Syllabus Objective: 3.1 – The student will solve problems using the unit circle.

Review:

a) Convert 1.2 hours into hours and minutes.
Solution: 1 hour + (0.2)(60) = 1 hour and 12 minutes

b) Convert 3 hours and 20 minutes into hours.
Solution: 3 hours + $\frac{20}{60}$ hours = 3.3 hours

Babylonian Number System: based on the number 60; 360 approximates the number of days in a year. Circles were divided into 360 degrees.

A degree can further be divided into 60 **minutes (60')**, and each minute can be divided into 60 **seconds (60'')**.

Converting from Degrees to DMS (Degrees – Minutes – Seconds): **multiply** by 60

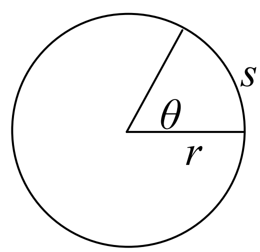
Ex1: Convert 114.59° to DMS.

114.59° = 114° + 0.59(60)' = 114° 35.4' = 114° 35' +0.4(60)" = $\boxed{114°35'\ 24"}$

Converting to Degrees (decimal): **divide** by 60

Ex2: Convert 72° 13' 52" to decimal degrees.

72° 13' 52" = $72 + \frac{13}{60} + \frac{52}{60^2} = \boxed{72.23^\circ}$



Radian: a measure of length; $\theta = 1$ radian when $r = s$
r = radius, s = length of arc

Recall: Circumference of a Circle; $C = 2\pi r$

So, there are 2π radians around the circle 2π radians = 360°, or $\boxed{\pi \text{ radians} = 180^\circ}$

Converting Degrees (D) to Radians (R): $D \cdot \frac{\pi}{180^\circ} = R$ (Note: $\frac{\pi}{180^\circ} = 1$)

Ex3: Convert 540° to radians.

$540^\circ \cdot \frac{\pi}{180^\circ} = \boxed{3\pi \text{ radians}}$

Note: We multiply by $\frac{\pi}{180^\circ}$ so that the degrees cancel. This will help you remember what to multiply by.

<u>Converting from Radians (*R*) to Degrees (*D*):</u> $D = R \cdot \frac{180^\circ}{\pi}$

Or use the proportion: $\frac{\theta_d}{180^\circ} = \frac{\theta_r}{\pi}$

Ex4: Convert $\frac{3\pi}{4}$ radians to degrees.

$\frac{3\pi}{4} \cdot \frac{180^\circ}{\pi} = \boxed{135^\circ}$

Note: We multiply by $\frac{180^\circ}{\pi}$ so that the π's cancel.

<u>Special Angles to Memorize</u> (Teacher Note: Have students fill this in for practice.)

Degrees	30°	45°	60°	90°	120°	135°	150°	180°
Radians	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π

<u>Arc Length:</u> $s = r\theta$ (θ must be measured in **radians**)

Ex5: A circle has an 8 inch diameter. Find the length of an arc intercepted by a 240° central angle.

Step One: Convert to radians. $\theta = 240^\circ \cdot \frac{\pi}{180^\circ} = \frac{4\pi}{3}$

Step Two: Find the radius. $r = \frac{d}{2} = \frac{8}{2} = 4$

Step Three: Solve for *s*. $s = 4 \cdot \frac{4\pi}{3} = \boxed{\frac{16\pi}{3} \text{ in}}$

<u>Linear Speed</u> (example: miles per hour): $l = \frac{s}{t} = \frac{r\theta}{t} = \frac{\text{length}}{\text{time}}$

<u>Angular Speed</u> (example: rotations per minute): $A = \frac{\theta}{t} = \frac{\text{angle}}{\text{time}}$

Ex6: Find the linear and angular speed (per second) of a 10.2 cm second hand.

Note: Each revolution generates 2π radians.

Linear Speed: A second hand travels half the circumference in 30 seconds:

$$l = \frac{10.2\pi}{30} = 0.34\pi \approx \boxed{1.068\,\text{cm/sec}} \quad \text{or} \quad \frac{10.2(2\pi)}{60} = 0.34\pi \approx 1.068 \text{ cm / sec}$$

Angular Speed: $A = \frac{2\pi}{60} = \boxed{\frac{\pi}{30}\text{radians/sec}}$ or $\frac{180^\circ}{30} = \boxed{6^\circ\text{ / sec}}$

Note: The angular speed does not depend on the length of the second hand!

For example: The riders on a carousel all have the same angular speed yet the riders on the outside have a greater linear speed than those on the inside due to a larger radius.

Ex7: Find the speed in mph of 36 in diameter wheels moving at 630 rpm (revolutions per minute).

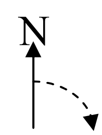
Unit Conversion: $\frac{630\ \cancel{\text{rev}}}{1\ \cancel{\text{min}}} \cdot \frac{60\ \cancel{\text{min}}}{1\,\text{hr}} \cdot \frac{36\pi\ \cancel{\text{in}}}{1\ \cancel{\text{rev}}} \cdot \frac{1\ \cancel{\text{ft}}}{12\ \cancel{\text{in}}} \cdot \frac{1\,\text{mi}}{5280\ \cancel{\text{ft}}} = \frac{945\pi}{44} \approx \boxed{67.473\text{ mph}}$

1 statute (land) mile = 5280 feet   Earth's radius $\approx$ 3956 miles

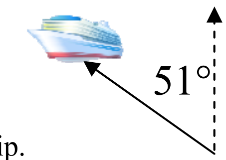
1 **nautical mile** = 1 minute of arc length along the Earth's equator

Ex8: How many statute miles are there in a nautical mile?

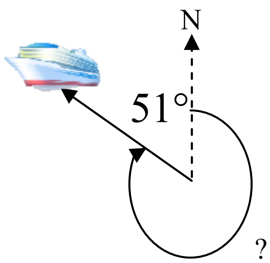
$s = r\theta$ $r = 3956$ $\theta = \frac{\pi}{180^\circ} \cdot \frac{1}{60}^\circ = \frac{\pi}{10800}$ $s = 3956\left(\frac{\pi}{10800}\right) \approx \boxed{1.151\,\text{miles}}$



<u>Bearing</u>: the **course** of an object given as the angle measured clockwise from due north



Ex9: Use the picture to find the bearing of the ship.



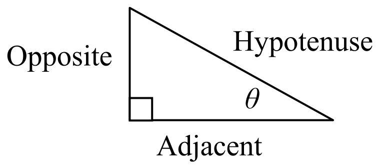
$? = 360 - 51 = \boxed{309^{\circ}}$

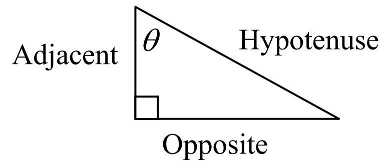
<u>You Try</u>:

1. A lawn roller with 10 inch radius wheels makes 1.2 revolutions/second. Find the linear and angular speed.
2. How many nautical miles are in a statute mile? Show your work.

<u>QOD</u>: How are radian and degree measures different? How are they similar?

Syllabus Objectives: 3.2 – The student will solve problems using the inverse of trigonometric functions. 3.5 – The student will solve application problems involving triangles.





Note: The adjacent and opposite sides are always the legs of the right triangle, and depend upon which angle is used.

Trigonometric Ratios of θ:

Sine: $\sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}}$ Cosine: $\cos\theta = \dfrac{\text{adjacent}}{\text{hypotenuse}}$ Tangent: $\tan\theta = \dfrac{\text{opposite}}{\text{adjacent}}$

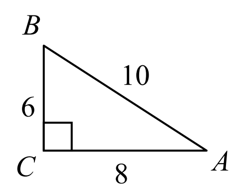
Memory Aid: SOHCAHTOA

Reciprocal Functions:

Cosecant: $\csc\theta = \dfrac{\text{hypotenuse}}{\text{opposite}}$ Secant: $\sec\theta = \dfrac{\text{hypotenuse}}{\text{adjacent}}$ Cotangent: $\cot\theta = \dfrac{\text{adjacent}}{\text{opposite}}$

Think About It: Which trigonometric ratios in a triangle must always be less than 1? Why?

