exponents.

$y=e^{x^2+C} \Rightarrow y=e^{x^2}\cdot e^C$ $\quad$ e^C is just a constant, so our solution is $\boxed{y=Ce^{x^2}}$.

Ex2: Solve the differential equation $\frac{dy}{dx}+5y=20,\ y(0)=2$.

Separate the variables. $\frac{dy}{dx}=20-5y \Rightarrow dy=(20-5y)dx \Rightarrow \frac{dy}{20-5y}=dx$

Integrate both sides. Don't forget the constant! $\int\frac{dy}{20-5y}=\int dx \Rightarrow -\frac{1}{5}\ln|20-5y|=x+C$

WORK: For the left side: $u=20-5y \Rightarrow du=-5dy \Rightarrow dy=-\frac{du}{5}$ $\quad$ $\int\frac{-du}{5u}=-\frac{1}{5}\ln|u|+C$

Rewrite the solution as a function of *y* in terms of *x*.

$$-\frac{1}{5}\ln|20-5y|=x+C \Rightarrow \ln|20-5y|=-5x+C \Rightarrow 20-5y=e^{-5x+C} \Rightarrow -5y=e^C\cdot e^{-5x}-20 \Rightarrow -5y=Ce^{-5x}-20 \Rightarrow y=Ce^{-5x}+4$$

Use the initial condition to solve for *C*. $\quad y=Ce^{-5x}+4 \quad y(0)=2$

$2=Ce^{-5(0)}+4 \Rightarrow C=-2$ $\quad$ $\boxed{y=-2e^{-5x}+4}$

Law of Exponential Change: If y changes at a rate proportional to the amount present $\left(\frac{dy}{dt} = ky\right)$ and $y = y_0$ when $t = 0$, then $y = y_0 e^{kt}$, where $k > 0$ represents growth and $k < 0$ represents decay. The number k is the **rate constant** of the equation.

Solve the differential equation $\frac{dy}{dt} = ky$ to show the relationship above.

$\frac{dy}{dt} = ky \Rightarrow \frac{dy}{y} = kdt$ $\quad$ $\int \frac{dy}{y} = \int kdt \Rightarrow \ln|y| = kt + C$ $\quad$ $y = e^{kt+C} \Rightarrow y = Ce^{kt}$

Radioactivity: The decay of a radioactive element is described by the equation $\frac{dy}{dt} = -ky,\ k > 0$. If y_0 is the number of radioactive nuclei present at time zero, the number still present at any later time t will be $y = y_0 e^{-kt},\ k > 0$.

Half-Life: the time required for half of the radioactive nuclei present in a sample to decay

Ex3: Carbon-14 is radioactive and decays at a rate proportional to the amount present. Its half-life is 5730 years. If 10 grams were present originally, how much will be left after 2000 years?

Use the equation $y = y_0 e^{-kt}$ and the half-life information given. $\quad$ $0.5 = 1e^{-k(5730)}$

Note: Any value can be chosen for y_0. To find y, it will equal half of y_0.

Solve for k. $\quad$ $\ln(0.5) = -5730k \Rightarrow k = -\frac{\ln(0.5)}{5730} \Rightarrow k = \frac{\ln 2}{5730}$

Substitute the given information and solve for t. $\quad$ $y = 10e^{-\frac{\ln 2}{5730}(2000)} \Rightarrow y \approx \boxed{7.851 \text{ grams}}$

Continuously Compounded Interest

Suppose that A_0 dollars are invested at a fixed annual interest rate r (expressed as a decimal). If interest is added to the account k times a year, the amount of money present after t years is $A(t) = A_0\left(1 + \frac{r}{k}\right)^{kt}$.

If the interest is added continuously at a rate proportional to the amount in the account, then the amount of money in the account after t years is $A(t) = A_0 e^{rt}$. Interest paid according to this formula is **compounded continuously**, with r as the **continuous interest rate**.

Ex4: Suppose an amount of \$1200 is deposited into an account that pays 5.7% annual interest. How much will the account be worth 10 years later if the interest is compounded monthly? if the interest is compounded continuously?