Sine and cosine, because the lengths of the opposite and adjacent legs are always smaller than the length of the hypotenuse.



Ex1: For $\triangle ABC$, find the six trig ratios of $\angle A$.

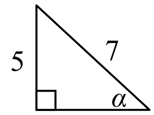
$\sin A = \dfrac{6}{10} = \dfrac{3}{5}$ $\cos A = \dfrac{8}{10} = \dfrac{4}{5}$ $\tan A = \dfrac{6}{8} = \dfrac{3}{4}$

$\csc A = \dfrac{5}{3}$ $\sec A = \dfrac{5}{4}$ $\cot A = \dfrac{4}{3}$

Teacher Note: It may help students to label the sides as Opp, Adj, and Hyp first. Have students try finding the six trig ratios for $\angle B$.

Special Relationship: $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

Ex2: Find the other trig ratios given $\sin\alpha = \frac{5}{7}$.

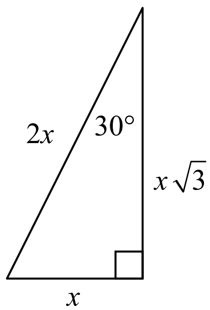


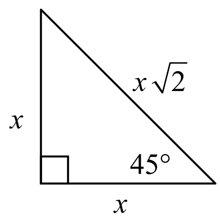
$\sin\alpha = \frac{5}{7} = \frac{\text{opp}}{\text{hyp}}$

Find the missing leg (a): $a^2 + 5^2 = 7^2 \Rightarrow a = \sqrt{24}$ This is the adjacent leg of angle α.

$\cos\alpha = \frac{\sqrt{24}}{7}$, $\tan\alpha = \frac{5}{\sqrt{24}}$, $\csc\alpha = \frac{7}{5}$, $\sec\alpha = \frac{7}{\sqrt{24}}$, $\cot\alpha = \frac{\sqrt{24}}{5}$

<u>Special Right Triangles</u>: 30-60-90 & 45-45-90





θ (Degrees)	**30°**	**45°**	**60°**
θ (Radians)	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$
sinθ	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$
cosθ	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$
tanθ	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$

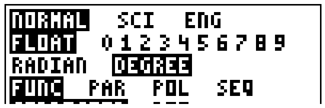
Teacher Note: Have students fill in the table (un-bolded values). This table must be memorized!

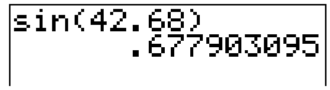
Evaluating Trigonometric Ratios on the Calculator

Note: Check the MODE on your calculator and be sure it is correct for the question asked (radian/degree). Unless an angle measure is shown with the degree symbol, assume the angle is in radians.

Ex3: Evaluate the following using a calculator.

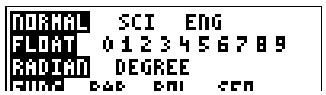
1. $\sin 42.68^\circ$ Mode: degree

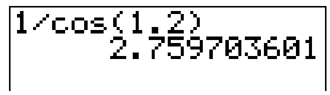




$\sin 42.68^\circ \approx \boxed{0.678}$

2. $\sec 1.2$ Mode: radian



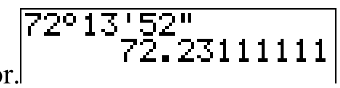


Note: There is not a key for secant. We must use $1/(\cos x)$ because secant is the reciprocal of cosine. $\sec 1.2 \approx \boxed{2.760}$

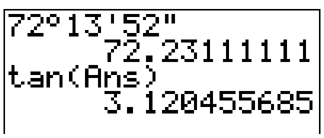
Also recall: $\csc x = \dfrac{1}{\sin x}$ and $\cot x = \dfrac{1}{\tan x}$

3. $\tan 72^\circ 13' 52''$ Mode: degree

Convert DMS to decimal degrees on calculator.



Note: The degree and minute symbols can be found in the ANGLE menu. The seconds symbol can be found above the + sign (alpha +).

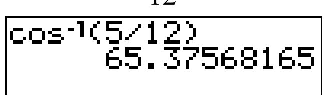


$\tan 72^\circ 13' 52'' \approx \boxed{3.120}$

Inverse Trigonometric Functions: use these to find the angle when given a trig ratio

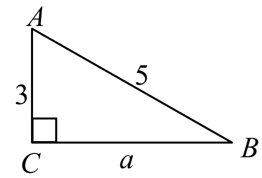
Ex4: Find θ (in degrees) if $\cos\theta = \dfrac{5}{12}$.

$\theta = \cos^{-1}\left(\dfrac{5}{12}\right)$ Mode: degrees



$\boxed{\theta \approx 65.376^\circ}$

<u>Solving a Triangle</u>: find the missing angles and sides with given information



Ex5: Solve the triangle (find all missing sides and angles).

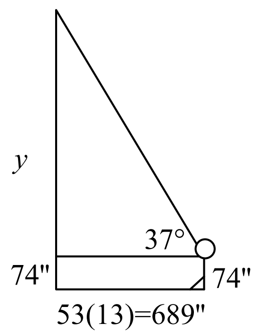
$\cos A = \frac{3}{5} \Rightarrow A = \cos^{-1}\left(\frac{3}{5}\right) \approx 53.130^\circ$ $\qquad m\angle B \approx 90 - 53.130 = 36.87^\circ$

$a = 4$ (Pythagorean Triple: 3-4-5; or use the Pythagorean Thm) $\boxed{m\angle A \approx 53.13^\circ,\ m\angle B \approx 36.87^\circ,\ a = 4}$

<u>Application Problem</u>

Ex6: A 6 ft 2 in man looks up at a 37° angle to the top of a building. He places his heel to toe 53 times and his shoe is 13 in. How tall is the building?

Draw a picture:

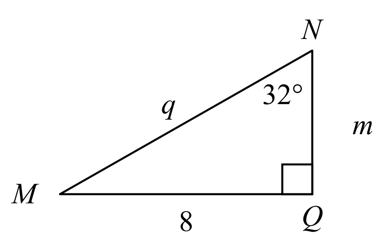


The height of the building is (y + 74) inches.

Use trig in the right triangle to solve for y: $\tan 37^\circ = \frac{y}{689} \Rightarrow y = 689 \tan 37^\circ \approx 519.199$

Height of the building = $y + 74 = 519.199 + 74 = 593.199$ inches

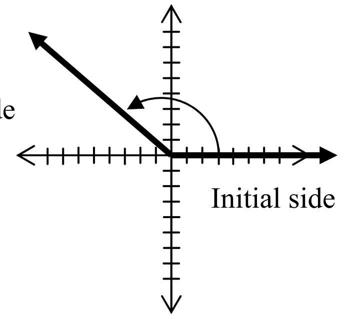
Convert to feet: The building is approximately $\frac{593.199}{12} \approx \boxed{49.433\text{ft}}$ tall.



<u>You Try</u>: Solve the triangle (find all missing sides and angles).

<u>QOD</u>: Explain how to find the inverse cotangent of an angle on the calculator.

Syllabus Objectives: 3.1 – The student will solve problems using the unit circle. 3.2 – The student will solve problems using the inverse of trigonometric functions.



Standard Position (of an angle): initial side is on the positive x-axis; positive angles rotate counter-clockwise; negative angles rotate clockwise

Coterminal Angles: angles with the same initial and terminal sides, for example $\pm 360^\circ$ or $\pm 2\pi$

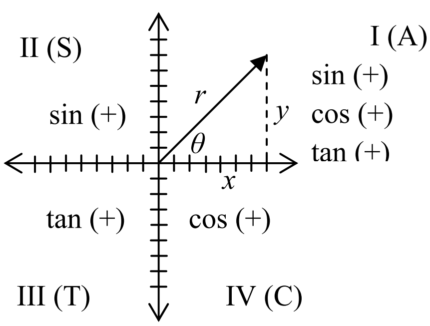
Ex1: Find a positive and negative coterminal angle for each.

1. 130° Positive: $130^\circ + 360^\circ = \boxed{490^\circ}$ Negative: $130^\circ - 360^\circ = \boxed{-230^\circ}$
2. $\frac{17\pi}{2}$

Positive: $\frac{17\pi}{2} - 2\pi = \boxed{\frac{13\pi}{2}} - 2\pi = \boxed{\frac{9\pi}{2}} - 2\pi = \boxed{\frac{\pi}{2}}$ (Note: Any of these are acceptable answers.)

Negative: $\frac{\pi}{2} - 2\pi = \boxed{-\frac{3\pi}{2}}$

Trigonometric Functions of any Angle



$x^2 + y^2 = r^2 \Rightarrow r = \sqrt{x^2 + y^2}$

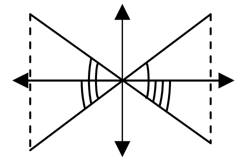
$\sin\theta = \frac{y}{r}$ $\csc\theta = \frac{r}{y}$

$\cos\theta = \frac{x}{r}$ $\sec\theta = \frac{r}{x}$

$\tan\theta = \frac{y}{x}$ $\cot\theta \frac{x}{y}$

Memory Aid: To remember which trig functions are positive in which quadrant, remember **A**wesome **S**tudents **T**ake **C**alculus. A – all are positive in QI, S – sine (and cosecant) is positive in QII, T – tangent (and cotangent) is positive in QIII, C – cosine (and secant) is positive in QIV.

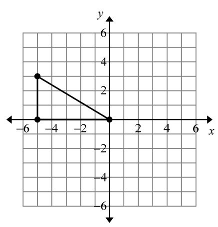
Reference Angle: every angle has a **reference angle**, with initial side as the x-axis



Evaluating Trig Functions Given a Point

- Use the ordered pair as x and y
- Find r: $r=\sqrt{x^2+y^2}$
- Check the signs of your answers by the quadrant

Ex2: Find the six trig functions of θ in standard position whose terminal side contains the point $(-5,3)$.



Note: The point is in QII, so sine (and cosecant) will be positive.

$$x=-5,\ y=3,\ r=\sqrt{(-5)^2+3^2}=\sqrt{34}$$

$$\sin\theta=\frac{3}{\sqrt{34}} \qquad \csc\theta=\frac{\sqrt{34}}{3}$$

$$\cos\theta=-\frac{5}{\sqrt{34}} \qquad \sec\theta=-\frac{\sqrt{34}}{5}$$

$$\tan\theta=-\frac{3}{5} \qquad \cot\theta=-\frac{5}{3}$$

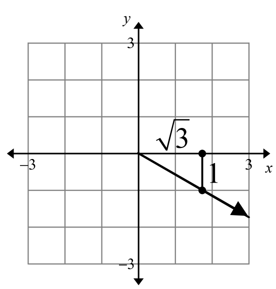
Evaluating Trig Functions Given an Angle

- Find the reference angle
- Label the sides (if the reference angle creates a special right triangle)
- Check the signs of your answers by the quadrant

Ex3: Find the 6 trig functions of 330°.

To find the reference angle, start at the positive x-axis and go counter-clockwise 330°.

Reference Angle = 30° $x=\sqrt{3},\ y=-1,\ r=2$ (30-60-90 triangle)



$$\sin\theta=-\frac{1}{2} \qquad \csc\theta=-2$$

$$\cos\theta=\frac{\sqrt{3}}{2} \qquad \sec\theta=\frac{2}{\sqrt{3}}$$

QIV, so $\cos\theta$ is positive.

$$\tan\theta=-\frac{1}{\sqrt{3}} \qquad \cot\theta=-\sqrt{3}$$

Evaluating Trig Functions Given One Trig Function

- Determine which quadrant by the signs of the trig functions given
- Label the sides corresponding to the given trig functions
- Check the signs of your answers by the quadrant

Ex: Find the six trig functions if $\sin\theta = \frac{5}{7}$ and $\tan\theta > 0$.

Since both sine and tangent are positive, we know that θ is in the first quadrant.

$\sin\theta = \frac{5}{7} = \frac{\text{opp}}{\text{hyp}} = \frac{y}{r}$ $\quad r^2 = x^2 + y^2 \Rightarrow 7^2 = x^2 + 5^2 \Rightarrow x = \sqrt{24}$

$\sin\theta = \frac{5}{7}$ $\quad \csc\theta = \frac{7}{5}$ $\qquad \sin\theta = \frac{5}{7}$ $\quad \csc\theta = \frac{7}{5}$

$\cos\theta = \frac{\sqrt{24}}{7}$ $\quad \sec\theta = \frac{7}{\sqrt{24}}$ $\quad$ or $\quad \cos\theta = \frac{2\sqrt{6}}{7}$ $\quad \sec\theta = \frac{7}{2\sqrt{6}}$

$\tan\theta = \frac{5}{\sqrt{24}}$ $\quad \cot\theta = \frac{\sqrt{24}}{5}$ $\qquad \tan\theta = \frac{5}{2\sqrt{6}}$ $\quad \cot\theta = \frac{2\sqrt{6}}{5}$

Quadrantal Angles: angles with the terminal side on the axes, for example, $0^\circ, 90^\circ, 180^\circ, 270^\circ$

Degrees	**0°/360°**	**30°**	**45°**	**60°**	**90°**	**180°**	**270°**
Radians	$0/2\pi$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
sinθ	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0	-1
cosθ	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1	0
tanθ	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	und	0	und

Teacher Note: Have students fill in the table (unbolded cells). Needs to be memorized!

Ex4: Find $\csc 13\pi$.

Find the reference angle: $13\pi = 12\pi + \pi = 6(2\pi) + \pi \Rightarrow \pi$

$\csc 13\pi = \csc\pi = \frac{1}{\sin\pi} = \frac{1}{0} \Rightarrow \boxed{\text{undefined}}$

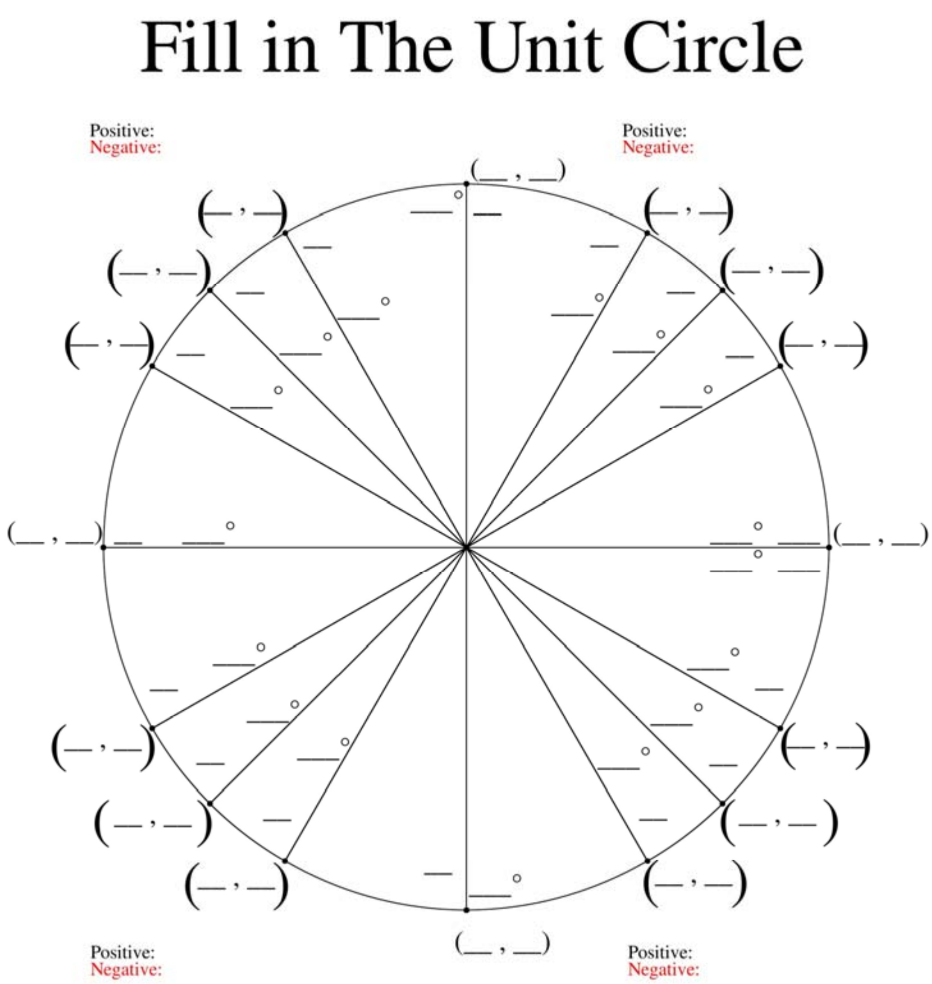
Periodic Functions: A function $y = f(t)$ is **periodic** if there is a positive number c such that $f(t+c) = f(t)$ for all values of t in the domain of f.