Compounded Monthly: Use $A(t) = A_0\left(1 + \frac{r}{k}\right)^{kt}$ with $k = 12$. $\quad$ $A(10) = 1200\left(1 + \frac{0.057}{12}\right)^{12(10)} \approx \boxed{\$2119.06}$

Compounded Continuously: Use $A(t) = A_0 e^{rt}$. $\quad$ $A(10) = 1200e^{0.057(10)} \approx \boxed{\$2121.92}$

Newton's Law of Cooling

The rate at which an object's temperature is changing at any given time is roughly proportional to the difference between its temperature and the temperature of the surrounding medium. T is the temperature of the object at time t, and T_S is the temperature of the surrounding medium: $\frac{dT}{dt} = -k\left(T - T_S\right)$. And so it follows that $\boldsymbol{T - T_S = \left(T_0 - T_S\right)e^{-kt}}$, where T_0 is the temperature at time $t = 0$.

Ex5: A hot potato at 100°C is put in a pan under running 20°C water to cool. After 6 minutes, the potato's temperature is found to be 40°C. How much longer will it take the potato to reach 25°C?

Substitute the given values into $\boldsymbol{T - T_S = \left(T_0 - T_S\right)e^{-kt}}$. $\quad 100 - 20 = \left(40 - 20\right)e^{-6k}$

Solve for k. $\quad 20 = 80e^{-6k} \Rightarrow e^{-6k} = \frac{1}{4} \Rightarrow -6k = \ln\left(\frac{1}{4}\right) \Rightarrow k = -\frac{1}{6}\ln\left(\frac{1}{4}\right)$

Substitute k and $T = 90$ into $\boldsymbol{T - T_S = \left(T_0 - T_S\right)e^{-kt}}$. $\quad 25 - 20 = \left(100 - 20\right)e^{-\left(-\frac{1}{6}\ln\left(\frac{1}{4}\right)\right)t}$

Solve for t.

$$5 = 80e^{\frac{1}{6}\ln\left(\frac{1}{4}\right)t} \Rightarrow e^{\frac{1}{6}\ln\left(\frac{1}{4}\right)t} = \frac{1}{16} \Rightarrow \frac{1}{6}\ln\left(\frac{1}{4}\right)t = \ln\left(\frac{1}{16}\right) \Rightarrow t = \frac{6\ln\left(\frac{1}{16}\right)}{\ln\left(\frac{1}{4}\right)} \approx 12$$

It takes 12 minutes *total* for the potato to reach this temperature. This is $\boxed{\text{6 minutes}}$ *longer*.

You Try: Find the half-life of a radioactive substance with decay equation $y = y_0e^{-kt}$ and show that the half-life depends only on k.

QOD: Explain how the first and second derivatives relate to the graph of a function.

Sample AP Calculus AB Exam Question(s):

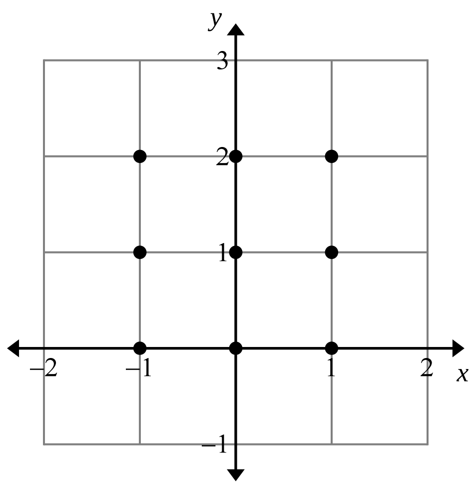
A pizza, heated to a temperature of 350 degrees Fahrenheit (°F), is taken out of an oven and placed in a 75°F room at time $t = 0$ minutes. The temperature of the pizza is changing at a rate of $-110e^{-0.4t}$ degrees Fahrenheit per minute. To the nearest degree, what is the temperature of the pizza at time $t = 5$ minutes?

(A) 112°F
(B) 119°F
(C) 147°F
(D) 238°F
(E) 335°F

Sample AP Free Response Question:

Consider the differential equation given by $\frac{dy}{dx} = \frac{xy}{2}$.

(a) On the axes provided, sketch a slope field for the given differential equation at the nine points indicated.



(b) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(0) = 3$. Use your solution to find $f(0.2)$.