The **period** of sine and cosine are 2π and the **period** of tangent is π.

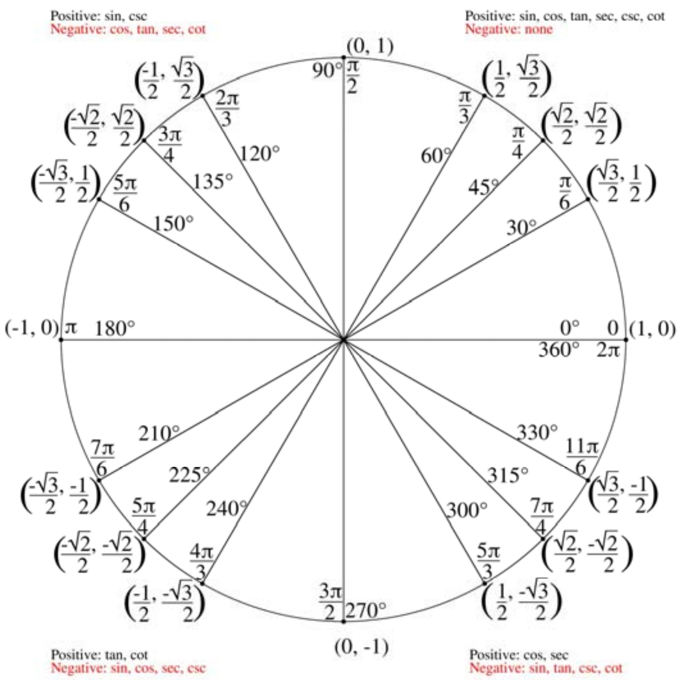
Ex5: Evaluate $\sin\frac{3601\pi}{2}$ without a calculator.

$$\sin\frac{3601\pi}{2} = \sin\left(\frac{3600\pi}{2} + \frac{\pi}{2}\right) = \sin\left(1800\pi + \frac{\pi}{2}\right) = \sin\frac{\pi}{2} = \boxed{1}$$

The Unit Circle: $r = 1$



The Unit Circle



<u>You Try</u>: Find the following without a calculator.

1. $\cos 990^{\circ}$
2. $\cot \dfrac{5\pi}{3}$
3. $\sin 225°$

<u>QOD</u>: How is the unit circle used to evaluate the trigonometric functions? Explain.

Syllabus Objectives: 4.1 – The student will sketch the graphs of the six trigonometric functions. 4.3 – The student will graph transformations of the basic trigonometric functions. 4.6 – The student will model and solve real-world application problems involving sinusoidal functions.

Teacher Note: Have students fill out the table as quickly as they can. Discuss patterns they can use to be able to memorize these.

$\boldsymbol{\theta}$	**0°**	**30°**	**45°**	**60°**	**90°**	**120°**	**135°**	**150°**	**180°**	**210°**	**225°**	**240°**	**270°**	**360°**
radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	2π
sin$\boldsymbol{\theta}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1	0
cos$\boldsymbol{\theta}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0	1
tan$\boldsymbol{\theta}$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	und	$-\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	und	0

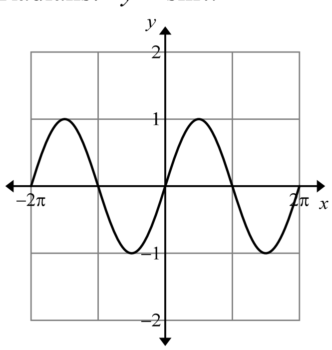
Sinusoid: a function whose graph is a sine or cosine function; can be written in the form $y = a\sin(bx + c) + d$

Sinusoidal Axis: the horizontal line that passes through the middle of a sinusoid

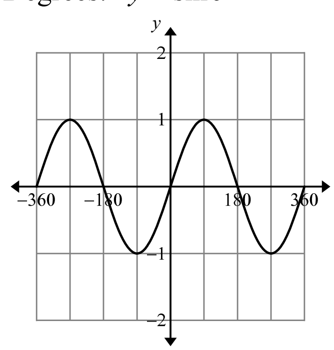
Graph of the Sine Function

Use the table above to sketch the graph.

Radians: $y = \sin x$



Degrees: $y = \sin\theta$



Sine is **periodic**, so we can extend the graph to the left and right.

Characteristics:

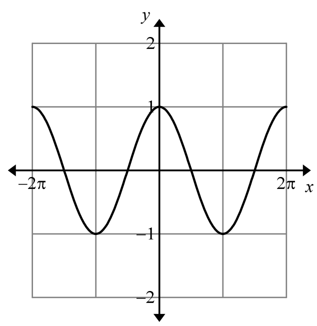
Domain: $(-\infty, \infty)$ Range: $[-1, 1]$ y-intercept: $(0, 0)$ x-intercepts: $(0 + k\pi, 0)$; $k \in \mathbb{Z}$

Absolute Max = 1 Absolute Min = −1 Decreasing: $\left(\frac{\pi}{2},\frac{3\pi}{2}\right)+2\pi$

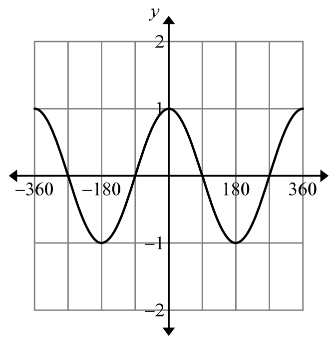
Increasing: $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)+2\pi$ Period = 2π Sinusoidal Axis: $y=0$

<u>Graph of the Cosine Function</u>
Use the table above to sketch the graph.

Radians: $y=\cos x$



Degrees: $y=\cos\theta$



Cosine is **periodic**, so we can extend the graph to the left and right.

<u>Characteristics:</u>
Domain: $(-\infty,\infty)$ Range: $[-1,1]$ *y*-intercept: $(0,1)$

x-intercepts: $\left(\frac{\pi}{2}+k\pi,0\right)$; $k\in\mathbb{Z}$ Absolute Max = 1 Absolute Min = −1

Decreasing: $(\pi,2\pi)+2\pi$ Increasing: $(0,\pi)+2\pi$ Period = 2π Sinusoidal Axis: $y=0$

<u>Note</u>: We will work mostly with the graphs in radians, since it is the graph of the function in terms of *x*.

<u>Recall</u>: $f(x)=a(x-h)^2+k$ is a transformation of the graph of $f(x)=x^2$. a represents the vertical stretch/shrink, *h* is a horizontal shift and *k* is a vertical shift.

<u>Transformations of Sinusoids:</u>

General Form $y=a\sin\left[b(x-h)\right]+k$

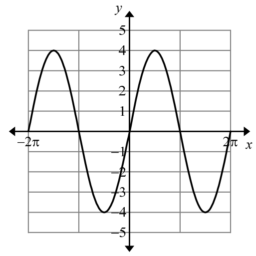
a: vertical stretch and/or reflection over *x*-axis **amplitude** = $|a|$

h: horizontal shift **phase shift** = *h* *k*: vertical shift

period = $\frac{2\pi}{|b|}$ **frequency** = $\frac{|b|}{2\pi}$ $|b|$ = number of cycles completed in 2π

<u>Amplitude:</u> the distance from the sinusoidal axis to the maximum value (half the height of a wave)

Ex1: Sketch the graph of $y = 4\sin x$.

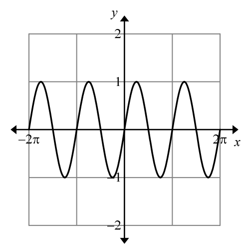


Vertical stretch: 4 Amplitude = 4

<u>Period</u>: the length of time taken for one full cycle of the wave $P = \frac{2\pi}{|b|}$

<u>Frequency</u>: the number of complete cycles the wave completes per unit of time freq = $\frac{|b|}{2\pi}$

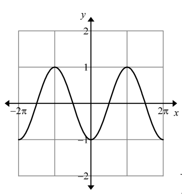
Ex2: Sketch the graph of $y = \sin(2x)$.



Period = $\frac{2\pi}{2} = \pi$ Frequency = $\frac{2}{2\pi} = \frac{1}{\pi}$

<u>Reflection</u>: If $a < 0$, the sinusoid is reflected over the x-axis.

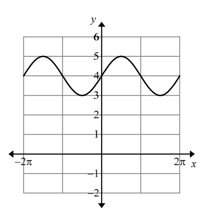
Ex: Sketch the graph of $y = -\cos x$.



Note: Period, amplitude, etc. all stay the same.

<u>Vertical Translation:</u> in the sinusoid $y = a\sin\left[b(x-h)\right]+k$, the line $y = k$ is the sinusoidal axis

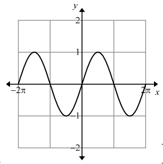
Ex3: Sketch the graph of $y = 4 + \sin x$.



Sinusoidal Axis: $y = 4$

<u>Phase Shift:</u> the horizontal translation, h, of a sinusoid $y = a\sin\left[b(x-h)\right]+k$

Ex4: Sketch the graph of $y=\cos\left(x-\frac{\pi}{2}\right)$.

Shift $y=\cos x$ to the right $\frac{\pi}{2}$.  Does this graph look familiar? It is $y=\sin x$!

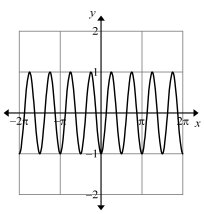
<u>Note:</u> Every cosine function can be written as a sine function using a phase shift.

$$y=\cos x \Leftrightarrow y=\sin\left(x+\frac{\pi}{2}\right)$$

Ex5: Sketch the graph of $y=\cos(4x+\pi)$. Then rewrite the function as a sine function.

The coefficient of x must equal 1. Factor out any other coefficient to find the actual horizontal shift.

$y=\cos(4x+\pi) \Rightarrow y=\cos\left[4\left(x+\frac{\pi}{4}\right)\right]$ Phase Shift: $-\frac{\pi}{4}$ Period: $\frac{2\pi}{4}=\frac{\pi}{2}$

Graph: 

Write as a sine function:

$$y=\cos(4x+\pi) \Rightarrow y=\sin\left(4x+\pi+\frac{\pi}{2}\right) \Rightarrow y=\sin\left(4x+\frac{3\pi}{2}\right) \Rightarrow y=\sin\left[4\left(x+\frac{3\pi}{8}\right)\right]$$

<u>Writing the Equation of a Sinusoid</u>

Ex6: Write the equation of a sinusoid with amplitude 4 and period $\frac{\pi}{3}$ that passes through $(6,0)$.

Amplitude: $a=4$; Period: $\frac{\pi}{3}=\frac{2\pi}{b} \Rightarrow b=6$;

Phase Shift: normally passes through $(0,0)$, so shift right 6 units

$$y=4\sin(6x-6) \text{ or } \boxed{y=4\sin[6(x-1)]}$$

<u>You Try:</u> Describe the transformations of $y=7\sin\left(2x+\frac{\pi}{4}\right)+0.5$. Then sketch the graph.

<u>QOD</u>: How do you convert from a cosine function to a sine function? Explain.

Syllabus Objectives: 4.1 – The student will sketch the graphs of the six trigonometric functions. 4.3 – The student will graph transformations of the basic trigonometric functions.

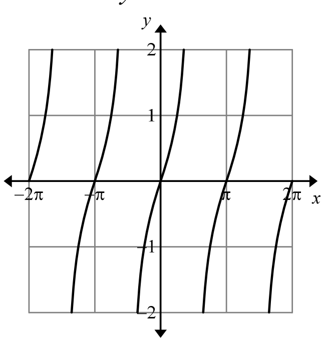
Teacher Note: Have students fill out the table as quickly as they can.

θ	0°	30°	45°	60°	90°	120°	135°	150°	180°	210°	225°	240°	270°	360°
radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	2π
sinθ	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1	0
cosθ	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0	1
tanθ	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	und	$-\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	und	0

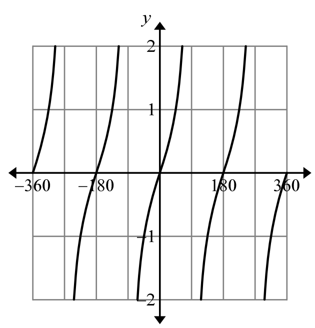
Graph of the Tangent Function

Use the table above to sketch the graph.

Radians: $y = \tan x$



Degrees: $y = \tan\theta$



Characteristics:

Domain: $x \neq \frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}$ Range: $(-\infty, \infty)$ Intercepts: $(k\pi, 0)$ Always Increasing

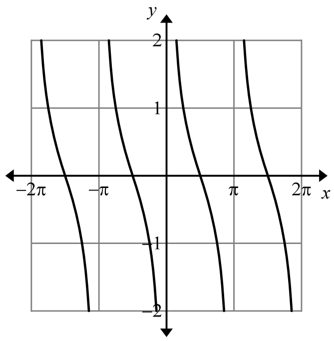
Period: π Vertical Asymptotes: $x = \frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}$

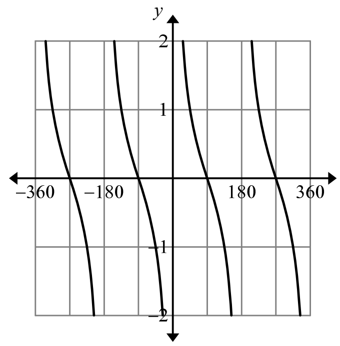
Note: $\tan x = \frac{\sin x}{\cos x}$, so the zeros of sine are the zeros of tangent, and the zeros of cosine are the vertical asymptotes of tangent.

Graph of the Cotangent Function

Use the table above to sketch the graph. (Cotangent is the reciprocal of tangent.)

Radians: $y = \cot x$ Degrees: $y = \cot\theta$





Characteristics:

Domain: $x \neq \dfrac{\pi}{2} + k\pi,\ k \in \mathbb{Z}$ Range: $(-\infty, \infty)$ Intercepts: $\left(\dfrac{\pi}{2} + k\pi, 0\right)$

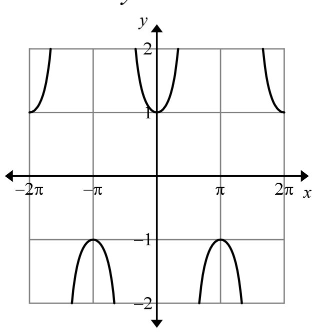
Always Decreasing Period: π Vertical Asymptotes: $x = k\pi,\ k \in \mathbb{Z}$

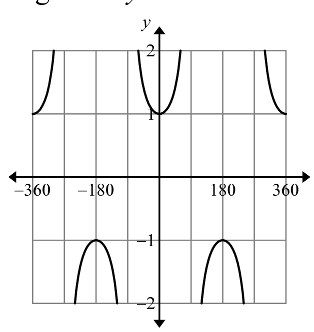
Note: $\cot x = \dfrac{\cos x}{\sin x}$, so the zeros of cosine are the zeros of cotangent, and the zeros of sine are the vertical asymptotes of cotangent.

Graph of the Secant Function

Use the table above to sketch the graph. (Secant is the reciprocal of cosine.)

Radians: $y = \sec x$ Degrees: $y = \sec\theta$





Note: It may help to graph the cosine function first.

Characteristics:

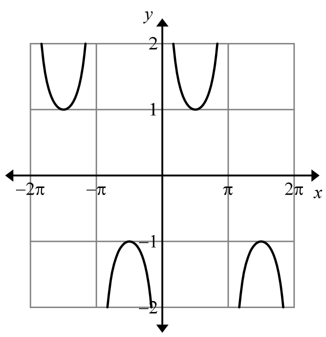
Domain: $x \neq \dfrac{\pi}{2} + k\pi,\ k \in \mathbb{Z}$ Range: $(-\infty, -1] \cup [-1, \infty)$ Intercept: $(0, 1)$

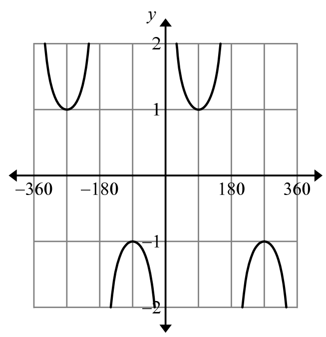
Local Max: -1 Local Min: 1 Period: 2π Vertical Asymptotes: $x = \dfrac{\pi}{2} + k\pi,\ k \in \mathbb{Z}$

Graph of the Cosecant Function

Use the table above to sketch the graph. (Cosecant is the reciprocal of sine.)

Radians: $y = \csc x$ Degrees: $y = \csc\theta$





Note: It may help to graph the sine function first.

Characteristics:

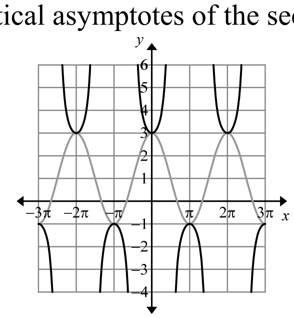
Domain: $x \neq k\pi,\, k \in \mathbb{Z}$ Range: $(-\infty,-1] \cup [-1,\infty)$ Intercept: none

Local Max: -1 Local Min: 1 Period: 2π Vertical Asymptotes: $x = k\pi,\, k \in \mathbb{Z}$

Transformations: Sketch the graph of the reciprocal function first!

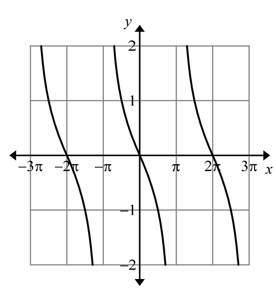
Ex1: Sketch the graph of $y = 1 + 2\sec x$.

Graph $y = 1 + 2\cos x$ (gray). Shift up 1, amplitude 2. Use the zeros of the cosine function to determine the vertical asymptotes of the secant function. Sketch the secant function (black).



Note: Amplitude is not applicable to secant. The number 2 represents a vertical stretch.

Ex2: Sketch the graph of $y = -\tan\left(\dfrac{x}{2}\right)$.



Period: $P = \dfrac{\pi}{1/2} = 2\pi$ Reflect over x-axis

Solving a Trigonometric Equation Algebraically

Ex3: Solve for x in the given interval using reference triangles in the proper quadrant.

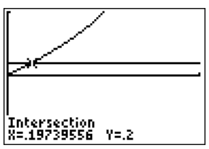
$\sec x = -2, \frac{\pi}{2} \le x \le \pi$

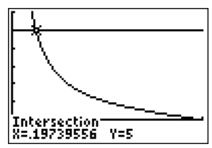
Quadrant II; $\cos x = -\frac{1}{2}$ Reference Angle $= \frac{\pi}{3}$ In QII: $x = \pi - \frac{\pi}{3} \Rightarrow \boxed{x = \frac{2\pi}{3}}$

Solving a Trigonometric Equation Graphically

Ex4: Use a calculator to solve for x in the given interval: $\cot x = 5, 0 \le x \le \frac{\pi}{2}$

The equation $\cot x = 5$ is equivalent to the equation $\tan x = \frac{1}{5}$. Graph each side of the equation and find the point of intersection for $0 \le x \le \frac{\pi}{2}$. Restrict the window to only include these values of x.





Note: We could have also graphed $y_1 = \frac{1}{\tan x}$ and $y_2 = 5$. (Shown in the second graph above.)

Solution: $\boxed{x \approx 0.197}$

You Try: Graph the function $y = -\csc(2x - \pi)$.

QOD:

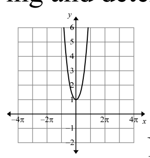
1. What trigonometric function is the slope of the terminal side of an angle in standard position? Explain your answer using the unit circle.
2. Which trigonometric function(s) are odd? Which are even?

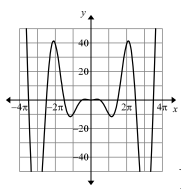
Syllabus Objective: 4.4 – The student will sketch the graph of combined trigonometric functions (sum, difference, product or composition of two trigonometric or polynomial functions) with or without technology.

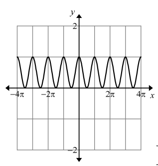
Trigonometric Functions Combined with Algebraic Functions

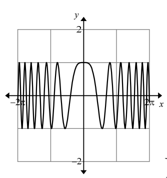
Recall: Periodic Functions: A function $y = f(t)$ is **periodic** if there is a positive number *c* such that $f(t+c) = f(t)$ for all values of *t* in the domain of *f*.


Graph the following and determine which function(s) are periodic.

1. $y = \cos x + x^2$  Not Periodic

2. $y = x^2 \cos x$  Not Periodic

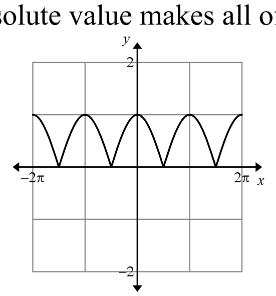
3. $y = (\cos x)^2$  Periodic

4. $y = \cos\left(x^2\right)$  Not Periodic

Absolute Value of Trig Functions

Ex1: Sketch the graph of $f(x) = |\cos x|$.

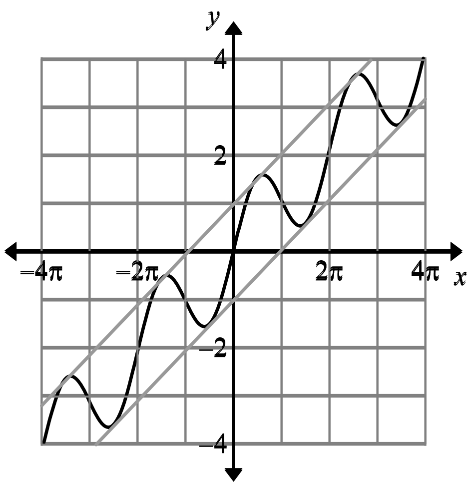
The absolute value makes all of the *y*-values of the function positive.



Note: The period of $f(x) = |\cos x|$ is π.

Adding a Sinusoid to a Linear Function

Ex2: Analyze the graph of $y=\frac{1}{3}x+\sin x$. Is it periodic?



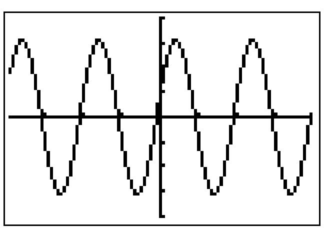
The function oscillates between the lines $y=\frac{1}{3}x-1$ and $y=\frac{1}{3}x+1$. It is NOT periodic.

Are the Sums of Sinusoid Function Sinusoids?

Teacher Note: Have students graph various sums to determine a pattern.

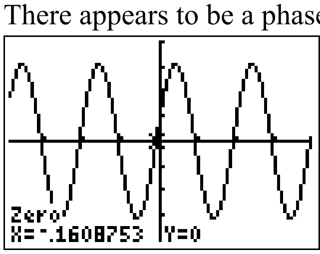
Conclusion: The sum of two sinusoids is a sinusoid if and only if the two sinusoids being added have the same period. The sum has this same period as well.

Ex3: Graph the function $f(x)=3\sin 2x+\cos 2x$. Write a sinusoid function that estimates $f(x)$.

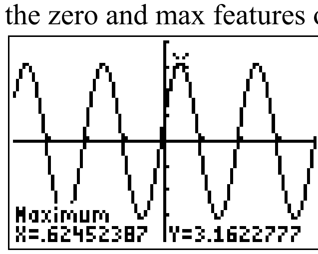


The period of $f(x)$ is the same as each of the addends.

There appears to be a phase shift and an amplitude. Use the zero and max features on the calculator.



Phase Shift: left 0.161



Amplitude: $a = 3.162$

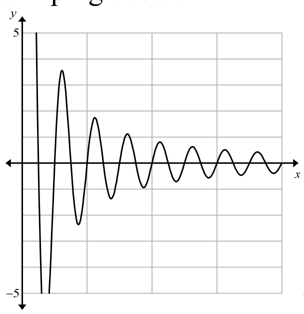
Sinusoid: $\boxed{f(x)=3.162\sin 2(x+0.161)}$

Note: Check your answer by graphing.

<u>Damped Oscillation</u>: the amplitude is reduced in the wave of a graph (the amplitude decreases as time increases

The graph of the product of a function, $f(x)$, and a sinusoid oscillates between the graph of the function, $f(x)$ and its opposite, $-f(x)$. The function $f(x)$ is called the **damping factor**.

Ex4: Graph the function $f(x)=2x^{-1.2}\sin(4\pi x)$. Identify the damping factor. Where does the damping occur?



Damping Factor: $2x^{-1.2}$ Occurs as $x \to \infty$

<u>You Try</u>: Does the function $f(x)=\sin(3x+3)+\cos(3x+2)$ represent a sinusoid? If yes, write an equation for the sinusoid.

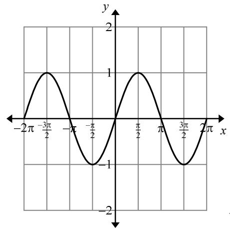
<u>QOD</u>: Are all sinusoids periodic? Are all periodic functions sinusoids? Explain.

Syllabus Objective: 4.2 – The student will sketch the graphs of the principal inverses of the six trigonometric functions.

Recall: In order for a function to have an inverse function, it must be **one-to-one** (must pass both the horizontal and vertical line tests).

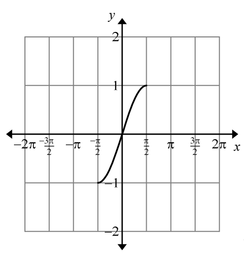
Notation: The inverse of $f(x)$ is labeled as $f^{-1}(x)$.

Inverse of the Sine Function

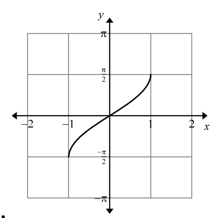
Graph of $f(x) = \sin x$ 

Domain: $(-\infty, \infty)$ Range: $[-1, 1]$

In order for $f(x) = \sin x$ to have an inverse function, we must restrict its domain to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.



To graph the inverse of sine, reflect about the line $y = x$.



Domain of $f^{-1}(x) = \sin^{-1} x$: $[-1, 1]$ Range of $f^{-1}(x) = \sin^{-1} x$: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Notation: Inverse of Sine $f^{-1}(x) = \sin^{-1} x$ or $y = \arcsin x$ (arcsine)

Note: $y = \sin^{-1} x$ denotes the inverse of sine (arcsine). It is NOT the reciprocal of sine (cosecant).

Evaluating the Inverse Sine Function

Ex1: Find the exact values of the following.

1. $\arcsin \frac{\sqrt{2}}{2}$ What value of x makes the equation $\sin x = \frac{\sqrt{2}}{2}$ true? $x = \boxed{\frac{\pi}{4}}$

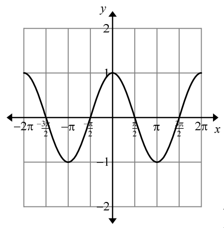
Note: The range of arcsine is restricted to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, so $\frac{\pi}{4}$ is the only possible answer.

2. $\sin^{-1} 3$ What value of x makes the equation $\sin x = 3$ true?

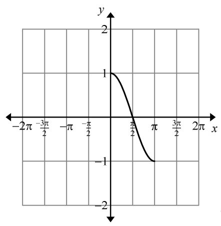
No solution, because 3 > 1 and the domain of arcsine = the range of sine = $[-1, 1]$

3. $\sin^{-1}\left(\sin \frac{2\pi}{3}\right)$ Taking the inverse sine of the sine function results in the argument. $\boxed{\frac{2\pi}{3}}$

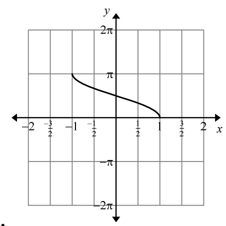
Inverse of the Cosine Function

Graph of $f(x) = \cos x$  Domain: $(-\infty, \infty)$ Range: $[-1, 1]$

In order for $f(x) = \cos x$ to have an inverse function, we must restrict its domain to $[0, \pi]$.



To graph the inverse of cosine, reflect about the line $y = x$.



Domain of $f^{-1}(x) = \cos^{-1} x$: $[-1, 1]$ Range of $f^{-1}(x) = \cos^{-1} x$: $[0, \pi]$

Notation: Inverse of Cosine $f^{-1}(x) = \cos^{-1} x$ or $y = \arccos x$ (arccosine)

Note: $y = \cos^{-1} x$ denotes the inverse of cosine (arccosine). It is NOT the reciprocal of cosine (secant).

Evaluating the Inverse Cosine Function

Ex2: Find the exact values of the following.

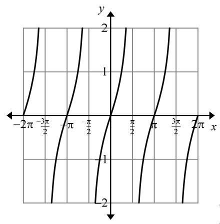
1. $\arccos\left(-\frac{\sqrt{2}}{2}\right)$ What value of x makes the equation $\cos x = -\frac{\sqrt{2}}{2}$ true? $x = \boxed{\frac{3\pi}{4}}$

   Note: The range of arcsine is restricted to $[0, \pi]$, so $\frac{3\pi}{4}$ is the only possible answer.

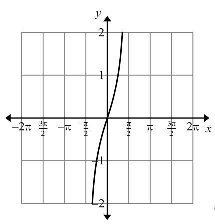
2. $\sin\left(\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)\right)$ $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$, so $\sin\left(\frac{\pi}{6}\right) = \boxed{\frac{1}{2}}$

3. $\cos^{-1}\left(\cos\frac{11\pi}{6}\right)$ Taking the inverse cosine of the cosine function results in the argument. $\boxed{\frac{11\pi}{6}}$

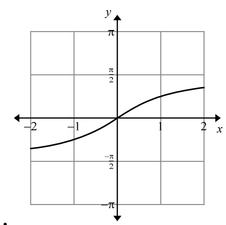
Inverse of the Tangent Function

Graph of $f(x)=\tan x$  Domain: $x \neq \frac{\pi}{2}+k\pi$ Range: $(-\infty,\infty)$

In order for $f(x)=\tan x$ to have an inverse function, we must restrict its domain to $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$.



To graph the inverse of tangent, reflect about the line $y=x$.



Domain of $f^{-1}(x)=\tan^{-1}x$: $(-\infty,\infty)$ Range of $f^{-1}(x)=\tan^{-1}x$: $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

Notation: Inverse of Tangent $f^{-1}(x)=\tan^{-1}x$ or $y=\arctan x$ (arctangent)

Note: $y=\tan^{-1}x$ denotes the inverse of tangent (arctangent). It is NOT the reciprocal of tangent (cotangent).

Evaluating the Inverse Tangent Function

Ex3: Find the exact values of the following.

1. $\sin\left(\tan^{-1}\frac{\sqrt{3}}{3}\right)$ $\qquad \sin\left(\tan^{-1}\frac{\sqrt{3}}{3}\right)=\sin\left(\frac{\pi}{6}\right)=\boxed{\frac{1}{2}}$

Note: The range of arctangent is restricted to $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, so $\frac{\pi}{6}$ is the only possible answer for $\tan^{-1}\frac{\sqrt{3}}{3}$.

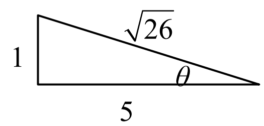
2. $\cos\left(\tan^{-1}1\right)$ $\qquad \cos\left(\tan^{-1}1\right)=\cos\left(\frac{\pi}{4}\right)=\boxed{\frac{1}{\sqrt{2}}}$

3. $\arccos\left(\tan\frac{\pi}{3}\right)$ $\qquad \arccos\left(\tan\frac{\pi}{3}\right)=\arccos\left(\sqrt{3}\right)$ No Solution, because $\sqrt{3}>1$.

Right Triangle Trigonometry and Inverse Trigonometric Functions: the trigonometric functions can be evaluated without having to find the angle

- Label the sides of the right triangle based upon the inverse trig function given
- Evaluate the length of the missing side (Pythagorean Theorem)
- Evaluate the trig function – be sure to choose the correct sign!

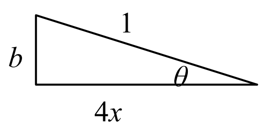
Ex4: Evaluate $\cos\left(\arctan\frac{1}{5}\right)$ without a calculator.



Right Triangle Hypotenuse: $\sqrt{5^2+1^2}=\sqrt{26}$

Let $\theta=\arctan\frac{1}{5}$. Since the range of arctangent is $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, and the tangent is positive, θ must be in Quadrant I. Therefore, cosine is **positive**. So $\cos\theta=\boxed{\frac{5}{\sqrt{26}}}$.

Ex5: Find an algebraic expression equivalent to $\sin\left[\arccos(4x)\right]$.



$b=\sqrt{1-(4x)^2}=\sqrt{1-16x^2}$ $\sin\left[\arccos(4x)\right]=\frac{\sqrt{1-16x^2}}{1}=\boxed{\sqrt{1-16x^2}}$

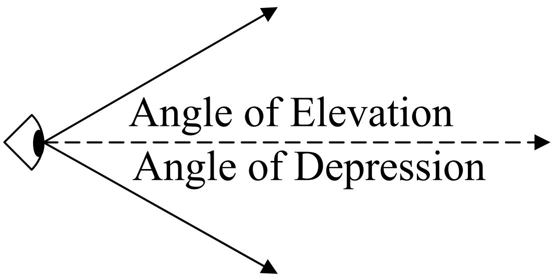
You Try: Evaluate $\cos^{-1}\left(\cos\left(-\frac{\pi}{4}\right)\right)$. Be careful!

QOD: Explain how the domains of sine, cosine, and tangent must be restricted in order to create an inverse function for each.

Syllabus Objective: 4.5 – The student will model real-world application problems involving graphs of trigonometric functions.

Angle of Elevation: the angle through which the eye moves up from horizontal to look at something above

Angle of Depression: the angle through which the eye moves down from horizontal to look at something below

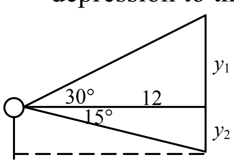



Solving Application Problems with Trigonometry:

- Draw and label a diagram (Note: Diagrams shown are not drawn to scale.)
- Find a right triangle involved and write an equation using a trigonometric function
- Solve for the variable in the equation

Note: Be sure your calculator is in the correct Mode (degrees/radians).

Ex1: If you stand 12 feet from a statue, the angle of elevation to the top is 30°, and the angle of depression to the bottom is 15°. How tall is the statue?

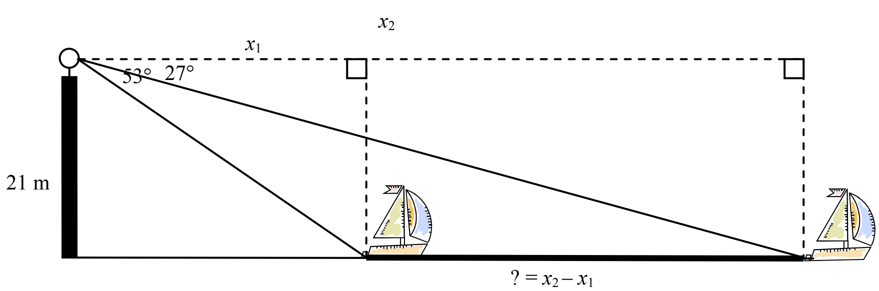


$$\tan 30^\circ = \frac{y_1}{12} \qquad \tan 15^\circ = \frac{y_2}{12}$$

$$y_1 = 12\left(\frac{1}{\sqrt{3}}\right) \approx 6.928 \qquad y_2 = 12\left(\tan 15^\circ\right) \approx 3.215$$

Height of the statue is approximately $6.928 + 3.215 = \boxed{10.143 \text{ ft}}$

Ex2: Two boats lie in a straight line with the base of a cliff 21 meters above the water. The angles of depression are 53° to the nearest boat and 27° to the farthest boat. How far apart are the boats?

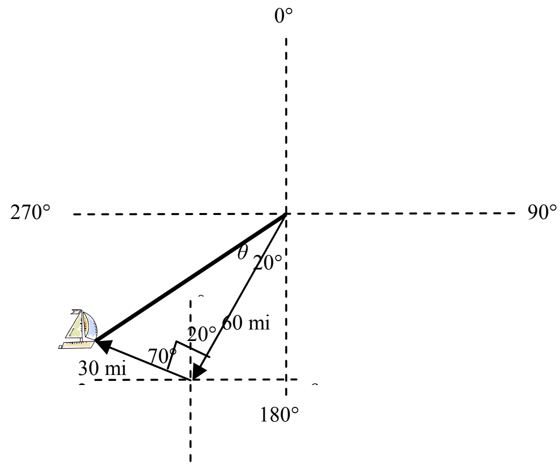


$\tan 53^\circ = \dfrac{21}{x_1}$ $\tan 27^\circ = \dfrac{21}{x_2}$

$x_1 = \dfrac{21}{\tan 53^\circ} \approx 15.825$ $x_2 = \dfrac{21}{\tan 27^\circ} \approx 41.215$

Distance between the boats is approximately $41.215 - 15.825 = \boxed{25.39\,\text{m}}$

Ex3: A boat leaves San Diego at 30 knots (nautical mph) on a course of 200°. Two hours later the boat changes course to 290° for an hour. What is the boat's bearing and distance from San Diego?



Remember: bearing starts N, clockwise

At 30 knots for 2 hours, the boat travels 60 nautical miles. At 30 knots for 1 hour, the boat travels 30 nautical miles.

Bearing: $\tan\theta = \dfrac{30}{60} \Rightarrow \tan^{-1}\left(\dfrac{1}{2}\right) \approx 26.565^\circ$ Bearing = $200^\circ + 26.565^\circ = \boxed{226.565^\circ}$

Distance: $d = \sqrt{30^2 + 60^2} \approx \boxed{67.082 \text{ nautical miles}}$

<u>Simple Harmonic Motion</u>: describes the motion of objects that oscillate, vibrate, or rotate; can be modeled by the equations $d = a\sin(bt)$ or $d = a\cos(bt)$.

Frequency = $\dfrac{b}{2\pi}$; the number of oscillations per unit of time

Ex4: A mass on a spring oscillates back and forth and completes one cycle in 3 seconds. Its maximum displacement is 8 cm. Write an equation that models this motion.

Period = 3: $3 = \dfrac{2\pi}{b} \Rightarrow b = \dfrac{2\pi}{3}$ Amplitude = 8 $\boxed{d = 8\sin\left(\dfrac{2\pi}{3}t\right)}$

You Try: You observe a rocket launch from 2 miles away. In 4 seconds, the angle of elevation changes from 3.5° to 41°. How far did the rocket travel and how fast?

QOD: What is the difference between an angle of depression and an angle of elevation?